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CONSTRUCTION OF CANTOR SETS

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Abstract

This thesis is motivated by the general construction of Cantor sets and the Hausdorff measure of Cantor sets corresponding to different series. We provide background knowledge on the Cantor set and Hausdorff measure, and apply it to the p-series and geometric series on Cantor space, extending the previous work by Cabrelli et al [2]. In the last chapter, we exhibit a construction of a Cantor set associated with Kolmogorov complexity which has Hausdorff dimension 1.

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Chapter 1

Introduction

The notion of Cantor sets was introduced by the German mathematician Georg Cantor (1845-1918) in his paper *On infinite, linear point-manifolds (sets)* [3]. Although the Cantor set has a geometric flavor, Cantor's discovery was motived by a purely arithmetic problem. Cantor's investigation of the uniqueness of trigonometric series representations had led him to consider sets both topologically and measure-theoretically negligible [9]. However, Cantor is not the first to discover Cantor set. Two problems in the point set topology arose during the period 1870-1885, (1) conditions under which a function can be integrated, (2) uniqueness of trigonometric functions [6]. Hermann Hankel (1839-1873) conjectured that a function is Riemann-integrable if and only if it contains countable removable discontinuous points[8]. He claimed that all nowhere dense sets could be enclosed in intervals of arbitrarily small total length, i.e. outer measure zero in modern terminology. We will see it is not the case at the end of this chapter. To understand the notion of infinity and nowhere dense was partly improved by construction of Cantor sets.The Cantor set C has many surprising properties such as uncountable, compact, nowhere dense. It is a remarkable construction that is employed as an illustrating examples in analysis. For example, the Cantor set indicates there exists an uncountable set with Lebesgue measure zero.

The binary Cantor space 2^ω consists of all infinite binary sequences, i.e.

$$2^\omega = \{x = x_0x_1x_2... \mid x_i \in \{0,1\}, \forall i \in \mathbb{N}\}$$

Let $N = \inf\{n : x_n \neq y_n\}$, a metric in this space is given by

$$d(x,y) = \begin{cases} 2^{-N} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The Cantor space is a topological abstraction of the Cantor set. The middle-third Cantor set is homeomorphic to the Cantor space. We can define a mapping ϕ that maps from 2^ω to $[0,1]$. Given an infinite binary sequence, $x = x_0x_1x_2...$,

$$\phi : x \longrightarrow \sum_{i=0}^{\infty} \frac{2x_i}{3^{i+1}}$$

The mapping ϕ gives a homeomorphism from Cantor space to the middle third Cantor set. The common construction is that of the middle third Cantor set, by deleting the open middle thirds of the interval recursively, also known as Cantor ternary set. Here is how to construct the middle third Cantor set. First, the set contains the closed interval $[0,1]$. By deleting $(\frac{1}{3},\frac{2}{3})$, we obtain $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$. Then follow this procedure, delete the middle third of remaining closed intervals repeatedly. Let C_n denote the union of the 2^n closed intervals of length $\frac{1}{3^n}$ at stage n, i.e. $C^0=[0,1], C^1=[0,\frac{1}{3}]\cup[\frac{2}{3},1], C^2=[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$. Let C_i^k be i-th closed interval at the k-th stage, $i=0,1,2,...,2^n-1$. Then the middle third Cantor set C is $\cap_{i=1}^{\infty}C^i$,the infinite intersection of C^n.

Definition 1 (perfect)**.** *A set E is perfect if all points in E are limit points.*

Definition 2 (nowhere dense)**.** *A set E is nowhere dense if it is dense in no intervals. That is, every interval has a subinterval contained in the complement of E.*

We can show E is nowhere dense if its complement E^c contains a dense open set.

The middle-third Cantor set has the following topological properties,

1. *Compact*

   Proof. By definition, the Cantor set is the complement of a union of open sets. The union is open, hence C is closed. C is bounded in $[0,1]$. Therefore, C is compact from the Heine-Borel Theorem. □

2. *Uncountable*

   Proof. Define a mapping ψ that maps from the Cantor set 2^ω to $[0,1]$ on real line. Given an infinite binary sequence, $x=x_0x_1x_2...$,

$$\psi : x \longrightarrow \sum_{i=0}^{\infty}\frac{x_i}{2^{i+1}}$$

   The map is surjective. This can be seen by that every number in $[0,1]$ has a binary expansion. Hence, the cardinality of C is larger than that of $[0,1]$. □

3. *Perfect*

   Proof. Let $x\in C$, $\epsilon>0$, choose n such that $\frac{1}{3^n}<\epsilon$. Let E be the set of endpoints of the F^n intervals. The interval $(x-\epsilon,x+\epsilon)$ contains one of the intervals of C^n, and its endpoints are in the Cantor set. Hence every point in C is a limiting point. □

4. *Totally Disconnected*

   Proof. Pick $x\in C$ and $\epsilon>0$, choose n such that $\frac{1}{3^n}<\epsilon$. Let I be the closed interval in C^n, which contains x, and is of length $\frac{1}{3^n}$. Then $J=C^n\backslash I$ is closed.

$$C=(C\cap I)\cup(C\cap J)$$

   C is the disjoint union of 2 clopen sets. $C\cap I$ is a subset of $(x-\epsilon,x+\epsilon)$. Therefore C is totally disconnected. □

5. *Nowhere Dense*

   Proof. For any interval (a, b), choose n such that $\frac{1}{3^n} < b-a$. Then $(a, b) \cap \bigcup_{i=0}^{2^k-1} C_i^k \neq \emptyset$, it follows that $(a, b) \cap C \neq \emptyset$. □

6. *Lebesgue Measure* 0

   Proof. At stage n, C^n contains the union of the 2^n closed intervals of length $\frac{1}{3^n}$. Therefore the total length L_n is $(\frac{2}{3})^n$. Let $n \to \infty$, then $L_n \to 0$. □

Definition 3 (Cantor space)**.** *A topological space is a Cantor space if it is non-empty, perfect, compact, and totally disconnected, denoted as* 2^ω*.*

In general, we say a set $C \subseteq \mathbb{R}$ is a *Cantor set* if it is homeomorphic to a Cantor space. There are many different constructions of Cantor sets. We will further discuss a general construction of Cantor sets in Chapter 3.

The Property 6 indicates that Lebesgue measure is a not suitable measure for the middle-third Cantor set and generalizations we introduce in Chapter 3. It does not let us distinguish between these sets measure theoretically. This motives us to define Hausdorff measure, introduced in Chapter 2.

Note that the Property 6 does not hold for a "fat Cantor set", which is constructed by removing intervals of length less than $\frac{1}{3}$ of each closed intervals of a "fat Cantor set".
construction. We can define a Cantor set by removing from $[0, 1]$ the middle interval of length $\frac{1}{4}$. From the remaining 2 intervals in C^1, remove the middle intervals of length $\frac{1}{16}$. From the remaining 4 intervals in C^2, remove the middle intervals of length $\frac{1}{64}$. Repeat this process, C^n is constructed from the 2^{n-1} closed intervals of C^{n-1}. The total length of the removed intervals is

$$\frac{1}{4} + 2 * \frac{1}{16} + 4 * \frac{1}{64} + 8 * \frac{1}{256} + ... = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ... = \frac{1}{2}$$

This "fat Cantor set" has positive Lebesgue measure.

Chapter 2

Fractal Geometry

In classical geometry, the dimension of a object is integer, such as 0-dimensional point, 1-dimensional line, 2-dimensional plane. The innovative discussion started from the measure of length of coastlines of England in a paper of Benoit B. Mandelbrot (1924-2010). If the kilometer is used as the unit for measure, it is not accurate; however smaller and smaller unit for measure gives makes the length of coastlines arbitrary large due to the irregularity and infinity of small particles. Later, mathematicians saw that fractal dimension is coastline's invariant property. The dimension of the object does not require to be integer, that is why it is called "fractal". Natural fractals include the shapes of mountains, coastlines and river basins, the structures of plants, blood vessels and lungs; the clustering of galaxies, and Brownian motion. [11]

Hausdorff measure and dimension measures the invariant property of the object, named after Felix Hausdorff. Hausdorff (1868-1942) was a German mathematician who is considered to be one of the founders of modern topology, who selected the neighborhood and a few characteristics to define topological space. Hausdorff introduced Hausdorff dimension and Hausdorff measure in 1919 to study Koch snowflake and other fractals in his paper *Dimension und äusseres Mass* [7].

Further properties of Hausdorff measures have been developed by A. S. Besicovitch (1891-1970). If the part retains the structure or shape of the whole, we say it is self-similar. Many famous fractals have the characteristic of self-similarity. Many self-similar fractals can be obtained through iterated function systems. In this case, the fractal dimensions can often be easily computed [5]. The most famous examples includes a middle third Cantor set, the Sierpinski triangle and the von Koch curve. A part of such an object can be used to recreate the whole object. For example, given the first $\frac{1}{3}$ of the middle third Cantor set, we can scale it by 3 and obtain the original Cantor set.

2.1 Hausdorff Measure

Hausdorff measure is a type of outer measure, i.e. it is monotonic, countable subadditive, and it assigns empty set a measure 0. Hausdorff dimension measures allows for a measure-theoretic investigation of many sets of Lebesgue measure 0. it generalizes the dimension notion, and gives irregular sets a dimension. Unlike classical geometric dimension, the topological Hausdorff dimension can be non-integer.

Claim 1. *The Hausdorff measure is an outer measure.*

Proof of the claim. Need to verify three properties of outer measure. For any $\delta > 0$, we can cover the empty set by a set of diameter ϵ, such that $\epsilon \leqslant \delta$. Therefore $\mathcal{H}^s(\emptyset) \leqslant \epsilon^s$. $\mathcal{H}^s(\emptyset) = \lim_{\delta\to 0} \mathcal{H}^s \leqslant \lim_{\delta\to 0} \delta^s = 0$.
To show monotonicity, let $E \subset F$. Every δ-cover of F is also a δ-cover of E. So $\mathcal{H}^s_\delta(E) \leqslant \mathcal{H}^s_\delta(F), \forall\delta$. Letting $\delta \to 0$, $\mathcal{H}^s(E) \leqslant \mathcal{H}^s(F)$.
To show countable subadditivity, let F_i be an open cover of $E \subset k^\omega$. For $\epsilon > 0$, there is a sequence of open cylinders $I_{i,j}, j = 1, 2, ...$ that covers F_i such that $\sum_j d[I_{i,j}]^s \leqslant \mathcal{H}^s_\delta(F_i) + \frac{\epsilon}{2^i}$. Then $I_{i,j}, i = 1, 2, ... j = 1, 2, ...$ is a δ-cover of $\cup F_i$.

$$\mathcal{H}^s_\delta(\bigcup_{i=1}^{\infty} F_i) \leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} d[I_{i,j}]^s \leqslant \sum_{i=1}^{\infty} \mathcal{H}^s_\delta(F_i) + \frac{\epsilon}{2^i} = \sum_{i=1}^{\infty} \mathcal{H}^s_\delta(F_i) + \epsilon$$

Letting $\epsilon \to 0$, $\mathcal{H}^s_\delta(\bigcup_{i=1}^{\infty} F_i) \leqslant \sum_{i=1}^{\infty} \mathcal{H}^s_\delta(F_i)$ □

Let E be any non-empty subset of $\mathbb{R}^n$, n-dimensional Euclidean Space. The diameter of E, denoted as $\mid E \mid$, is $\sup\{|x - y| : x, y \in E\}$. We say $\{F_i\}$ a δ-cover of E if $E \subset \bigcup F_i$ and $|F_i| < \delta$ for each i. Define

$$\mathcal{H}^s_\delta(E) = \inf\{\sum |F_i|^s : \{F_i\} \text{ is any } \delta\text{-cover of E}\}$$

Let $\delta \to 0$, $\mathcal{H}^s_\delta$ is non-decreasing, since less coverings are admissible. The s-dimensional Hausdorff measure of E is

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E)$$

For $s = 1$, the Hausdorff measure coincides with the Lebesgue measure.

2.2 Hausdorff Dimension

Hausdorff dimension generalizes the concept of dimension in a way such that points have Hausdorff dimension 0, lines have Hausdorff dimension 1, but in general, the Hausdorff dimension is not necessarily an integer. Fractals are defined as sets whose Hausdorff dimension is greater than its topological dimension[5]. It is clear that for any given set $E \subseteq k^\omega$ and $\delta < 1$, $\mathcal{H}^s_\delta(E)$ is non-increasing with s, so $\mathcal{H}^s(E)$ is also non-increasing.

Proposition 1. *Let $E \subseteq \mathbb{R}$. If $\mathcal{H}^s(E) < \infty$ and $\{U_i\}$ is a δ-cover of E, then $\mathcal{H}^t(E) = 0, \forall t > s$.*

Proof. If t>s and $\{U_i\}$ is a δ-cover of E, we have

$$\sum_i \mid U_i \mid^t = \sum_i \mid U_i \mid^{t-s} \mid U_i \mid^s \leqslant \delta^{t-s} \sum_i \mid U_i \mid^s$$

Letting $\delta \to 0$, we see that if $\mathcal{H}^s(E) < \infty$, then $\mathcal{H}^t(E) = 0$, $\forall t > s$. □

Let E be a non-empty set in $\mathbb{R}$. There is a critical value of s where the Hausdorff measure jumps from ∞ to 0. The critical value is Hausdorff dimension for E, which is defined as

$$\dim_H(E) = \inf\{s : \mathcal{H}^s(E) = 0\}$$

Hausdorff dimension satisfies the following properties:

1. *Monotonicity* $E \subseteq F \Longrightarrow \dim_H E \leqslant \dim_H F$. Assume $\dim_H E = k$, we also know $\mathcal{H}^s(E) \leqslant \mathcal{H}^s(F)$ for all s. $0 \leqslant \mathcal{H}^k(E) \leqslant \mathcal{H}^k(F)$. Hence $k \leqslant dim_H F$.

2. *Countable Stability* If $E_1, E_2, ...$ is a countable sequence of sets, then $\dim_H \bigcup F_i = \sup \dim_H F_i$. It implies that countable set has Hausdorff dimension zero.

Proof of Countable Stability. Certainly $\dim_H \bigcup F_i \geqslant \dim_H F_j$ by monotonicity. For the opposite inequality, if $s > \dim_H F_i$ for all i, then $\mathcal{H}^s(F_i) = 0$, so that $\mathcal{H}^s(\bigcup_{i=1}^{\infty} F_i) = 0$. □

2.3 Self-Similarity

The following theorem facilitates our calculation of Hausdorff dimension if the set is self-similar.

Definition 4 (contraction mapping)**.** *Let D be a closed subset of $\mathbb{R}^n$. A mapping $S : D \to D$ is called contraction mapping if there exists a number $0 < c < 1$ such that $\mid S(x) - S(y) \mid \leqslant c \mid x - y \mid, \forall x, y \in D$.*

A finite family of contractions $\{S_1, S_2, ..., S_m\}$ with $m \leqslant 2$, is called an iterated function system (IFS). Any family of contractions as above has a *limit set.*

Theorem 1. *Let $S_1, S_2, \ldots, S_m$ be contractions on D. Then there exists a unique non-empty compact set F such that F is invariant for the S_i, i.e. it satisfies*

$$F = \bigcup_{i=1}^{m} S_i(F).$$

A proof can be found in Falconer [5], Theorem 9.1.

Definition 5 (similarities)**.** $S_1, S_2, ..., S_m : \mathbb{R}^n \to \mathbb{R}^n$ *are similarities if*

$$|S_i(x) - S_i(y)| = c_i|x - y|$$

where $0 < c_i < 1$. Each S_i transforms subsets of $\mathbb{R}^n$ into geometrically similar sets.

It is clear that every similarity as defined above is a contraction, but not necessarily vice versa. If a family of similarities satisfies a certain topological condition, one has a nice formula to compute the Hausdorff dimension of the limit set.

Definition 6 (open set condition)**.** *A finite family of contractions $\{S_i\}$ satisfies the open set condition if there exists a non-empty bounded open set V such that $V \supseteq \cup_{i=1}^{m} S_i(V)$, with the union disjoint.*

Theorem 2. *Suppose the open set condition holds for the similarities S_i on $\mathbb{R}^n$ with ratios $0 < c_i < 1$ for $1 \leqslant i \leqslant m$. If $F = \cup_{i=1}^m S_i(F)$, then $\dim_H F = s$, where s is given by*

$$\sum_{i=1}^{m} c_i^s = 1$$

A proof is given in Theorem 9.3 of Falconer[5]. Now I will calculate the Hausdorff dimension of the classical Cantor set by the theorem.

Example 1 (Hausdorff dimension of the middle-third Cantor set)**.** *There are different sets of similarity transformations on $\mathbb{R}$ which the middle third Cantor set is the attractor. For example*

$$S_1(x) = \frac{x}{3} \quad S_2(x) = \frac{x}{3} + \frac{2}{3} \tag{2.1}$$

$$S_1(x) = \frac{x}{9} \quad S_2(x) = \frac{x}{9} + \frac{2}{9} \quad S_3(x) = \frac{x}{3} + \frac{2}{3} \tag{2.2}$$

By the Theorem 1, S has a unique invariant set F. So we can pick set (1) *to compute. For the first set of transformation, $c_1 = c_2 = \frac{1}{3}$, $2(\frac{1}{3})^s = 1$, so that $s = \frac{\log 2}{\log 3}$.*

Chapter 3

General Cantor Sets

After providing the necessary background of the Hausdorff measure, we want to apply it to general Cantor sets.

A general Cantor space is a generalization of the canonical Cantor space 2^ω. We name it h-Cantor space, denoted as h^ω, associated with the assignment function h. Let $h : \mathbb{N}\backslash\{0\} \longrightarrow \mathbb{N}\backslash\{0\}$ be a function that assigns the number of cuts at each stage i, $i = 1, 2, 3, ...$ $(h(i)+1)$. More intuitively, we can see this as a branching process. Start from 1 individual, this individual produces $(h(1) + 1)$-many branchings, which are individuals of generation 1. At stage $(i - 1)$, each individual of generation $(i - 1)$ produces $(h(i) + 1)$-many new branches.

The topological nature does not change for an h-Cantor space, i.e. it is non-empty, perfect, compact, totally disconnected. Every Cantor space is homeomorphic to the middle-third Cantor set, which was shown by Moore and Kline (see Theorem 9.1 of [12]). Hence, a general Cantor set also is homeomorphic to 2^ω.

To specify Cantor *sets*, i.e. subsets of the real line, we additionally provide the lengths of the intervals that are cut out at each stage. This is given by a sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ of positive numbers so that $\sum_i \lambda_i = L < \infty$.

Stage 0 start with a closed interval I_0^0 of length L, $[0, L]$.

Stage 1 remove $h(1)$-many disjoint open intervals from left to right with length $\lambda_1, \lambda_2, ..., \lambda_{h(1)}$ from I_0^0. We will see later that the location of each cut is uniquely determined by the sequence $\{\lambda_k\}$ and the function h. This leaves $(h(1) + 1)$-many closed intervals, denoted by $I_0^1, I_1^1, ...I_{h(1)}^1$.

Stage k remove $h(k)$-many open intervals from each of the $\prod_{i=1}^{k-1}(h(i) - 1)$ closed intervals at the end of stage $(k - 1)$. We have $\prod_{i=1}^{k}(h(i) + 1)$-many closed intervals.

Let C_λ^k be the union of the closed intervals at stage k,

$$C_\lambda^k = \bigcup_{l=0}^{(\prod_{i=1}^{k}(h(i)+1))-1} I_l^k$$

The general Cantor set with sequence λ is

$$C_\lambda = \bigcap_{k=0}^{\infty} C_\lambda^k$$

Through this general construction, the resulting set is compact, uncountable, nowhere dense, and totally discounted.

Proposition 2. *A general Cantor set is nowhere dense.*

Proof. For any interval $(a, b), b > a$, choose k such that $L - \sum_{i=0}^{k} \lambda_i < b - a$, Then $(a, b) \cap \bigcup_{i=0}^{2^k-1} I_i^k \neq \emptyset$, it follows that $(a, b) \cap C \neq \emptyset$. We showed every interval has a subinterval completely contained in C^c. □

We now define the Hausdorff measure in Cantor space. Recall the construction of general Cantor sets, we fix an assignment function $h(i)$ constant. Here the corresponding topological space is homeomorphic to k^ω. Let $x, y \in k^\omega, M = \min\{i : x(i) \neq y(i)\}$. The corresponding metric is given as

$$d(x, y) = \begin{cases} k^{-M} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

To check the d is a valid metric funtion, that satisfies (i)$d(x, y) \geqslant 0$, (ii)$d(x, y) = d(y, x)$, (iii)$d(x, y) \leqslant d(x, z) + d(z, y)$. Properties (i) and (ii) are trivial by the definition of d.

Claim 2. *That triangular inequality is satisfied.*

proof of the claim. Let $x, y, z \in k^\omega$, $x \neq y$, otherwise it is trivial. Let $M = \min\{i : x(i) \neq y(i)\}$, $N = \min\{i : x(i) \neq z(i)\}$. If $N < M$, $k^{-M} < k^{-N}$. If $N \geqslant M$, it implies $d(y, z) = d(x, y) = 2^{-M}$. So $d(x, y) \leqslant d(x, z) + d(y, z)$ since $d(x, z) \leqslant 0$. □

In Cantor space, the basis of the topology is formed by the open cylinders. In order to apply Hausdorff measure to the generalized Cantor space k^ω, we further define the cylinder on k^ω.

Definition 7 (Cylinder)**.** *Let σ be a finite binary sequence, i.e. $\sigma = \sigma_0\sigma_1\sigma_2...\sigma_{n-1}$. We define the cylinder $[\sigma]$ as*

$$[\sigma] = \{x \in k^\omega : \forall i < n, x_i = \sigma_i\}$$

The diameter of the cylinder, $d[\sigma]$, is the maximum distance between any two elements in $[\sigma]$. The length of σ is n, denoted as $|\sigma|$. Note that d$[\sigma]$=$k^{-|\sigma|}$.

Let E be a non-empty subset of k^ω. E is covered by $\{F_i\}$ of finite binary strings if and only if $E \subset \bigcup [F_i]$. Let $\delta = k^{-n}$, then $\{F_i\}$ is a $\delta(n)$-cover of E is equivalent to $|F_i|$>n for all i.

$$\mathcal{H}^s_{\delta(n)}(E) = \{\sum |F_i|^s : E \subset \bigcup F_i, \mid F_i \mid > n\}$$

Then the s-dimensional Hausdorff measure of E on the Cantor space is

$$\mathcal{H}^s(E) = \lim_{n\to\infty} \mathcal{H}^s_{\delta(n)}(E)$$

Chapter 4

Hausdorff Measure of p-series in Cantor Space

We have shown that Hausdorff measure is suitable for Cantor sets associated with various sequences. In this chapter, we focus on Hausdorff measure of Cantor sets with p-series and extend work by Cabrelli, Molter, Paulauskas, and Shonkwiler [2]. We generalized the case from $h(k) = 1$ to $h(k)$ arbitrary positive constant integer function. Sets of similar structure arise when studying the set of extremal points of the boundaries of the so-called random stable zonotopes[2]. They were motived by finding limited distribution for normed sums $b_n^{-1} \sum_{i=1}^{n} \psi_i$, where ψ_i are i.i.d random convex compact set in a given separable Banach space, and summation is defined in a sense of Minkowski [4].

We will consider $h(k) = r - 1, k = 1, 2, ...,$ where r is a constant natural number greater than 1. First we will show that the location of each cut is unqiuely determined by $\{\lambda_k\}$. By observation,

$$| I_l^k |= \sum_{i=0}^{r-1} | I_{rl+i}^{k+1} | + \sum_{j=0}^{r-2} \lambda_{r^k+l(r-1)r^{n-k}+j}$$

Generally, for $k = 1, 2, 3, ..., l = 0, 1, 2, ..., r^k - 1$,

$$| I_l^k | = \sum_{h=k}^{\infty} \sum_{j=l(r-1)r^{n-k}}^{(l+1)(r-1)r^{n-k}-1} \lambda_{r^n+j} \tag{4.1}$$

$$= \sum_{h=0}^{\infty} \sum_{j=0}^{(r-1)r^h-1} \lambda_{r^{k+h}+l(r-1)r^h+j} \tag{4.2}$$

It means the length of each interval at stage k is determined. So we see the positions of gaps removed from intervals are also determined.

Let's consider the sequence $\lambda = \{k^{-p}\}$, where $p \geqslant 1$.

Lemma 1. *For k=1,2,... and l=0,1,..., $r^k - 1$,*

$$\frac{r^p(r-1)}{r^p - r} \lambda_{r^k+(l+1)(r-1)} \leqslant | I_l^k | \leqslant \frac{r^p(r-1)}{r^p - r} \lambda_{r^k+l(r-1)}$$

From this, for $l' \geqslant l$,

$$1 \leqslant \left| \frac{I_k^l}{I_k^{l'}} \right| \leqslant r^p$$

Proof. By the equation (2),

$$\begin{aligned}
\mid I_l^k \mid &= \sum_{h=0}^{\infty} \sum_{j=0}^{(r-1)r^h-1} \lambda_{r^{k+h}+l(r-1)r^h+j} \\
&\leqslant \sum_{h=0}^{\infty} \frac{(r-1)r^h}{r^{(k+h} + l(r-1)r^h)^p} \\
&= \sum_{n=0}^{\infty} \frac{r^h(r-1)}{r^{hp}(r^k + l(r-1))^p} \\
&= \frac{r-1}{(r^k + l(r-1))^p} \sum_{h=0}^{\infty} \frac{1}{r^{(p-1)h}} \\
&= \frac{r^p(r-1)}{r^p - r} \frac{1}{(r^k + l(r-1))^p} \\
&= \frac{r^p(r-1)}{r^p - r} \lambda_{r^k+l(r-1)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mid I_l^k \mid &= \sum_{h=0}^{\infty} \sum_{j=0}^{(r-1)r^h-1} \lambda_{r^{k+h}+l(r-1)r^h+j} \\
&\geqslant \sum_{h=0}^{\infty} \frac{(r-1)r^h}{(rk + h + (l+1)(r-1)r^h - 1)^p} \\
&= \sum_{h=0}^{\infty} \frac{(r-1)r^h}{r^{hp}(r^k + (l+1)(r-1) - \frac{1}{r^h})^p} \\
&\geqslant \frac{r^p(r-1)}{r^p - r} \frac{1}{(r^k + (l+1)(r-1))^p} \\
&= \frac{r^p(r-1)}{r^p - r} \lambda_{r^k+(l+1)(r-1)}
\end{aligned}$$

Note that

$$I_l^k \geqslant \frac{r^p(r-1)}{r^p - r} \lambda r^k + (l+1)(r-1) \geqslant I_{l+1}^k$$

Inductively, $I_l^k \geqslant I_{l'}^k$, for any $l' \geqslant l$. For the upper bound of $\left|\frac{I_l^k}{I_{l'}^k}\right|$,

$$\left|\frac{I_l^k}{I_{l'}^k}\right| \leqslant \left|\frac{I_0^k}{I_{r^k-1}^k}\right| \leqslant \frac{\frac{r^p(r-1)}{r^p-r}\lambda_{r^k}}{\frac{r^p(r-1)}{r^p-r}\lambda_{r^k+r^k(r-1)}} = r^p$$

$\square$

With the same technique, we can show the following lemma, which is useful in later theorm proof.

Lemma 2. *For k=1,2,... and l=0,1,..., r^k-1,*

$$\frac{r^p}{r^p-r}\sum_{i=0}^{r-2}\lambda_{r^k+(r-1)(l+1)+i} \leqslant \left|I_l^k\right| \leqslant \frac{r^p}{r^p-r}\sum_{i=0}^{r-2}\lambda_{r^k+l(r-1)+i}$$

Lemma 3. *Let $x=a_0+a_1+a_3+...+a_{2n-1}+a_{2n}, n\in\mathbb{N}$, a_i arbitrary positive number, $p>1$. Then*

$$x\leqslant\frac{(n+1)^p}{(n+1)^p-(n+1)}(a_1+a_2+...+a_{2n-1})\Longrightarrow x^{\frac{1}{p}}\geqslant a_0^{\frac{1}{p}}+a_2^{\frac{1}{p}}+...+a_{2n}^{\frac{1}{p}}$$

Proof.

$$\begin{aligned} x\leqslant\frac{(n+1)^p}{(n+1)^p-(n+1)}\sum_{i=1}^{n}a_{2i-1} \quad & \Rightarrow \quad (x-\sum_{i=1}^{n}a_{2i-1})(n+1)^p\leqslant(n+1)x^n \\ & \Rightarrow \quad \frac{\sum_{i=0}^{n}a_{2i}}{n+1}\leqslant\frac{x}{(n+1)^p} \\ & \Rightarrow \quad (\frac{\sum_{i=0}^{n}a_{2i}}{n+1})^{\frac{1}{p}}\leqslant\frac{x^{\frac{1}{p}}}{n+1} \end{aligned}$$

By the convexity of $x^{\frac{1}{p}}$, which means for any $x,y\in\mathbb{R}, 0<\theta<1$,

$$f[\theta x+(1-\theta)y]\leqslant\theta f(x)+(1-\theta)f(y)$$

we can show the inequality inductively

$$\begin{aligned} (\frac{a_0+a_2+...+a_{2n}}{n+1})^{\frac{1}{p}} & \geqslant \frac{a_0^{\frac{1}{p}}}{n+1}+\frac{n}{n+1}(\frac{a_2+a_4+...+a_{2n}}{n})^{\frac{1}{p}} \\ & \geqslant \frac{a_0^{\frac{1}{p}}}{n+1}+\frac{n}{n+1}(\frac{a_2^{\frac{1}{p}}}{n}+\frac{n-1}{n}(\frac{a_4+...+a_{2n}}{n-1})^{\frac{1}{p}} \\ & ... \\ & \geqslant \frac{a_0^{\frac{1}{p}}}{n+1}+\frac{a_2^{\frac{1}{p}}}{n+1}+...\frac{a_{2n}^{\frac{1}{p}}}{n+1} \end{aligned}$$

□

Lemma 4. *For all k=1,2,3,... and l=0,1,...,r^k-1,*

$$\mid I_l^k\mid\geqslant\sum_{i=0}^{r-1}\mid I_{rl+i}^{k+1}\mid^{\frac{1}{p}}$$

Proof. We know by observation,

$$\mid I_l^k \mid = \sum_{i=0}^{r-1} \mid I_{rl+i}^{k+1} \mid + \sum_{i=0}^{r-2} \lambda_{r^n+l(r-1)+i}$$

By Lemma 2, we have

$$\left|I_l^k\right| \leqslant \frac{r^p}{r^p - r} \sum_{i=0}^{r-2} \lambda_{r^k+l(r-1)+i}$$

Then by the previous lemma, we get the desired result. □

Lemma 5. *Let J be an arbitrary open interval in I_0, then let k_1 fixed, then*

$$(r+2) \mid J \mid^{\frac{1}{p}} \geqslant \sum_{l: I_l^{k_1} \subset J} \mid I_l^{k_1} \mid^{\frac{1}{p}}$$

Proof. Let $k_0 = \min\{k \in \mathbb{N} : I_l^k \subset J \text{ for some } 0 \leqslant l \leqslant r^k - 1\}$. We observe that J can at most intersect $(2r-2)$-many intervals at stage k_0. We will consider the case that J contains exactly r intervals at k_0, and not all r intervals come from the same I_m^k for some m. We denote the first interval contained in J at stage k_0 be $I_l^{k_0}$. Let $\tilde{I}$ be the smallest closed interval that contains those r intervals, $I_l^{k_0}, I_{l+1}^{k_0}, ..., I_{l+r-1}^{k_0}$. We want to first show that

$$\mid \tilde{I} \mid^{\frac{1}{p}} \geqslant \sum_{i=0}^{r-1} \mid I_{l+i}^{k_0} \mid^{\frac{1}{p}}$$

Given the claim is true, we have

$$\mid J \mid^{\frac{1}{p}} \geqslant \mid \tilde{I}^{\frac{1}{p}} \geqslant \sum_{i=0}^{r-1} \mid I_{l+i}^{k_0} \mid^{\frac{1}{p}}$$

By the Lemma 1, $\mid I_{l-1}^{k_0} \mid \leqslant r^p \mid I_l^{k_0} \mid \leqslant r^p \mid J \mid$, which implies

$$\mid I_{l-1}^{k_0} \mid^{\frac{1}{p}} \leqslant r \mid I_l^{k_0} \mid \leqslant r \mid J \mid^{\frac{1}{p}}$$

$$\mid I_{l+r}^{k_0} \mid^{\frac{1}{p}} \leqslant \mid I_{l+r-1}^{k_0} \mid^{\frac{1}{p}} \leqslant \mid J \mid^{\frac{1}{p}}$$

Sum the equations above, the result is obtained,

$$(r+2) \mid J \mid^{\frac{1}{p}} \geqslant \sum_{i=0}^{r-1} \mid I_{l+i}^{k_0} \mid^{\frac{1}{p}} + \mid I_{l-1}^{k_0} \mid^{\frac{1}{p}} + \mid I_{l+r}^{k_0} \mid^{\frac{1}{p}}$$

For all $k_1 < k_0$, there is no intervals at stage k_1 contained in J, so that the summation is zero. For all $k_1 \geqslant k_0$, by Lemma 4, we have

$$(r+2) \mid J \mid^{\frac{1}{p}} \geqslant \sum_{l: I_l^{k_1} \subset J} \mid I_l^{k_1} \mid^{\frac{1}{p}}$$

□

Theorem 3. *Let $\lambda = \{\lambda_k\}$ be defined by $\lambda_k = (\frac{1}{k})^p, p > 1$. Then*

$$\mathcal{H}^{\frac{1}{p}}(C_\lambda) \geqslant \frac{1}{r+2} \mid \frac{r-1}{r^p - r} \mid^{\frac{1}{p}}$$

Proof. Let $\{F_i\}$ be a δ-cover of C_λ, i.e. $\cup F_i \supset C_\lambda$ and $\mid F_i \mid < \delta$. Since C_λ is compact, let $\{F_{h_i}\}_{i=1}^m$ be a finite subscover of C_λ. Let $\epsilon > 0, \mathbb{R}\backslash C_\lambda$ is dense so we can construct open intervals E_j such that

$$F_{h_j} \subset E_j \text{and} \mid E_j \mid^{\frac{1}{p}} < \mid F_{h_j} \mid^{\frac{1}{p}} + \frac{\epsilon}{m}$$

Hence

$$\sum_{j=1}^{m} \mid E_j \mid^{\frac{1}{p}} < \sum_{j=1}^{m} \mid F_{h_j} \mid^{\frac{1}{p}} + \epsilon$$

Since ϵ is arbitrary, using Lemma 5

$$\begin{aligned} \sum_{j=1}^{m} \mid E_j \mid^{\frac{1}{p}} &\geqslant \sum_{j=1}^{m} \frac{1}{r+2} \sum_{i=0}^{r^k-1} \mid I_l^k \mid^{\frac{1}{p}} \\ &\geqslant \frac{1}{r+2} \sum_{l=0}^{r^k-1} \mid I_l^k \mid^{\frac{1}{p}} \\ &\geqslant \frac{r^k}{r+2} \mid \frac{r^p(r-1)}{r^p - r} \lambda_{r^k+(r^k-1+1)(r-1)} \mid^{\frac{1}{p}} \\ &= \frac{1}{r+2} \mid \frac{r-1}{r^p - r} \mid^{\frac{1}{p}} \end{aligned}$$

□

We can find the upper bound of Hausdorff dimension by some trick. The following two propositions are given by Cabrelli, Molter, Paulauskas, and Shonkwiler [2]. It holds true for any any assignment function $h : \mathbb{N}\backslash\{0\} \to \mathbb{N}\backslash\{0\}$, i.e. h does not need to be constant.

Proposition 3. *Let $\lambda = \{\lambda_k\}$ be defined by $\lambda_k = (\frac{1}{k})^p, p > 1$. Then*

$$\mathcal{H}^{\frac{1}{p}}(C_\lambda) \leqslant (\frac{1}{p-1})^{\frac{1}{p}}$$

Proof. Let $\delta > 0$, consider n large enough such that $\sum_{i=n+1}^{\infty} \lambda_i \leqslant \delta$. It means we removed n open intervals from I_0, and still have $(n+1)$ closed intervals that cover general Cantor set C_λ. Let $E_j, j = 0, 1, ..., n$ be those $(n+1)$ intervals, then $\{E_j\}$ forms a δ-covering of C_λ.

$$\begin{aligned} \sum_{j=0}^{n} \mid E_j \mid^s &\leqslant (\sum_{j=0}^{n} \mid E_j \mid)^s (n+1)^{1-s} && \text{by Holder's inequality} \\ &= (\sum_{i=n+1}^{\infty} j^{-p})^s (n+1)^{1-s} \\ &\leqslant (\frac{n^{1-p}}{p-1})^s (n+1)^{1-s} && \text{by integral comparison test} \\ &= (\frac{1}{p-1})^s \frac{(n+1)^{1-s}}{n^{sp-s}} \end{aligned}$$

Let $n \to \infty$, which implies $\delta \to 0$, if $s > \frac{1}{p}$, $\mathcal{H}^s(C_\lambda) \leqslant (\frac{1}{p-1})^s \frac{(n+1)^{1-s}}{n^{sp-s}} = 0$. Therefore $\mathcal{H}^{\frac{1}{p}}(C_\lambda) \leqslant (\frac{1}{p-1})^{\frac{1}{p}}$. □

Together with Theorem 1, $\frac{1}{r+2} \mid \frac{r-1}{r^p - r} \mid^{\frac{1}{p}} \leqslant \mathcal{H}^{\frac{1}{p}}(C_\lambda) \leqslant (\frac{1}{p-1})^{\frac{1}{p}}$, we have shown that $\dim C_\lambda = \frac{1}{p}$ by the definition of Hausdorff dimension.

The next proposition shows any Cantor sets associated with a sequence of geometric decay has Hausdorff dimension 0. The proposition implies all Cantor set associated with geometric decay has Hausdorff dimension 0.

Definition 8 (Geometric Decay). *A sequence of positive numbers x_k has geometric decay if there exist $c > 0$ and $0 < d < 1$ such that $a_k \leqslant cd^k$ for all k.*

Proposition 4. *Let $\{\lambda_k\}$ be a sequence that satisfies geometric decay, then C_λ on h-Cantor space has Hausdorff dimension 0.*

Proof. Let $\epsilon > 0$ and $0 < \lambda_k \leqslant cd^k, d < 1$. Let $\delta > 0$, consider n large enough such that $\sum_{i=n+1}^{\infty} \lambda_i \leqslant \delta$. Let $E_j, j = 0, 1, ..., n$ be those $(n+1)$ intervals, then $\{E_j\}$ forms a δ-covering of C_λ.

$$\begin{aligned} \sum_{j=0}^{n} \mid E_j \mid^\epsilon &\leqslant (\sum_{j=0}^{n} \mid E_j \mid)^s (n+1)^{1-\epsilon} \qquad \text{by Holder's inequality} \\ &= (\sum_{i=n+1}^{\infty} cd^i)^\epsilon (n+1)^{1-\epsilon} \\ &\leqslant (\frac{cd^{n+1}}{1-d})^\epsilon (n+1)^{1-\epsilon} \end{aligned}$$

Let $n \to \infty$, $(\frac{cd^{n+1}}{1-d})^\epsilon (n+1)^{1-\epsilon} < \infty$. Since it is true for every ϵ, $\dim \mathcal{H}_\lambda = 0$. □

Hence for geometric sequences, Hausdorff dimension cannot distinguish between various rates of decay anymore. However, one can generalize Hausdorff measure to obtain another measure that is more informative. Let $p : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and continuous function. We call it a dimension function. Let E be a set in Cantor space. Define

$$\mathcal{H}_\delta^p = \inf\{\sum p(|F_i|) : \{F_i\} \text{is a } \delta\text{-cover of } E\}$$

This induces a measure associated with p

$$\mathcal{H}^p(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^p(E)$$

The next claim asserts that one can also generalize the notion of Hausdorff dimension to some extent.

Claim 3. *p and q are dimension functions such that $\frac{p(t)}{q(t)} \to 0$ as $t \to 0$. If $\mathcal{H}^q(E) \leq \infty$, then $\mathcal{H}^p(E) = 0$*

Proof. Let $\delta > 0$ and $\{F_i\}$ be a δ-cover of E.

$$\sum_i p(|F_i|) = \sum_i \frac{p(|F_i|)}{q(|F_i|)} q(|F_i|)$$

Since $\lim_{t \to 0} \frac{p(t)}{q(t)}$, $\sup_{0<t<\delta} \frac{p(t)}{q(t)} \to 0$ as $\delta 0$. Let $k(\delta)$=$\sup_{0<t<\delta} \frac{p(t)}{q(t)}$, then

$$\sum_i p(\mid F_i \mid) \leqslant k(\delta) \sum_i q(\mid F_i \mid) \Longrightarrow \mathcal{H}^p_\delta(E) \leqslant k(\delta)\mathcal{H}^q_\delta(E)$$

Let $\delta \to 0, \mathcal{H}^p = 0$ □

For the desired Hausdorff measure, $h(x) = x^s$. As seen above, this family of dimension functions does not capture Cantor sets defined by sequences of geometric decay. However, Cabrelli, Mendivil, Molter, and Shonkwiler [1] showed that *any* Cantor set constructed through a positive non-increasing sequence has finite, positive $\mathcal{H}^p$ measure for *some* continuous, concave dimension function p.

Chapter 5

Kolmogorov Complexity

We have shown in the Chapter 4 that the Cantor set associated with p-series has Hausdorff dimension $\frac{1}{p}$, where p is larger than 1, in which way the series converges. Now one can ask whether there is any series that yields a Cantor set of Hausdorff dimension 1. The answer is yes. We will consider a Cantor set constructed using Kolmogorov complexity, and prove it has Hausdorff dimension 1.

The notion of Kolmogorov complexity has its roots in probability theory, information theory, and philosophical notions of randomness and became mature with the development of the theory of algorithm. A. N. Kolmogorov (1903-1987) used the theory of effective computability to resolve foundational problems concerning the notion of a random sequence. [10]. The theory of Kolmogorov complexity can be regarded as an effective version of entropy [13]. More recently, a deep relationship between Kolmogorov complexity and effective Hausdorff dimension has been discovered.

Plain Kolmogorov complexity C measures the randomness of an individual binary string. Intuitively, $C(x)$ is the length of the shortest program for a universal Turing machine that outputs x. However, plain complexity has some technical shortcomings: it is not subadditive and is nonmonotonic over prefixes.

Therefore, in many applications, plain complexity is replaced by *prefix free complexity*. We denote the prefix-free Kolmogorov complexity by K.

A universal Turing machine is a Turing machine that can simulate all other Turing machines. We can construct such a Turing machine since there exists an effective enumeration of all Turing machines $T_1, T_2, T_3, \ldots$ (see Example 1.7.3 of Li and Vitanyi [10]. There are infinitely many universal Turing machines.

A machine is *prefix-free* if no two converging inputs are prefixes of one another. One can construct a prefix-free Turing machine that is universal for all other prefix-free machines in way similar to the construction of a standard universal machine. For example, we can construct a Turing machine that expects inputs of format 1^i0p. First, it calls the ith prefix-free Turing machine T_i, and then runsTuring machine T_i on input p. For further details, see again [10].

Definition 9 (prefix-free Kolmogorov complexity)**.** *Given a prefix-free universal Turing ma-*

chine U, then the prefix-free Kolmogorov complexity K *of a binary string x is defined as*

$$K(x) = \min\{|p| : U(p) = x\}$$

A sequence (a_n) of real numbers is *approximable from below* if there exists a computable function $f(n, m)$ such that for each n, m, $f(n, m) \leqslant f(n, m + 1)$ and $\lim_{m\to\infty} f(n, m) = a_n$.

Given a natural number n, let x_n be the binary representation of n. This way we can define the (prefix-free) Kolmogorov complexity $K(n)$ of the natural number n as $K(x_n)$. The positive sequence $m(n) = 2^{-K(n)}$ has two important properties:

1. $m(n)$ is approximable from below.

2. $\sum_{n\in\mathbb{N}} m(n) < 1$.

It is fundamental result independently due to Levin, Gacs, and Chaitin (see e.g. [10]), that $m(n)$ is optimal among the sequences with this property, in the following sense. Say that a sequence $\{a_n\}$ is of *lower order* than a sequence $\{b_n\}$, denoted as $a \prec b$ if $\exists k > 0$ s.t. $\frac{a_n}{b_n} < k, \forall n \in \mathbb{N}$.

Theorem 4 (Levin; Gacs; Chaitin)**.** *If $\{a_n\}$ is so that $\sum_{n\in\mathbb{N}} a_n < \infty$ and $\{a_n\}$ is approximable from below, then $\{a_n\}$ is of lower order than $\{m(n)\}$.*

The connection with Hausdorff dimension of Cantor sets is now given by following theorem due to Cabrelli et al [1].

Theorem 5. *Let C_a and C_b be Cantor sets associated to the sequences $\{a_n\}$ and $\{b_n\}$, then we have*

$$a \prec b \Longrightarrow \dim_H(C_a) \leqslant \dim_H(C_b)$$

We can combine both results to construct a Cantor set with Hausdorff dimension 1.

Theorem 6. *Let $m(n) = 2^{-K(n)}$. The Cantor set associated with m, C_m, has Hausdorff dimension 1.*

Proof. Let $p > 1, p \in \mathbb{Q}$, $\alpha = \{(\frac{1}{n})^p\}_{n\in\mathbb{N}}$. The sum of the p-series α is convergent, and α_n can be approximated from below from for each n. By the Theorem 4, $m(x)$ is of higher order then α, $m(n) \succ \alpha_n$. By the results presented in Chapter 4, $\dim_H(C_\alpha) = \frac{1}{p}$.

Hence $1 \geqslant \dim_H(C_m) \geqslant \frac{1}{p}$ for any rational $p > 1$. It follows that $\dim_H(C_m) = 1$. □

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Time Series Analysis STAT 463H Fall 2011
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- Apply ARCH and GARCH modeling to daily stock index data. Analysis based on ACF, Box-Pierce, Jarque-Bera and other tests. Write code with R to fit the data into ARCH and GARCH modeling, which gives excellent account for financial data.

Linear Algbra MATH 441H Fall 2011
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- Write report on determinant from the aspect of tensor and multilinear map. Give new proofs of determinant properties from that view.
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- Derive an algorithm to compute all roots of the polynomial with nested multiplication.
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CISCO Systems, Inc. Engineer-Summer Intern Summer 2011
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- Established the Premium SmartNet wiki site for internal use, modified the site based on feedback from colleague
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