THE PENNSYLVANIA STATE UNIVERSITY
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SCHOOL OF ENGINEERING

A COMPUTATIONAL ANALYSIS OF THE FRICTION PENDULUM BASE ISOLATION SYSTEM

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Abstract

Each year, earthquakes occur globally and at times cause serious damage to buildings, bridges, and other structures. As a result, seismic base isolation devices have been developed to dampen vibration caused by seismic activity and prevent serious structural damage by changing the stiffness of the structure. One type of isolation device is the friction pendulum bearing. The curved sliding surfaces of the friction pendulum bearing are one of the defining features of the friction pendulum. The friction between the sliding surfaces of the pendulum provides the necessary damping while the curved surface alters the natural frequency of the structure.

However, the effect of each of the design parameters on the isolation system response is not well understood. These design parameters include the radius of curvature, coefficient of friction, the mass of the structure being supported, and the design displacement for the pendulum. A one dimensional mathematical model is developed to further understand how these parameters affect the response. To simplify the modeling, an energy approach is utilized to create an equivalent spring and viscous damper system that will mimic the friction pendulum bearing system response with coulomb damping.

From the model, several important conclusions can be made including the realization that the effective spring constant is a function of not only the radius of curvature but the coefficient of friction. Also, the natural frequency of the isolation system is independent from the mass of the supporting structure. The understanding of the design parameters can then be utilized to optimally design a friction pendulum bearing isolation system for a specific application that will minimize the shearing displacement of the structure during an earthquake.
# Table of Contents

Abstract ............................................................................................................................... i

List of Figures .................................................................................................................... iv

List of Tables ...................................................................................................................... vii

Acknowledgments ........................................................................................................... viii

Introduction ......................................................................................................................... 1
  Modern Seismic Base Isolation Systems ................................................................. 1
  Seismic Base Isolation System Comparisons ......................................................... 2
  Friction Pendulum Applications............................................................................. 3
    Base Isolation for Bridges ...................................................................................... 3
    Retrofit to Buildings ............................................................................................... 4
  Project Overview ......................................................................................................... 4

Mathematical Modeling of Friction Pendulum Bearing ........................................... 4
  Mathematical Modeling of Coulomb Damping ....................................................... 4
  Linear System with Coulomb Damping .................................................................. 5
  Rotational System with Coulomb Damping for Free Vibration .................... 8
  Equivalent Viscous Damping System for Free Vibration .................................. 13
  Equivalent Viscous Damping System for Forced Vibration .......................... 17

Mathematical Modeling of Friction Pendulum Bearing ........................................... 20
  Friction Pendulum Bearing Free Body Diagram and Equations of Motion .... 20
  Friction Pendulum Bearing Effective Spring Constant .................................... 22
  Friction Pendulum Bearing Effective Damping Coefficient ............................ 25
  Effects of Friction Pendulum Design Parameters on the System Response .... 26
  Selecting Friction Pendulum Bearing Parameters for Design Applications .... 41
  Effective Spring Constant and Damping Coefficient Restrictions ................. 44
Analysis of the Effects of a Friction Pendulum Bearing Isolation System on the Dynamic Response of a Five Story Building

Model Parameters for Five Story Building without a Base Isolation System

Five Story Building with a Friction Pendulum Bearing Base Isolation System

Conclusion

Bibliography
List of Figures

Figure 1: Cross Section of a Friction Pendulum Bearing ........................................ 2
Figure 2: Two Degree of Freedom Model with Coulomb Damping ........................... 5
Figure 3: Block Diagram for 2 Degree of Freedom System with Coulomb Damping ....... 7
Figure 4: Displacement Response for 2 Degree of Freedom System with Coulomb Damping
M_1=M_2=98870kg, K_1=K_2=5.097*10^6 kN/m, µ=0.3 ........................................... 7
Figure 5: Pendulum System with Coulomb Damping .............................................. 8
Figure 6: Free Body Diagram for Pendulum System with Coulomb Damping ............... 9
Figure 7: Simulink Block Diagram of Simple Pendulum with Coulomb Damping ............ 11
Figure 8: Free Vibration Response of Simple Pendulum with Coulomb Damping l=100m, µ=0.3
................................................................................................................................. 11
Figure 9: Free Vibration Response of Simple Pendulum with Coulomb Damping l=100m, µ=0.3
................................................................................................................................. 12
Figure 10: Free Body Diagram for Pendulum System with Rotational Viscous Damper .... 14
Figure 11: Simulink Model of Simple Pendulum with Viscous Damping .................... 15
Figure 12: Free Vibration Response of Simple Pendulum with Viscous Damping l=100m, µ=0.3
ζ=0.023 ......................................................................................................................... 16
Figure 13: Free Vibration Response of Simple Pendulum with Coulomb and Viscous Damping
l=100m, µ=0.3 ζ=0.023 ............................................................................................... 17
Figure 14: Forced Vibration for Coulomb Damping and Equivalent Viscous Damping System 19
Figure 15: Forced Vibration Response of Simple Pendulum with Coulomb and Viscous Damping
l=100m, µ=0.3 c=0.47 ................................................................................................. 19
Figure 16: Free Body Diagram for Friction Pendulum Bearing .................................. 21
Figure 17: Friction Component of Force .................................................................... 23
Figure 18: Restoring Linear Component of Force ..................................................... 23
Figure 19 Hysteresis Loop for System with Frictional Damping ................................. 24
Figure 20: Friction Pendulum Bearing Hysteresis for Parameter Set 0 ....................... 28
Figure 21: Friction Pendulum Bearing Hysteresis for Parameter Set 1 ....................... 28
Figure 22: Friction Pendulum Bearing Hysteresis for Parameter Set 2 ....................... 28
Figure 23: Friction Pendulum Bearing Hysteresis for Parameter Set 3 ....................... 29
Figure 24: Friction Pendulum Bearing Hysteresis for Parameter Set 4 ........................................ 29
Figure 25: Friction Pendulum Bearing Hysteresis for Parameter Set 5 ........................................ 29
Figure 26: Effect of Mass on the Friction Pendulum Displacement ............................................. 30
Figure 27: Friction Pendulum Bearing Hysteresis for Parameter Set 6 ........................................ 31
Figure 28: Friction Pendulum Bearing Hysteresis for Parameter Set 7 ........................................ 32
Figure 29: Friction Pendulum Bearing Hysteresis for Parameter Set 8 ........................................ 32
Figure 30: Friction Pendulum Bearing Hysteresis for Parameter Set 9 ........................................ 32
Figure 31: Friction Pendulum Bearing Hysteresis for Parameter Set 10 ....................................... 33
Figure 32: Friction Pendulum Bearing Hysteresis for Parameter Set 11 ...................................... 33
Figure 33: Effect of Friction on the Friction Pendulum Displacement ........................................... 34
Figure 34: Friction Pendulum Bearing Hysteresis for Parameter Set 12 ....................................... 35
Figure 35: Friction Pendulum Bearing Hysteresis for Parameter Set 13 ...................................... 35
Figure 36: Friction Pendulum Bearing Hysteresis for Parameter Set 14 ....................................... 36
Figure 37: Friction Pendulum Bearing Hysteresis for Parameter Set 15 ....................................... 36
Figure 38: Friction Pendulum Bearing Hysteresis for Parameter Set 16 ....................................... 36
Figure 39: Effect of Radius of Curvature on the Friction Pendulum Displacement ......................... 37
Figure 40: Friction Pendulum Bearing Hysteresis for Parameter Set 17 ....................................... 38
Figure 41: Friction Pendulum Bearing Hysteresis for Parameter Set 18 ....................................... 39
Figure 42: Friction Pendulum Bearing Hysteresis for Parameter Set 19 ....................................... 39
Figure 43: Friction Pendulum Bearing Hysteresis for Parameter Set 20 ....................................... 39
Figure 44: Friction Pendulum Bearing Hysteresis for Parameter Set 21 ....................................... 40
Figure 45: Effect of Design Displacement on the Friction Pendulum Displacement ....................... 40
Figure 46: Effect of Radius of Curvatures and Coefficient of Friction on the Natural Frequency for a Design Displacement of 0.2m ........................................................................ 42
Figure 47: Ratio of Natural Frequency of the Friction Pendulum Bearing to the Forcing Frequency of an Earthquake as a Function of the Radius of Curvature and the Coefficient of Friction for a Design Displacement of 0.2m and Forcing Frequencies 1 Hz, 2Hz, and 4Hz ...... 43
Figure 48: Effect of Radius of Curvature and Coefficient of Friction on the Damping Ratio for a Friction Pendulum for a Design Displacement of 0.2m ........................................................................ 44
Figure 49: Diagram of Five Story Building without A Friction Pendulum Base Isolation System ................................................................................................................................. 47
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>Simulink Block Diagram for the State Space Model of a Five Story Building without a Base Isolation System</td>
</tr>
<tr>
<td>51</td>
<td>Acceleration Response Without a Base Isolation System for a 1.0Hz Input Signal</td>
</tr>
<tr>
<td>52</td>
<td>Displacement Response Without a Base Isolation System for a 1.0Hz Input Signal</td>
</tr>
<tr>
<td>53</td>
<td>Acceleration Response Without a Base Isolation System for a 2.0Hz Input Signal</td>
</tr>
<tr>
<td>54</td>
<td>Displacement Response Without a Base Isolation System for a 2.0Hz Input Signal</td>
</tr>
<tr>
<td>55</td>
<td>Acceleration Response of Floor 1 With a Friction Pendulum Bearing Base Isolation System for a 1.0Hz Input Signal</td>
</tr>
<tr>
<td>56</td>
<td>Acceleration Response of Floors 2-5 With a Friction Pendulum Bearing Base Isolation System for a 1.0Hz Input Signal</td>
</tr>
<tr>
<td>57</td>
<td>Displacement Response With a Friction Pendulum Bearing Base Isolation System for a 1.0 Hz Input Signal</td>
</tr>
<tr>
<td>58</td>
<td>Acceleration Response for Floor 1 With a Friction Pendulum Bearing Base Isolation System for a 2.0Hz Input Signal</td>
</tr>
<tr>
<td>59</td>
<td>Acceleration Response for Floors 2-5 With a Friction Pendulum Bearing Base Isolation System for a 2.0Hz Input Signal</td>
</tr>
<tr>
<td>60</td>
<td>Displacement Response with a Friction Pendulum Bearing Base Isolation System for a 2.0Hz Input Signal</td>
</tr>
</tbody>
</table>
List of Tables

Table 1: Equivalent Damping Constants ................................................................. 14
Table 2: Friction Pendulum Bearing Design Parameters [11] .................................... 27
Table 3: Case A, Effect of Mass on Effective Spring Constant and Damping Coefficient .... 27
Table 4: Case B, Effect of Friction on Effective Spring Constant and Damping Coefficient ..... 31
Table 5: Case C, Effect of Radius of Curvature on Effective Spring Constant and Damping Coefficient ......................................................................................................................... 35
Table 6: Case D, Effect of Design Displacement on Effective Spring Constant and Damping Coefficient ......................................................................................................................... 38
Table 7: Structure Parameters for Five Story Building .................................................. 47
Table 8: Five Story Building Model Parameter Constants .............................................. 48
Table 9: State Space Model Parameters ....................................................................... 50
Table 10: Friction Pendulum Bearing Design Parameters ............................................ 55
Table 11: Effective Friction Pendulum Constants ........................................................ 56
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Introduction

Throughout history, vibration damping has been critical to the design of many devices, from the design of an engine to the design of bridges. One area in particular where vibration damping is important is seismic base isolation. Seismic base isolation refers to the protection of structures from the harmful effects of seismic activity by adding flexibility to the system while dissipating energy [1]. Ultimately, a base isolation system shifts the frequency of the structure away from the frequency of the ground motion caused by an earthquake.

Base isolation is a relatively old concept, with known applications dating back to the early 1900’s[2]. However, in the United States, it is still fairly new. The first known application occurred in 1985 for the Law and Justice Center in Rancho Cucamonga, California [3].

Modern Seismic Base Isolation Systems

There are two commonly used modern types of seismic base isolation systems, elastomeric bearings and sliding bearings. Elastomeric bearings refer to a multitude of base isolation devices consisting of alternating thin layers of metal and rubber disks bonded and stacked upon one another. This type of system is one of the oldest types of base isolation systems, originating in New Zealand over 30 years ago [2]. The number of layers determines the maximum lateral displacement that the bearing can handle. Normally elastomeric bearings can handle displacements up to twice the initial height. One drawback of this design, however, is a lack of energy dissipation [2]. As a result, another type of elastomeric bearing was developed which has a lead core. This lead cored adds damping to the system.

A sliding bearing system is another common device used for base isolation. Specifically there are two designs that fit this category, sliding plate bearings and friction pendulum bearings. The sliding plate bearing consists of two plates centered over one another. One plate is usually made from stainless steel and is either attached to the structure or the supporting foundation. On the other side, which may either be the structure or supporting foundation depending on the placement of the first plate, is a Teflon impregnated pad [2]. In addition to the two plates, there is a limiting mechanism that constrains the maximum horizontal movement of the device.
The other type of sliding bearing is the friction pendulum bearing. This type of sliding bearing is a modern development invented in 1985 by Dr. Victor Zayas. As the name suggests, this base isolation device operates like a pendulum. Unlike the sliding plate, the friction pendulum has a polished, curved bottom surface known as the bearing bottom plate shown in Figure 1. The bearing bottom plate is connected to the ground and moves with the ground during seismic excitation. Resting on this surface is a concave slider. Finally, on top of this concave slider is a top plate with a curved section that will partially encase the concave slider. The top plate is connected to the structure but is not directly connected to the concave slider bearing [4]. This allows the surfaces to slide freely over one another.

![Figure 1: Cross Section of a Friction Pendulum Bearing](image)

The defining feature of a friction pendulum is the ability to shift the natural frequency of the building away from the frequencies of the ground motion. This is a direct result of the free sliding curved surfaces. In turn, the building is better protected from the damaging effects of large ground accelerations during an earthquake.

**Seismic Base Isolation System Comparisons**

The two main types of seismic base isolation devices, sliding bearings and elastomeric bearings, both have advantages and disadvantages. Elastomeric bearings have a proven history of success and are very reliable. They are also easily manufactured and are resistant to wear from the environment [5]. However, they have limited use for applications that involve large displacements normally seen near the epicenter of an earthquake. In such applications, the maximum displacement for the elastomeric bearing is exceeded. It is important to note that the range of acceptable displacements can be broadened if a lead core is added, adding hysteretic damping to the system [2].
Sliding bearings have similar advantages and disadvantages. The major advantage to sliding bearings is the relatively low cost of the base isolation system. Also, compared to elastomeric bearings, sliding bearings inherently have damping built-in, a result of the sliding between the contact surfaces, and can achieve much higher levels of displacement. Despite these advantages, there are multiple disadvantages. For both sliding plate bearings and friction pendulum bearings, there is a trade-off between an increase in energy dissipation through friction and the excitation of higher modes of the structure [2]. Also, the device is highly dependent on friction and more susceptible to environmental changes such as temperature and wear, leading to more or less damping. The unpredictability is undesirable although it can more easily be modeled than elastomeric rubber bearing [3]. Also, sliding plate bearings do not have a built-in centering mechanism. This leads to the unpredictability of the ending location following an earthquake. However, this issue is corrected with the friction pendulum bearing which uses the force of gravity to re-center the device. Also, due to the curvature of the friction pendulum, the period of the friction pendulum bearing is independent from the mass of the structure [3].

For a versatile, minimal-cost base isolation device, the friction pendulum is a leading option. Although it is slightly more costly than an elastomeric bearing, it can handle a large range of displacements, has both damping and re-centering capabilities, and improves upon some of the down-falls of sliding plate bearings. As a result, this modern development is becoming a widely used device for seismic base isolation.

**Friction Pendulum Applications**

**Base Isolation for Bridges**

One important area where friction pendulums are used is base isolation for bridges. Due to the cost effectiveness of the device, friction pendulums are often chosen for bridge construction projects. One example is the retrofitting of a friction pendulum isolation system to the Benicia-Martinez Bridge. To date, this is one of the largest retrofit projects. It has been said that without the time constants that can be obtained with frictions pendulums, the project may not have been possible [6].
**Retrofit to Buildings**

One of the key benefits of the friction pendulum bearing is the ability to easily retrofit it to existing structure in order to minimize damage during an earthquake. With minimal changes, historical buildings can be saved. One example of this is the US Court of Appeals building in San Francisco, California. The courthouse was the largest building in the world to be retrofitted with seismic isolators. Also, due to the use of a friction pendulum device, roughly $7.6 million was saved on the project. Also, due to the compact design of the friction pendulum, which combines energy dissipation and flexibility into one mechanism, 80,000 square feet were saved [6].

**Project Overview**

In this paper, the friction pendulum bearing, one example of a vibration damping device will be modeled. The analysis will begin by modeling the critical components that affect the response of the friction pendulum bearing, starting with the coulomb damping. Upon completion of this modeling, the understanding of these components will be applied to the analysis of the friction pendulum bearing. A series of equations will be derived that will equate the response of the friction pendulum to the response of a system with a viscous damper and spring. This equivalent system will be utilized to understand the impact of the friction pendulum bearing design parameters, including the coefficient of friction, radius of curvature, and design displacement of the friction pendulum, on the response of the bearing. This understanding will then be used to optimize a friction pendulum bearing for a structural application, a five story building.

**Mathematical Modeling of Friction Pendulum Bearing**

**Mathematical Modeling of Coulomb Damping**

All vibration damping devices utilize some method of removing energy from the system, usually by converting mechanical energy to thermal energy through friction. Friction pendulum bearings also remove energy from the system utilizing friction which is known as coulomb damping. While oscillating, the system dissipates energy between the two sliding surfaces, the bearing bottom plate and the concave slider bearing.
To begin the mathematical analysis of the friction pendulum bearing, an understanding of coulomb damping must be obtained. However, this mathematical analysis is limited because an exact solution cannot be directly obtained. As a result, an equivalent system will need to be created.

**Linear System with Coulomb Damping**

The first system that will be analyzed is a two degree of freedom system with two masses connected by springs. One of the masses in this model is grounded and has coulomb damping. The friction on the other mass will be neglected in this model. A schematic of this setup can be seen in Figure 2. This model was selected to mimic a two story building with a base isolation system with coulomb damping, similar to a friction pendulum bearing. The spring, $K_1$, which connects that mass of the ground floor, $M_1$, acts as the restoring force on a friction pendulum base isolation system. Also, the friction acting on the mass represents the friction between the sliding surfaces in a friction pendulum system. The second mass was added to represent the second story of the building. The spring connecting the two masses results from the stiffness of the steel framing between the floors of the building.

![Figure 2: Two Degree of Freedom Model with Coulomb Damping](image)

For the first mass, the equation of motion can be derived to be:

$$M_1 \ddot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 = F_{\text{applied}} + -\mu M_1 g = 0$$

**Equation 1**

For the second mass, the equation of motion can be derived to be:

$$M_2 \ddot{x}_2 - (K_2)x_1 + K_2x_2 = 0$$

**Equation 2**
The model described captures many key features of a friction pendulum system. As a result, it is an appropriate model to begin the analysis of coulomb damping. In order to visually demonstrate the response of the system, appropriate values will applied to the masses and spring constants in the system. The weight of each floor of a skyscraper can be approximated to be 969kN which is based off of a dead weight of 90lb/ft$^2$ for the floor of a building[7].

Also, the steel beams between floors can be modeled as springs with an approximate spring stiffness of 5097kN/m. This stiffness is derived using Equation 3 for a 3m long W8 X 40 wide-flange beam made from steel. The modulus of elasticity for steel is 30*10$^6$ psi while the moment of inertia of the column is 133.2 in$^4$.

$$k_{beam} = 12 * E * \frac{I_x}{L_{beam}^3}$$

Equation 3

With the equations derived earlier and the approximate numerical values selected, Matlab Simulink can be utilized to create a graphical representation of the model. For the purpose of the model, an initial displacement for $M_1$ of 1m will be used. The friction coefficient will be approximated to be 0.3. The Simulink block diagram is presented in Figure 3 while the system response for $M_1$ and $M_2$ is found in Figure 4.
Figure 3: Block Diagram for 2 Degree of Freedom System with Coulomb Damping

Displacement Response for 2 Degree of Freedom System with Coulomb Damping

Figure 4: Displacement Response for 2 Degree of Freedom System with Coulomb Damping $M_1 = M_2 = 98870\text{kg}$, $K_1 = K_2 = 5.097 \times 10^6 \text{kN/m}$, $\mu = 0.3$
The main point of the model in Figure 4 is to demonstrate the impact of coulomb damping on the system response. The coulomb damping is capable of removing a significant portion of the energy from the system in slightly over 5 seconds. By 6 seconds, the system stops. However, the system does not resume the initial equilibrium position. This is because the friction force is greater than the restoring force from the spring. As a result, the system is incapable of moving.

**Rotational System with Coulomb Damping for Free Vibration**

This understanding of how coulomb damping dissipates energy can be further explored with a model that oscillates like a pendulum. As the name suggests, the friction pendulum bearing operates like a simple pendulum [4].

![Pendulum System with Coulomb Damping](image)

*Figure 5: Pendulum System with Coulomb Damping*

The curved surface below the mass in Figure 5 above represents the curved surface of the bearing bottom plate. The length of the chord, l, corresponds to the radius of curvature of this plate. The mass, M, is considered to be a concentrated mass. For a friction pendulum bearing this mass would include the mass of the entire structure supported by the bearing. If the plate is considered frictionless, the system in Figure 5 will respond like a simple pendulum.

The equation of motion for this system without friction, Equation 5, can be derived using an energy approach. The potential energy is defined as $U = M g (l (1 - \cos(\theta)))$ while the kinetic energy is defined as $\frac{1}{2} M l^2 \dot{\theta}^2$. A small angle approximation will also be used where $\sin \theta = \theta$ and $\cos \theta = 1$. 
\[ U + T = \left( \frac{1}{2} M l^2 \dot{\theta}^2 + M g l (1 - \cos(\theta)) \right) = 0 \]

Equation 4

\[ \frac{d}{dt} (U + T) = M l^2 \ddot{\theta} + M g l \dot{\theta} = 0 \]

Equation 5

The natural frequency of the system can be derived from Equation 5 as \( \omega_n = \sqrt{\frac{g}{l}} \). This relationship is important not only for a simple pendulum, but also for a friction pendulum bearing. The natural frequency of the friction pendulum is independent from the mass of the structure.

The derivation of the equation of motion for the pendulum system is more complicated with the addition of friction. The free body diagram for this system is displayed below in Figure 6.

![Free Body Diagram for Pendulum System with Coulomb Damping](image)

**Figure 6: Free Body Diagram for Pendulum System with Coulomb Damping**

As the mass oscillates about point 0, the normal force and friction force vary in magnitude. Also, the friction force changes direction after each half cycle of motion so that it always opposes the motion of the pendulum. Both of these factors must be considered when deriving the equations of motion. In the derivation below, Equation 6, the summation of moments about point 0 will be taken. Again, a small angle approximation will be used,
simplifying Equation 6 to the form shown in Equation 7. Equation 7 demonstrates that the system natural frequency is still independent from the mass of the system.

\[ Ml^2\ddot{\theta} = Mg\sin(\theta) + \text{sgn}(\dot{\theta})Mgl\mu \cos(\theta) \]

**Equation 6**

\[ l^2\ddot{\theta} - g\theta = gl\mu \text{sgn}(\dot{\theta}) \]

**Equation 7**

The differential equation, Equation 7, does not have a simple solution. This is a result of the complexity of the \text{sgn}() or signum function which corresponds to the directional change of the friction force ever half period. Although a series of equations for each half period of motion will generate an analytical solution, an alternative approach will be developed [8].

For the analysis of the model in Figure 6, Matlab’s Simulink package will be utilized to generate the system response. An analysis of the system response will assist in the development of an alternative modeling approach.

In the model, constant values will be utilized, although unrealistic, that clearly demonstrate the motion. The radius of curvature, \( l \), will be 100 m while the coefficient of friction, \( \mu \), will be 0.3. The system will undergo free vibration with an initial displacement of 10 degrees. Figure 7 shows the block diagram while Figure 8 shows the system response for 100 seconds.
Figure 7: Simulink Block Diagram of Simple Pendulum with Coulomb Damping

Figure 8: Free Vibration Response of Simple Pendulum with Coulomb Damping l=100m, μ=0.3
There are several features of the response in Figure 8 to comment on. The first feature to note is the period of motion. With a chord length of 100m, the natural frequency of the system is 0.049849Hz and the period is 20.06 seconds. The calculation for this result can be found in Equation 8. As expected, the addition of friction to the system does not affect the natural frequency.

\[
T = \frac{2\pi}{\sqrt{g/l}} = 20.06s
\]

Equation 8

The decay of the system is also important to analyze. Although coulomb damping is complex, the amplitude of vibration decreases linearly for systems undergoing free vibration. For the system described thus far, the amplitudes for each cycle can be fitted with a linear curve, demonstrated in Figure 9.

\[y = -0.060x + 10.000\]

Figure 9: Free Vibration Response of Simple Pendulum with Coulomb Damping l=100m, \(\mu=0.3\) 

The linear decay can be predicted through a mathematical analysis of the system utilizing the relationship between the work done by the damping force and the loss of kinetic energy during each cycle. This is only applicable under the condition that the system responds in a harmonic fashion of the form \(x = x_0 \cos \omega t\). [9] The system described thus far falls under this
condition. The work done during each cycle by the frictional force is

\[ W = \int_0^{2\pi} f(\dot{x}) \cos \omega t \, d(\omega t) \]  

where \( f(\dot{x}) \) is the damping force [9]. The loss of kinetic energy between two cycles can be represented as

\[ \frac{1}{2}m\omega^2x_0^2 - \frac{1}{2}m\omega^2(x_0 - \Delta x)^2 \]  

where \( m \) is the mass, \( \omega \) is the natural frequency, and \( x \) is the amplitude [9]. The damping force for coulomb damping can be represented as \( f(\dot{x}) = sgn(\dot{x})\mu mg \). For oscillatory motion, this relationship will also apply under the assumptions that a small angle approximation is applicable. After setting the loss in kinetic energy equal to the work done by the damping force, in this case friction, the equation can be simplified to [9]. For the pendulum system with coulomb damping in Figure 6, this equation can be further simplified if the natural frequency is substituted with \( \sqrt{\frac{g}{l}} \). The change in amplitude becomes \( \Delta x = 4\mu l \). This directly corresponds to the slope of the line which intersects the peak of each cycle in Figure 9. According to the derivation presented, \( \Delta x = 4 \times 0.3 \times 100 = 120 \). For our scenario with a small angle approximation, \( \Delta \theta = \frac{\Delta x}{l} = 1.2 \) degrees. This corresponds to a slope of -.06 deg/sec.

**Equivalent Viscous Damping System for Free Vibration**

Despite the ability to predict the amplitude of each cycle for a system with coulomb damping with free vibration, modeling the full system response is still a daunting task. As a result, creating other systems that can produce a similar response will be attempted. These simplifications will assist with modeling the friction pendulum bearing. A pendulum system which utilizes a viscous damper in place of the coulomb damping is one example.

For the system in Figure 6, an equivalent damping constant can be derived from the energy dissipated during each oscillation of the system. The peaks for the four complete cycles shown in Figure 9 are displayed in Table 1. Utilizing these values, the damping ratio and the damping constant can be calculated. The calculations utilized to determine the damping ratio and coefficient are displayed in Equation 9 and Equation 10. These values can be used to develop an equivalent system with a viscous damper.
<table>
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<th>$\zeta$</th>
<th>c(kg m$^2$/s)</th>
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<td>8.80</td>
<td>N/A</td>
</tr>
<tr>
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<td>0.023326</td>
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<td>x3</td>
<td>6.40</td>
<td>0.025334</td>
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<tr>
<td>x4</td>
<td>5.20</td>
<td>0.027899</td>
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</table>

Table 1: Equivalent Damping Constants

$$\zeta = \frac{\frac{1}{x_2 - 1} \ln \left( \frac{x_1}{x_2} \right)}{\sqrt{4\pi^2 + \left[ \frac{1}{x_2 - 1} \ln \left( \frac{x_1}{x_2} \right) \right]^2}}$$

Equation 9

$$c = 2\zeta \sqrt{\frac{g}{l}}$$

Equation 10

Figure 10 depicts the simple pendulum system utilizing a rotational viscous damper with a damping constant of $c$ at point 0. The viscous damping replaces the coulomb or friction damping that occurred between the hanging mass and the curved surface in Figure 5.

![Free Body Diagram for Pendulum System with Rotational Viscous Damper](image)

Figure 10: Free Body Diagram for Pendulum System with Rotational Viscous Damper
For the system displayed in Figure 10, the equation of motion describing the system changes from the one used to describe the system with coulomb damping. An energy approach will be taken, as demonstrated in Equation 11, to derive the equation of motion, Equation 12. A small angle approximation will still be used.

\[
\frac{d}{dt}\left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos(\theta))\right) = -c\theta^2
\]

Equation 11

\[
ml^2\ddot{\theta} + c\dot{\theta} + mgl\theta = 0
\]

Equation 12

The response of the viscously damped system can be modeled using Matlab Simulink software. The block diagram for the model is shown in Figure 11. The damping constant calculated earlier of 0.0146 kg m^2/s will be used. All other model parameters will remain the same as those used to model the system with coulomb damping. The response generated by this system with an initial rotation of 10 degrees is shown in Figure 13.

![Simulink Model of Simple Pendulum with Viscous Damping](image-url)
However, this approach of using an equivalent damper does not adequately mimic the motion of a system with coulomb damping undergoing free vibration. The fact that the equivalent damping ratio, listed in Table 1, changes between each cycle indicated that the response would not be identical. Figure 13 shows this more clearly with the overlap of system responses for both the system with frictional damping and the system with an equivalent rotational viscous damper.
The reason for the discrepancy is a result of the types of decay each of the two systems have. The coulomb damping system has linear decay while the viscously damped system has exponential decay. The exponential curve fit to the peaks matches the expected decay for the viscously damped system, \( y = \theta_0 e^{-\zeta \omega_n t} = 10e^{-0.0233 \cdot \frac{0.81}{\sqrt{100}} t} = 10e^{-0.007t}. \)

**Equivalent Viscous Damping System for Forced Vibration**

Although an equivalent damping constant fails to produce an alike system response under free vibration, an equivalent damping constant approach can be used for harmonic forced vibration to predict the steady state response of a system \([8]\). To demonstrate this, a system similar to that in Figure 10 will be used. However, in this case, a force will be applied to the hanging mass.

For forced vibration, a different approach from the one discussed will be used to find the equivalent damping constant. In this approach, the energy dissipated by friction in one cycle will be equated to the energy dissipated in one full cycle of a viscous damper. The equations for
determining this energy dissipation for both friction and viscous dampers are presented in equations Equation 13 and Equation 14 respectively [8].

$$\Delta W = 4\mu N X$$

Equation 13

$$\Delta W = \pi c_{eq} \omega X^2$$

Equation 14

When these two equations are equated, the equivalent damping constant can be found to be $c_{eq} = \frac{4\mu N}{\pi \omega X}$. $N$ is the normal force, $\mu$ is the coefficient of friction, $\omega$ is the forcing frequency, and $X$ is the amplitude of motion under steady-state conditions. The amplitude can be solved for using the particular solution to the differential equation, Equation 15 [8].

$$X = \frac{F_0}{k} \left[ 1 - \left( \frac{4\mu N}{\pi F_0} \right)^2 \right]^2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2$$

Equation 15

The method described however is limited to systems where the frictional force is small compared to the applied force, $F_0$. The exact criteria is $1 - \frac{4\mu N^2}{\pi F_0} > 0$. This is necessary to avoid imaginary values for the amplitude $X$. As stated earlier, this new method for determining an equivalent damping constant for a system will be applied to the model in Figure 10, where a periodic force will be applied directly to the mass. For this system, $l$, the length of the chord in Figure 10, will be defined to be 100m, the coefficient of friction will remain 0.3, the magnitude of the applied force will be 100N, and the forcing frequency will be 2Hz. Assuming a small angle approximation, the normal force will be a constant at 0.981N. Also, for our pendulum system, the equivalent spring constant $k$ will be 0.981N. The equivalent spring constant is equal to $mg$ for a pendulum. Using these values, the predicted amplitude of the steady state response is 0.063 radians and the equivalent damping constant is 0.47 Ns. To confirm these approximations, Matlab Simulink will be utilized to model both the forced vibration system with friction and the equivalent system with viscous damping. The block diagram for these models is
shown in Figure 14. The responses generated are in Figure 15.

![Figure 14: Forced Vibration for Coulomb Damping and Equivalent Viscous Damping System](image)

**Forced Vibration Response with Coulomb and Viscous Damping**

![Graph showing forced vibration response with Coulomb and Viscous Damping](image)

**Figure 15: Forced Vibration Response of Simple Pendulum with Coulomb and Viscous Damping l=100m, µ=0.3 c=0.47**

The generated response shown in Figure 15 is between 995 seconds and 1000 seconds. This was necessary to show the steady state response of the system. During startup, the
viscously and coulomb damped system do not respond identically. However, once steady-state is reached the two systems with coulomb and viscous damping are identical. Also, it is important to note that for the given parameters discussed earlier, both systems respond with maximum amplitude of 0.063 radians and a frequency of 2Hz which was calculated earlier.

**Mathematical Modeling of Friction Pendulum Bearing**

Both free vibration and forced vibration systems have been modeled to demonstrate several key components of the friction pendulum bearing and to gain an understanding of how to appropriately model coulomb damping. Thus far, it has been demonstrated that the frequency of the pendulum is independent from the mass or weight of a structure. Also, it has been demonstrated that a viscous damper which has an exponential decay response cannot adequately model the linear decay of coulomb damping.

These conclusions that have been drawn will enable a complete mathematical model to be developed for the friction pendulum bearing by creating an equivalent system using springs and viscous dampers. A different modeling approach will be utilized to accommodate for the deficiencies of the previous models that have been developed. The end result will be an equivalent model that is capable of modeling both forced vibration and free vibration, from startup to steady state.

**Friction Pendulum Bearing Free Body Diagram and Equations of Motion**

To begin the friction pendulum bearing analysis, the free body diagram, Figure 16, of the system must be analyzed. In this diagram, the chord length \( l \) from the simple pendulum is replaced by \( R \), corresponding to the radius of curvature of the bearing bottom plate. \( U \) indicates the direction of motion and the displacement relative to the center point. \( W \) is the weight of the structure. The weight replaces the swinging mass of the pendulum system, although both the mass and the weight will be considered to act at a single point. \( N \) is the normal force on the slider. \( F_{\text{friction}} \) is the friction force that results from the two sliding surfaces moving against one
From Figure 16, the sum of the forces on the slider in the x direction can be obtained. Once again a small angle approximation will be utilized, assuming that the displacement $U$ is much smaller than the radius $R$. The exact criteria will be discussed later.

\[
\sum F_x = F_{friction} \cos \theta + N \sin \theta = \frac{W}{g} \ddot{u}
\]

**Equation 16**

In Equation 16, the normal force and friction force are defined as follows:

\[
N = W \cos \theta
\]

**Equation 17**

\[
F_{friction} = \mu N \text{sgn}(\dot{U})
\]

**Equation 18**

Combining Equation 17 and Equation 18, Equation 16 can be written as the following.
\[ \sum F_x = \mu W \cos \theta \text{sgn}(\dot{U}) \cos \theta + W \cos \theta \sin \theta = \frac{W}{g} \ddot{u} \]

**Equation 19**

Upon applying the small angle approximation where \( \sin \theta = \theta = \frac{u}{R} \) and \( \cos \theta = 1 \), Equation 19 can be further simplified to the following.

\[ F_x = \mu W \text{sgn}(\dot{U}) + \frac{WU}{R} = \frac{W}{g} \ddot{u} \]

**Equation 20**

It is important to note the similarity between Equation 20 and Equation 7. Equation 7 is in terms of rotation instead of horizontal displacement. However, the horizontal motion of the simple pendulum discussed in the sections earlier can be described by the following differential equation, applying the small angle approximation. To further demonstrate the similarity between Equation 20 and Equation 7, the chord length \( l \) is replaced by \( R \).

\[ m\ddot{u} - \frac{mg}{R} u = mg\text{sgn}(\ddot{u}) \]

**Equation 21**

This confirms the assumptions made earlier. The friction pendulum bearing responds like a pendulum sliding along a curved surface. In the case of the friction pendulum, the motion is dictated not by the length of the chord from which the mass is hanging but by the radius of curvature of the bottom plate.

**Friction Pendulum Bearing Effective Spring Constant**

The components of this motion have not been discussed thus far. Equation 20 shows that the resisting force acting on the pendulum is inversely related to the radius of curvature while proportional to the change in weight or mass. This is demonstrated in the component of the force, \( \frac{W}{R} \). This component was used earlier as the spring constant in both the simple pendulum system and the pendulum system with sliding. Unlike a simple pendulum, there is another component that must be included which is a result of the friction force acting on the sliding mass. This is the nonlinear component. The following hysteresis diagrams, Figure 17 and Figure 18,
demonstrate the role these two components, the restoring and frictional component, play in the overall force acting on the bearing.

When the components are combined they produce the system hysteresis response in Figure 19. This hysteresis loop is defined by two parallel lines with a slope of $\frac{W}{R}$ [10]. These lines are offset from one another by $\mu W$ which is a result of the directional change of the friction force.
By connecting the maximum and minimum points, points 1 and 2 in Figure 19, an effective spring constant for the system can be developed. The slope of the line connecting these two points corresponds to the effective spring constant shown in Equation 22[10]. D represents the maximum displacement of the system.

\[
K_{eff} = \frac{W}{R} + \frac{\mu W}{D}
\]

Equation 22

Although this equation appears to include effects of damping with the addition of friction, this addition is essential to account for the nonlinearity of the system due to the directional change of the friction force. A simple spring mass system would have a hysteresis loop similar to that shown in Figure 18. However, the resisting force acting on the system includes the directional change of the friction force as described before. The friction therefore plays a crucial role not only in the damping as will be discussed next, but also a role in the oscillation and natural frequency of the system. This is why the previous attempts to develop an equivalent pendulum system with a rotational damper failed to adequately mimic the motion of a pendulum system sliding along a curved surface during free vibration. These prior models attempted to model linear decay with the exponential decay of a rotational damper.
The equivalent viscously damped system for forced vibration under steady state conditions was adequate however because the system oscillated at the periodic forcing frequency. During steady-state oscillation, the energy removed during each oscillation remains constant.

**Friction Pendulum Bearing Effective Damping Coefficient**

An effective damping ratio and damping constant can be developed for the friction pendulum bearing using a similar technique to that used for developing the equivalent damping ratio for the simple pendulum system. This approach equates the work done by a system with viscous damping to the work done by a friction pendulum bearing. The following equations show the derivation. In these equations, \(W\) is the amount of work that is done by the system, \(\mu\) is the coefficient of friction, \(N\) is the normal force, \(\omega_n\) is the natural frequency of the system, \(D\) is the specified maximum system displacement, \(m\) is the total mass supported by the bearing, and \(R\) is the radius of curvature of the friction pendulum base plate. \(c_{eff}\) will be used to represent the equivalent damping ratio for a system with viscous damping.

\[
\Delta W = 4\mu N x
\]

**Equation 23 Frictional Work between Slider and Bottom Plate**

\[
\Delta W = \pi c_{eff} \omega_n x^2
\]

**Equation 24 Work by Viscous Damper**

\[
4\mu N x = \pi c_{eq} \omega_n x^2
\]

**Equation 25**

\[
\omega_n = \sqrt{\frac{W + \mu W}{R + \frac{D}{m}}} \quad x = D \quad c_c = 2\sqrt{mk}
\]

**Equation 26**

\[
4\mu N D = \pi c_{eff} \sqrt{\frac{W + \mu W}{R + \frac{D}{m}}} D^2
\]

**Equation 27**
There are several important developments that result from the derivation of Equation 28 and Equation 29. First, as expected the damping constant is proportional to the weight and friction between the two sliding surfaces. However, the damping is inversely related to the specified displacement, meaning that the larger the displacement of the system, the greater the damping is for the system.

**Effects of Friction Pendulum Design Parameters on the System Response**

The effects of each of the parameters of the friction pendulum, the system mass, the coefficient of friction, the radius of curvature of the friction pendulum, and the initial design displacement of the system will be analyzed in this next section. Except for this last parameter, each of these parameters can be controlled for a friction pendulum bearing. Depending on the size of a structure, the mass will vary. Also, the surface roughness can be altered to change the friction between the surfaces. Finally, the pendulum itself can be manufactured with a different curvature.

In the tables below, Table 3 through Table 6, the effective spring constants and damping coefficients have been calculated for systems with varying parameters. Each of these systems will be compared to a set of standard parameters obtained from a realistic friction pendulum [11]. These realistic values are shown in Table 2.
Table 2: Friction Pendulum Bearing Design Parameters [11]

Table 3, which will be referred to as Case A, shows parameter sets 0 through 5 which demonstrate the effect of mass change on the horizontal displacement of the pendulum. A 531,179.18 kg step size was used between each parameter set. As seen in the table, both the effective spring constant and damping coefficient increase linearly as a result. As expected, the system response does not change. This is demonstrated in Figure 26. This clearly shows that the system displacement is independent from the mass of the structure being supported. However, the hysteresis graphs in Figure 21 through Figure 25 demonstrate that the overall force acting on the system increases as the system mass is increased. As the spring constant increases, a larger force must be utilized to move the same distance as a system with a smaller spring constant.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>1.5m</td>
</tr>
<tr>
<td>m₁</td>
<td>5.312X10⁵kg</td>
</tr>
<tr>
<td>μ</td>
<td>0.06</td>
</tr>
<tr>
<td>D</td>
<td>0.2m</td>
</tr>
</tbody>
</table>

Table 3: Case A, Effect of Mass on Effective Spring Constant and Damping Coefficient
Figure 20: Friction Pendulum Bearing Hysteresis for Parameter Set 0

Figure 21: Friction Pendulum Bearing Hysteresis for Parameter Set 1

Figure 22: Friction Pendulum Bearing Hysteresis for Parameter Set 2
Figure 23: Friction Pendulum Bearing Hysteresis for Parameter Set 3

Figure 24: Friction Pendulum Bearing Hysteresis for Parameter Set 4

Figure 25: Friction Pendulum Bearing Hysteresis for Parameter Set 5
Table 4, which will be referred to as Case B, shows parameter sets 6 through 11 which demonstrate the effect of friction on the system response. Friction affects both the effective spring constant and the effective damping constant, shown mathematically in Equation 22. This is a result of the directional change of the friction force when the direction of motion changes. In the table, a system without friction is analyzed. Also, systems with an increase of friction are analyzed, utilizing a step size of 0.06 between each parameter set. Unlike changing the mass, changes in friction affect both the system response and the system hysteresis. As friction is increased, the natural frequency of the system increases. This is a result of the increase in the effective spring constant of the system. As expected, as the friction increases, the damping increases which results in quicker dissipation of energy in the system. The hysteresis responses for the parameters in Table 4 can be found in responses, Figure 27 through Figure 32, and the horizontal displacements can be found in, Figure 33.
### Table 4: Case B, Effect of Friction on Effective Spring Constant and Damping Coefficient

<table>
<thead>
<tr>
<th>Parameter Set</th>
<th>Standard Parameter</th>
<th>No Friction</th>
<th>Friction 1</th>
<th>Friction 2</th>
<th>Friction 3</th>
<th>Friction 4</th>
<th>Friction 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>K(_{eff})(N/m)</td>
<td>5,035,366.67</td>
<td>3,472,666.67</td>
<td>6,598,066.67</td>
<td>8,160,766.67</td>
<td>9,723,466.67</td>
<td>11,286,166.67</td>
<td>12,848,866.67</td>
</tr>
<tr>
<td>c(_{eff}) (N*s/m)</td>
<td>646,229.86</td>
<td>-</td>
<td>1,129,078.71</td>
<td>1,522,854.27</td>
<td>1,860,167.27</td>
<td>2,158,238.06</td>
<td>2,427,288.57</td>
</tr>
<tr>
<td>R(_{fps}) (m)</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
</tr>
<tr>
<td>m(_i)(kg)</td>
<td>531,170.18</td>
<td>531,170.18</td>
<td>531,170.18</td>
<td>531,170.18</td>
<td>531,170.18</td>
<td>531,170.18</td>
<td>531,170.18</td>
</tr>
<tr>
<td>u</td>
<td>0.06</td>
<td>-</td>
<td>0.12</td>
<td>0.18</td>
<td>0.24</td>
<td>0.30</td>
<td>0.36</td>
</tr>
<tr>
<td>D(m)</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>W(N)</td>
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<td>5,209,000.00</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
</tr>
</tbody>
</table>

**Figure 27:** Friction Pendulum Bearing Hysteresis for Parameter Set 6
Figure 28: Friction Pendulum Bearing Hysteresis for Parameter Set 7

Figure 29: Friction Pendulum Bearing Hysteresis for Parameter Set 8

Figure 30: Friction Pendulum Bearing Hysteresis for Parameter Set 9
Figure 31: Friction Pendulum Bearing Hysteresis for Parameter Set 10

Figure 32: Friction Pendulum Bearing Hysteresis for Parameter Set 11
Figure 33: Effect of Friction on the Friction Pendulum Displacement

Table 5, which will be referred to as Case C, shows parameter set 12 through 16 which demonstrate the effects of the radius of curvature of the bottom plate of the friction pendulum bearing. In the table a step size of 1.5 m is used. The table demonstrates that as the radius of curvature is increased, the effective damping coefficient increases while the effective spring constant decreases. Therefore, as the radius of curvature is increased, the natural frequency decreases. This is demonstrated in the horizontal displacement response in Figure 39. The hysteresis graphs for parameter set 12 through parameter set 16 are shown in Figure 34 through Figure 38. These hysteresis graphs demonstrate that as the radius of curvature increases, there is increased damping.

<table>
<thead>
<tr>
<th>Parameter Set</th>
<th>Parameter Set 0</th>
<th>Parameter Set 12</th>
<th>Parameter Set 13</th>
<th>Parameter Set 14</th>
<th>Parameter Set 15</th>
<th>Parameter Set 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Parameter</td>
<td>K_{eff}(N/m)</td>
<td>5,035,366.67</td>
<td>3,299,033.33</td>
<td>2,720,255.56</td>
<td>2,430,866.67</td>
<td>2,257,233.33</td>
</tr>
<tr>
<td>c_{eff}(N*s/m)</td>
<td>646,229.86</td>
<td>798,379.21</td>
<td>879,220.32</td>
<td>930,083.63</td>
<td>965,193.40</td>
<td>990,936.41</td>
</tr>
</tbody>
</table>
### Table 5: Case C, Effect of Radius of Curvature on Effective Spring Constant and Damping Coefficient

<table>
<thead>
<tr>
<th>$R_{fps}$ (m)</th>
<th>1.50</th>
<th>3.00</th>
<th>4.50</th>
<th>6.00</th>
<th>7.50</th>
<th>9.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_i$ (kg)</td>
<td>531,170.18</td>
<td>531,170.18</td>
<td>531,170.18</td>
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<td>531,170.18</td>
<td>531,170.18</td>
</tr>
<tr>
<td>$u$</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>$D$ (m)</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>$W$ (N)</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
</tr>
</tbody>
</table>

Figure 34: Friction Pendulum Bearing Hysteresis for Parameter Set 12

Figure 35: Friction Pendulum Bearing Hysteresis for Parameter Set 13
Figure 36: Friction Pendulum Bearing Hysteresis for Parameter Set 14

Figure 37: Friction Pendulum Bearing Hysteresis for Parameter Set 15

Figure 38: Friction Pendulum Bearing Hysteresis for Parameter Set 16
Table 6, which will be referred to as Case D, shows parameter set 17 through 21 which demonstrate the effect of the design displacement for the friction pendulum bearing on the system response. The design displacement is the parameter that corresponds to the predicted maximum displacement of the pendulum. Although this is not a physical design parameter, it is necessary in the derivation of the effective spring constant and damping constant. As indicated in the previous sections where the effective spring and damping coefficient formulas were derived, the design displacement affects both the spring constant and the damping of the system. However, as Figure 45 indicates, the effect of the design displacement on the natural frequency of the system is limited. Although the natural frequency decreases, it is a small change for each step size of 0.1 meters. Figure 40 through Figure 44 show the hysteresis response for parameter sets 17 through 21.
<table>
<thead>
<tr>
<th>Parameter Set 0</th>
<th>Parameter Set 17</th>
<th>Parameter Set 18</th>
<th>Parameter Set 19</th>
<th>Parameter Set 20</th>
<th>Parameter Set 21</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Standard Parameter</strong></td>
<td><strong>Displacement 1</strong></td>
<td><strong>Displacement 2</strong></td>
<td><strong>Displacement 3</strong></td>
<td><strong>Displacement 4</strong></td>
<td><strong>Displacement 5</strong></td>
</tr>
<tr>
<td>$K_{\text{eff}}$ (N/m)</td>
<td>5,035,366.67</td>
<td>6,598,066.67</td>
<td>4,254,016.67</td>
<td>3,993,566.67</td>
<td>3,863,341.67</td>
</tr>
<tr>
<td>$c_{\text{eff}}$ (N*s/m)</td>
<td>646,229.86</td>
<td>1,129,078.71</td>
<td>351,538.57</td>
<td>241,880.49</td>
<td>184,442.51</td>
</tr>
<tr>
<td>$R_{\text{fps}}$ (m)</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
</tr>
<tr>
<td>$m_1$ (kg)</td>
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<td>531,170.18</td>
<td>531,170.18</td>
<td>531,170.18</td>
<td>531,170.18</td>
</tr>
<tr>
<td>u</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>D(m)</td>
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<td>0.10</td>
<td>0.40</td>
<td>0.60</td>
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<td>W</td>
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<td>5,209,000.00</td>
<td>5,209,000.00</td>
<td>5,209,000.00</td>
</tr>
</tbody>
</table>

Table 6: Case D, Effect of Design Displacement on Effective Spring Constant and Damping Coefficient

![Friction Pendulum Bearing Hysteresis for Parameter Set 17](image)

Figure 40: Friction Pendulum Bearing Hysteresis for Parameter Set 17
Figure 41: Friction Pendulum Bearing Hysteresis for Parameter Set 18

Figure 42: Friction Pendulum Bearing Hysteresis for Parameter Set 19

Figure 43: Friction Pendulum Bearing Hysteresis for Parameter Set 20
Figure 44: Friction Pendulum Bearing Hysteresis for Parameter Set 21

Figure 45: Effect of Design Displacement on the Friction Pendulum Displacement
Selecting Friction Pendulum Bearing Parameters for Design Applications

Now that an understanding of how the design parameters affect the response of the friction pendulum has been obtained, it is important to analyze how each of the design parameters can be utilized to design a friction pendulum for a base isolation device for buildings and bridges.

To begin this analysis, the effect of the radius of curvature and the coefficient of friction on the natural frequency of the system will be looked at in Figure 46. The graph is limited to realistic values for a friction pendulum bearing. As seen in the figure, the natural frequency increases significantly for small radiiuses. However, for radiiuses greater than 5m, there is only a small decrease in natural frequency. Also, the coefficient of friction has a mild impact on the natural frequency of the system. As the coefficient of friction is increased, the natural frequency of the system increases.

Although Figure 46 shows the effects of the radius of curvature and coefficient of friction on the natural frequency, it is important to comment on the effect of the design displacement. The figure shown only demonstrates the natural frequencies that are capable with a design displacement of 0.2m. The natural frequency can also be shifted by altering the design displacement. As the design displacement increases, the natural frequency decreases. Different design displacements will result in surfaces that are nearly parallel to the surface shown in Figure 46.
The natural frequency of the friction pendulum is very important when selecting the design parameters for seismic base isolation. A friction pendulum base isolation system must be designed to protect the supported structure from the damaging effects, specifically from the shearing displacement, caused by seismic excitation. One part of this protection is to move the modes of vibration of the structure away from the frequencies of the earthquake. The forcing frequencies for earthquakes may fall anywhere in the range from 0.1Hz to 20Hz [12]. However, the frequencies commonly associated with earthquakes are between 1-4Hz like the El Centro Imperial Valley earthquake in 1940[13]. As a result, it is necessary to design the friction pendulum base isolation system so that neither the pendulum nor the natural frequencies of the structure are in this range. This can be accomplished by ensuring that the ratio between the natural frequency of the friction pendulum and the forcing frequency of the earthquake is not 1. Figure 47 shows this ratio as a function of radius of curvature and coefficient of friction. As can
seen for a design displacement of 0.2m, the ratio will only approach 1 for small radius of curvatures and a large coefficient of friction. In general, the friction pendulum should be designed to have a relatively large radius of curvature and a smaller coefficient of friction.

**Figure 47:** Ratio of Natural Frequency of the Friction Pendulum Bearing to the Forcing Frequency of an Earthquake as a Function of the Radius of Curvature and the Coefficient of Friction for a Design Displacement of 0.2m and Forcing Frequencies 1 Hz, 2Hz, and 4Hz

Although reducing friction will keep the natural frequency of the friction pendulum bearing away from the forcing frequencies of the earthquake, there is a trade-off with the damping. Figure 48 shows this more clearly. This figure shows that for coefficients of friction greater than 0.2, there is a less significant impact on the damping ratio. Also, increasing the radius more than 5m results in a less significant increase in the damping ratio.
Figure 48: Effect of Radius of Curvature and Coefficient of Friction on the Damping Ratio for a Friction Pendulum for a Design Displacement of 0.2m

As a result of these findings, it appears that when selecting a material for a friction pendulum bearing, the target value of the coefficient of friction should be near the point where there is a significant decrease in the slope of the surface. In Figure 48, this point occurs near a friction coefficient value of 0.2. If a value near this point is selected, the damping of the friction pendulum will be maximized without shifting the natural frequency of the pendulum near the likely forcing frequencies of an earthquake.

Effective Spring Constant and Damping Coefficient Restrictions

Although an effective spring constant and damping coefficient can be used to model the response of a friction pendulum bearing isolation system, there are several limitations to this modeling approach. The first limitation results from the small angle approximation that was
used to derive the equations of motion. The second limitation results from the fundamental difference between a viscous damper and a coulomb damper.

The modeling of the effective spring constant and effective damping constant is based on the assumption that the horizontal displacement of the system is relatively small when compared to the radius of curvature of the pendulum bearing. This enables \( \theta \approx \frac{x}{R} \). However, the error in this approximation can be calculated using Equation 30. Normally, the error in this approximation is relatively small.

\[
\%\text{error} = 1 - \frac{\text{Displacement}}{\text{Radius} \times \sin^{-1} \left( \frac{\text{Displacement}}{\text{Radius}} \right)}
\]

*Equation 30*

Unlike a viscously damped system, the motion of a system with coulomb damping is finite. A viscously damped system has exponential decay which was displayed earlier. As a result, a viscously damped system will theoretically never stop oscillating. However, a system with coulomb damping will stop when the restoring force is less than the friction force. Equation 31 shows the relationship for the friction pendulum system assuming a small angle approximation.

\[
mgu \geq \left( \frac{W}{R} + \frac{\mu W}{D} \right) x
\]

*Equation 31*

This understanding is not only important when creating a mathematical model of the system response. It is also important when designing a friction pendulum bearing. The bearing should be designed so that it is automatically self-centering. This means that after seismic activity has ceased, the building will return to its original position automatically.

**Analysis of the Effects of a Friction Pendulum Bearing Isolation System on the Dynamic Response of a Five Story Building**

The modeling approach utilizing an effective spring and damping constant to model the response of a friction pendulum bearing can be utilized to model a five story building during seismic activity with a friction pendulum base isolation system. Also, the understanding of the
impact of each of the design parameters of the friction pendulum will enable an optimal pendulum system to be selected for the structure. The considerations described earlier for picking an ideal friction pendulum will be utilized when designing a friction pendulum system for the five story building.

**Model Parameters for Five Story Building without a Base Isolation System**

Before the analysis of the effects of the friction pendulum base isolation system can begin, a model of a building without a seismic base isolation system will be analyzed. This model will serve as the basis of comparison for the model with a friction pendulum bearing isolation system. A multi degree of freedom model of the building will be developed to model the horizontal vibration of the five story building. A diagram of the five story building as well as the equivalent model for the five story building can be found in Figure 49. To simplify the modeling, each floor of the building will be modeled as a point mass. The steel beams between each floor will be represented by a spring between each point mass. For the model, it will be assumed that 12 steel beams are used to support each floor. The stiffness of these beams can be calculated utilizing the same approximation that was utilized in modeling the linear system with coulomb damping, Equation 3. The internal damping of the walls between each floor will also be modeled. A viscous damper between each point mass will be utilized to approximate this damping.

All the structure constants for the five story building are shown in Table 7 while the model parameters, such as effective spring constants and damping coefficients, can be found in Table 8. It is important to note that $K_1$ represents the attachment of the building to the ground. The large stiffness is a result of the 0.1m tall beams connecting the first floor to the ground. Later, this spring in the model will be replaced by the friction pendulum bearing.

<table>
<thead>
<tr>
<th>Structure Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dead Weight of Each Floor</td>
<td>$2.18 \times 10^5$ lbf</td>
</tr>
<tr>
<td>Type of Supporting Beams</td>
<td>W8 X 40 Wide-Flanged Beams</td>
</tr>
<tr>
<td>Steel Beam Modulus of Elasticity</td>
<td>$30 \times 10^6$ psi</td>
</tr>
<tr>
<td>Steel Beam Moment of Inertia</td>
<td>133.2 in$^4$</td>
</tr>
<tr>
<td>-------------------------------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\zeta$ (viscous damping ratio for building supports) [14]</td>
<td>0.010</td>
</tr>
<tr>
<td>Distance between First Floor and Ground</td>
<td>0.1m</td>
</tr>
<tr>
<td>Height of Each Floor</td>
<td>3m</td>
</tr>
</tbody>
</table>

**Table 7: Structure Parameters for Five Story Building**

---

**Figure 49: Diagram of Five Story Building without A Friction Pendulum Base Isolation System**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Constant Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>$1.651 \times 10^{12}$ N/m</td>
</tr>
</tbody>
</table>
Table 8: Five Story Building Model Parameter Constants

<table>
<thead>
<tr>
<th>$K_2, K_3, K_4, K_5$</th>
<th>$6.116 \times 10^7 \text{ N/m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1, C_2, C_3, C_4, C_5$</td>
<td>$2.459 \times 10^4 \text{ N*s/m}$</td>
</tr>
<tr>
<td>$M_1, M_2, M_3, M_4, M_5$</td>
<td>$9.887 \times 10^4 \text{ kg}$</td>
</tr>
</tbody>
</table>

To study the displacement response of the building to ground motion, an input ground acceleration ($\ddot{U}$) is used with a magnitude of 0.3g or $2.9 \text{ m/s}^2$. This value is based off of the peak acceleration for the El Centro Earthquake in 1940[13]. The acceleration will be input as a sinusoidal function. A 1 and 2 Hz signal will be utilized to analyze the response of the building to different frequencies.

Equations of Motion

The equations of motions for each of the floors of the building can be developed by drawing free body diagrams for each floor. The equations of motion for each floor are shown in Equation 32 through Equation 36 respectively.

$$m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 - c_2 (\dot{x}_2 - \dot{x}_1) - k_2 (x_2 - x_1) = c_1 \ddot{U} + k_1 U$$  \hspace{1cm} \text{Equation 32}

$$m_2 \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) - c_3 (\dot{x}_3 - \dot{x}_2) - k_3 (x_3 - x_2) = 0$$  \hspace{1cm} \text{Equation 33}

$$m_3 \ddot{x}_3 + c_3 (\dot{x}_3 - \dot{x}_2) + k_3 (x_3 - x_2) - c_4 (\dot{x}_4 - \dot{x}_3) - k_4 (x_4 - x_3) = 0$$  \hspace{1cm} \text{Equation 34}

$$m_4 \ddot{x}_4 + c_4 (\dot{x}_4 - \dot{x}_3) + k_4 (x_4 - x_3) - c_5 (\dot{x}_5 - \dot{x}_4) - k_5 (x_5 - x_4) = 0$$  \hspace{1cm} \text{Equation 35}

$$m_5 \ddot{x}_5 + c_5 (\dot{x}_5 - \dot{x}_4) + k_5 (x_5 - x_4) - c_6 (\dot{x}_6 - \dot{x}_5) - k_6 (x_6 - x_5) = 0$$  \hspace{1cm} \text{Equation 36}

Modal Analysis of Five Story Building without a Base Isolation System

To show the importance and impact of base isolation, a modal analysis will be performed on the five story building with a natural frequency near the frequencies commonly associated with seismic activity. Five story building usually have a period between 0.5s and 1s. The mass
and stiffness matrices for the five story building are listed in Equation 37 and Equation 38. The natural frequencies are listed in Equation 39.

It can be seen that the first mode of the building, referring to the horizontal motion of the structure, is 8.854 rad/sec. This corresponds to a period 0.888s. This frequency is in the middle of the range of frequencies for seismic signals. As a result, this five story building without a base isolation system is highly susceptible to earthquakes.

\[
K = \begin{bmatrix}
(k_1 + k_2) & -k_2 & 0 & 0 & 0 \\
-k_2 & (k_2 + k_3) & -k_3 & 0 & 0 \\
0 & -k_3 & (k_3 + k_4) & -k_4 & 0 \\
0 & 0 & -k_4 & (k_4 + k_5) & -k_5 \\
0 & 0 & 0 & -k_5 & k_5
\end{bmatrix}
\]

Equation 37

\[
M = \begin{bmatrix}
m_1 & 0 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 & 0 \\
0 & 0 & m_3 & 0 & 0 \\
0 & 0 & 0 & m_4 & 0 \\
0 & 0 & 0 & 0 & m_5
\end{bmatrix}
\]

Equation 38

\[
\omega_n = \begin{bmatrix}
8.854 \\
28.057 \\
43.796 \\
4.087 \times 10^3 \\
5.78 \times 10^3
\end{bmatrix}
\quad \text{rad/s} = \begin{bmatrix}
1.409 \\
4.465 \\
6.852 \\
650.466 \\
919.916
\end{bmatrix}
\]

Equation 39

**State Space Modeling of Five Story Building without a Base Isolation System**

The building with the friction pendulum will be analyzed using a state space model in Matlab Simulink. The state space format is as follows:

\[
[\dot{W}] = [A][W] + [B][\ddot{u}]
\]

\[
[X] = [C][W] + [D][\ddot{u}]
\]
The state space variable substitutions used can be found in Table 9 while the state space matrices can be found in Equation 40 through Equation 46. The state space matrices were developed from the equations of motion for the system.

<table>
<thead>
<tr>
<th>State Space Variable</th>
<th>Variable Substitution</th>
<th>Differential Equation Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$y_1$</td>
<td>$u - x_1$</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$y_1'$</td>
<td>$\dot{u} - \dot{x}_1$</td>
</tr>
<tr>
<td>$w_3$</td>
<td>$y_2$</td>
<td>$x_2 - x_1$</td>
</tr>
<tr>
<td>$w_4$</td>
<td>$\dot{y}_2$</td>
<td>$\dot{x}_2 - \dot{x}_1$</td>
</tr>
<tr>
<td>$w_5$</td>
<td>$y_3$</td>
<td>$x_3 - x_2$</td>
</tr>
<tr>
<td>$w_6$</td>
<td>$\dot{y}_3$</td>
<td>$\dot{x}_3 - \dot{x}_2$</td>
</tr>
<tr>
<td>$w_7$</td>
<td>$y_4$</td>
<td>$x_4 - x_3$</td>
</tr>
<tr>
<td>$w_8$</td>
<td>$\dot{y}_4$</td>
<td>$\dot{x}_4 - \dot{x}_3$</td>
</tr>
<tr>
<td>$w_9$</td>
<td>$y_5$</td>
<td>$x_5 - x_4$</td>
</tr>
<tr>
<td>$w_{10}$</td>
<td>$\dot{y}_5$</td>
<td>$\dot{x}_5 - \dot{x}_4$</td>
</tr>
</tbody>
</table>

Table 9: State Space Model Parameters

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-k_1/m_1 & -c_1/m_1 & -k_2/m_1 & -c_2/m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-k_1/m_1 & -c_1/m_1 & (k_2/m_1 + k_2/m_2) & -(c_2/m_1 + c_2/m_2) & k_3/m_2 & c_3/m_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & k_2/m_2 & c_2/m_2 & (k_3/m_2 + k_3/m_3) & -(c_2/m_2 + c_2/m_3) & k_4/m_3 & c_4/m_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}

[A]

Equation 40
The Simulink block diagram is shown in Figure 50. The output from the state space model is in Equation 47. The relative displacements of the system in response to a sinusoidal input of 1Hz and 2Hz with a magnitude of 0.3g can be found in Figure 52 and Figure 54. Only the relatively displacements between each of the floors of the building will be analyzed, and not the relative displacement between the first floor and the ground. The interest is in the shearing displacement of the building.
Figure 50: Simulink Block Diagram for the State Space Model of a Five Story Building without a Base Isolation System
Figure 51: Acceleration Response Without a Base Isolation System for a 1.0Hz Input Signal

Figure 52: Displacement Response Without a Base Isolation System for a 1.0Hz Input Signal
There are several observations that can be made regarding the system responses for the five story building without a base isolation system. First, the relative horizontal displacement between the first and fifth floor is large. However, the shearing displacement for the 2Hz signal
was much larger than the shearing displacement for the 1Hz signal. This demonstrates that the building is susceptible to frequencies which commonly occur during seismic activity, 1-4 Hz. Secondly, both the acceleration and displacement system responses are not smooth and sinusoidal in nature. This too is an undesirable characteristic of the response since it may increase the structural damage.

**Five Story Building with a Friction Pendulum Bearing Base Isolation System**

A friction pendulum bearing base isolation system can be added to the building in Figure 49 to reduce the shearing displacement of the structure. From the analysis of the effects of the design parameters in the previous sections, it is possible to design a friction pendulum base isolation system that will both increase the damping and shift the natural frequency of the building away from the common seismic forcing frequencies, 1-4 Hz. The selected pendulum design parameters are shown in the table below.

<table>
<thead>
<tr>
<th>Friction Pendulum Design Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of Curvature (R)</td>
<td>1.5m</td>
</tr>
<tr>
<td>Coefficient of Friction (∅)</td>
<td>0.3</td>
</tr>
<tr>
<td>Maximum Design Displacement (D)</td>
<td>0.2m</td>
</tr>
</tbody>
</table>

Table 10: Friction Pendulum Bearing Design Parameters

A relatively large radius of curvature was chosen to increase both the damping as well as shift the natural frequency of the structure away from the seismic frequencies. With the chosen parameter, the first natural frequency of the building shifts from 1.4 Hz to 0.6 Hz. This new natural frequency was calculated by completing a modal analysis of the five story building with the friction pendulum base isolation system. An effective spring constant for the friction pendulum bearing, $k_f$, replaced $k_1$ in the system shown in Figure 49. The effective spring constant value, $k_f$, is shown in Table 11.

Although a larger radius of curvature would decrease the natural frequency of the system further, it would result in a larger overall displacement of the building. This means that the design displacement would have to increase as well. In general, the movement of the base of the building should be minimized.

A coefficient of friction of 0.3 was chosen to increase the damping. Although this shifts the frequency closer to the frequency range of earthquakes, it is acceptable, as seen in the
response discussed later. It is important to note that the friction coefficient is limited by material selection. Normal friction coefficients values range between 0.03 and 0.20 although higher values may be obtained [15].

Finally, a maximum design displacement of 0.2m was chosen. This value was chosen after conducting research on commonly accepted values for friction pendulum bearings [11]. Also, after selecting this value, the maximum displacement of the friction pendulum bearing was confirmed by the response analysis for the friction pendulum bearing for inputs of 1Hz and 2Hz. Neither of these systems have a maximum displacement that exceeds the 0.2m.

With the selected design parameters, the effective spring constant and damping coefficient can be found. The equations developed in previous sections will be utilized to find these values, Equation 22 and Equation 28. In these equations, the weight on the friction pendulum will be the total weight of the structure, 3.878x10^3 kN. This results in the effective spring constant k_f and an effective damping constant value shown in the table below.

<table>
<thead>
<tr>
<th>Friction Pendulum Constant</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effective Spring Constant k_f</td>
<td>8.403x10^8 kg/s^2</td>
</tr>
<tr>
<td>Effective Viscous Damping Constant c_f</td>
<td>1.607x10^6 kg/s</td>
</tr>
</tbody>
</table>

Table 11: Effective Friction Pendulum Constants

A similar modeling approach to that used to analyze the response of the five story building without a base isolation system will be utilized to analyze the response of the building with a friction pendulum bearing. In the equations of motions and state space matrices developed earlier, k_1 and c_1 will be replaced with k_f and c_f. Once again Matlab Simulink will be utilized to generate the response from the state space model. The model will be run for both a 1Hz and 2Hz input signal with a magnitude of 0.3g. The acceleration and displacement response generated can be found in Figure 55 through Figure 60.
Figure 55: Acceleration Response of Floor 1 With a Friction Pendulum Bearing Base Isolation System for a 1.0Hz Input Signal

Figure 56: Acceleration Response of Floors 2-5 With a Friction Pendulum Bearing Base Isolation System for a 1.0Hz Input Signal
Displacement Response **With** a Friction Pendulum Bearing Base Isolation System for a 1.0 Hz Input Signal

![Displacement Response Graph](image)

**Figure 57:** Displacement Response With a Friction Pendulum Bearing Base Isolation System for a 1.0 Hz Input Signal

Acceleration Response **for Floor 1** **With** a Friction Pendulum Bearing Base Isolation System for a 2.0 Hz Input Signal

![Acceleration Response Graph](image)

**Figure 58:** Acceleration Response for Floor 1 With a Friction Pendulum Bearing Base Isolation System for a 2.0 Hz Input Signal
Figure 59: Acceleration Response for Floors 2-5 With a Friction Pendulum Bearing Base Isolation System for a 2.0Hz Input Signal

Figure 60: Displacement Response With a Friction Pendulum Bearing Base Isolation System for a 2.0Hz Input Signal
There are several important conclusions that can be drawn from a comparison between the response seen in the building without seismic base isolation and the system with seismic base isolation. First, as a result of shifting the natural frequency of the structure away from the forcing frequency of the earthquake, the deformation of the building is significantly reduced.

For a 1Hz input signal, the shearing displacement of each floor is almost reduced by half. For an input signal of 2Hz, the shearing displacement is reduced by almost a factor of 8. This is because the natural frequency of the structure with the friction pendulum bearing is further away from the 2Hz frequency than the non-isolated building modeled in the previous section. It is important to note that for even larger seismic frequencies such as 4Hz, 10Hz, and even 20Hz, the friction pendulum base isolation system is even more effective with reducing the deformation of the structure. It is important to also note that although the relative displacement between the first floor and the ground is not modeled in the figures, the displacement does not exceed the design displacement of 0.2m for any model in the range of 0.1Hz to 20Hz.

The addition of the friction pendulum base isolation system also causes a significant increase in acceleration of the first floor of the structure. This is a direct result from the reduced deformation of the structure. Therefore, the building behaves more like a rigid body. The acceleration is a result of the rigid body acceleration as opposed to the acceleration due to the deformation of the structure.
Conclusion

There are many types of seismic base isolation devices, from rubber bearings to sliding bearings, including the sliding friction pendulum bearing. No matter what type of bearing is used for base isolation, they all are designed for one purpose; to limit the damage to structures by earthquakes by adding flexibility to the structure while shifting the frequency of the building away from the frequency range of the ground motion.

To demonstrate the impact of the friction pendulum bearing on the response of a structure, a mathematical model can be developed. However, due to the complexity of the friction or coulomb damping between the sliding surfaces of the friction pendulum bearing, a mathematical solution cannot be directly obtained. As a result, an effective viscous damping system can be developed to approximate the response of the coulomb damping and simplify the modeling of the friction pendulum bearing. The development of the effective spring and viscous damper system demonstrates that the effective spring constant must include both the effects of the radius of curvature as well as the effects of the directional change of the frictional force. Without this inclusion, it is not feasible to model the linear decay of a system with coulomb damping with the exponential decay of a system with viscous damping, especially for free vibration.

The development of an effective spring constant and damping coefficient for the friction pendulum bearing demonstrates the effects each of the design parameters of the friction pendulum have on the system response. These design parameters are the radius of curvature, the coefficient of friction associated with the sliding surfaces of the friction pendulum, and the design displacement which corresponds to the maximum displacement of the friction pendulum bearing. An analysis of the friction pendulum bearing response shows:

- The natural frequency of the friction pendulum decreases as the radius of curvature is increased.
- Larger design displacements result in a lower natural frequency of the system.
- As friction increases, the natural frequency of the system increases.

Another parameter that can be analyzed is the damping ratio. As friction is increased, the damping in the system increases. However, as friction increases, the rate of increase of damping
decreases for higher coefficient of friction values. An increase of the radius of curvature also results in a small increase in damping.

The analysis of the effects of the friction pendulum bearing parameters can be used to design a friction pendulum bearing for a structural application. Ultimately the friction pendulum bearing must be designed to add damping to the system while shifting the natural frequency of the structure away from the frequency of the ground motion. To accomplish this, a large radius of curvature is desired although the value will be limited by the application. The design displacement will directly affect the design displacement, or the maximum horizontal movement of the structure. The other design parameter, friction, should be selected to be relatively large to increase damping. However, increasing friction will also result in a shifting of the building frequency closer to the frequency of the ground motion. As a result, the diminishing rate of increase in damping should be analyzed closely when selecting the friction value. The friction value will also be limited by material choices.

The analysis conducted in this paper is not all inclusive and is limited in scope. As a result, future work should be concentrated on expanding upon the modeling techniques discussed in this paper. The one dimensional model used throughout this thesis can be expanded to a three dimensional model which will take into account all degrees of freedom of the system, modeling both the shearing and vertical motion of the structure. In addition to this, a set of experimental tests could be conducted to compare the mathematical analysis with the real response of a system with a friction pendulum base isolation system [4]. For example, it may be shown that the friction coefficient cannot be treated as a constant, but is dependent on the velocity. Finally, the understanding of friction pendulum bearings could be further extended to include the double and triple friction pendulum bearings. Through the addition of curved surfaces, the responses of the system can be altered significantly as a result of the variable stiffness of the double and triple friction pendulum bearing [16]. The effects of the design parameters on these systems may be significantly different from the effects of the design parameters on the single friction pendulum bearing analyzed in this thesis.
Bibliography


Vita

Matthew Heid

Education

Penn State Erie, The Behrend College
Major: Mechanical Engineering
Minor: Computer Science
Expected Graduation: May 2012
Rank: First out of 122 ME Students

Work Experience

• GE Transportation Diesel Engine Testing Lab System Analyst Internship (September 2011-May 2012)
  o Developed and implemented a valve labeling system for the newly constructed diesel engine testing facility while revising the piping and instrumentation diagrams.
  o Evaluated test cell heat exchangers, ensuring that the test cell cooling systems were capable of meeting the requirements for the Tier 4 diesel engine upgrade.
  o Created a user-friendly preventative maintenance plan for the engine test lab equipment, reducing unwanted downtime caused by equipment failures.

• GE Transportation C-Class Sourcing Department Analyst Internship (May 2011-Present)
  o Implemented an automated forecasting system, improving accuracy for the department.
  o Utilized my programming skills, cross-functional communications, and sourcing knowledge to develop advanced department tools and reports, exposing an additional $900,000 in spend and ($12,000) in variance.
  o Organized training sessions for other members of the department, sharing my knowledge of the department and encouraging other interns to implement similar time-saving techniques in their daily tasks.

• GE Transportation C-Class Sourcing Department Project Support Internship (March 2010-January 2011)
  o Designed and created an automated reporting system, including the weekly financial report for C-class parts, saving over 10 hours of work each week.
  o Monitored $14 million contract and resolved pricing discrepancies with suppliers, reducing discrepancies by 90%.
  o Developed weekly reports for the C-Class Commodity, forecasting $120 million spend for 20,000 parts.
  o Gained valuable leadership experience while supporting the sourcing department team during their management transition.

Academic Experience

• Analyzed and tested the Cybersonics CyberWand™ Lithotripter and proposed a redesigned mechanism that will reduce probe breakages and wear during use while maintaining current performance standards.
• Designed critical components for a Vertical Axis Wind Turbine including the brakes, bearings, supporting structure, gears, and shafts while following the Engineering Design Process.
• Applied knowledge of signal conditioning, transient responses, data acquisition, and transducer operation to design experiments measuring vibrations, temperature changes, and stresses.
• Utilized knowledge of Strength of Materials and Dynamics to design a structurally sound and economical Play-Pump structure to pump water from a well, using common playground equipment.

Activities and Community Service

• **Tau Beta Nu Engineering Honors Society Treasurer (2010-2012)**
  o Worked with Electrical, Software, and Mechanical Engineers to expand our engineering knowledge beyond our respective majors, while serving the community.
• **Erie Children’s Museum Volunteer (2010)**
  o Organized museum activities and events while serving as a weekend volunteer.
• **Behrend Robotics Club Member (2009-2010)**
  o Mentored and organized a FIRST/4H Robotics team in Erie.
• **Lambda Sigma Honors Society (2009-2010)**
  o Worked to create a better Erie community through high-way cleanups, fundraisers for the needy, and numerous other service activities.
• **FIRST Robotics CIA Team 291 Mentor (2008-2010)**
  o Assisted high school students design and create a robot as part of the FIRST Robotics Challenge.

Awards

• Schreyer Honors College (2008-2012)
• The Evan Pugh Scholar Award (2012)
• The President Sparks Award (2010)
• Behrend Honors Program (2008-2010)
• The President’s Freshman Award (2009)
• First-Year Writing Award (2009)
• Dean’s List (2008-2012)

Other Work Experience

• Tutored college students in math and engineering as part of the Penn State Tutoring Program (2011-2012)
• Tutored at-risk students at Central Tech as part of PEPP (2008-2009)