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REPRESENTATION THEORY OF $\mathfrak{sl}(2,\mathbb{C})$ AND $\mathfrak{sl}(3,\mathbb{C})$

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Abstract

In this paper, we build the irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ from basic principles. We begin with the adjoint representation and develop the theory for any irreducible representation. We then generalize the approach for $\mathfrak{sl}(2, \mathbb{C})$ to show how to construct the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$. This includes generalizing the Cartan subalgebra, as well as the basic theory of root and weight space decompositions of representations.

Contents

1 Introduction

A *group* G is a non-empty set equipped with a binary operation on its elements such that the following properties are satisfied:

1. Closure: if x and y are in G then xy is in G.

2. Associativity: if x, y, and z are in G then $(xy)z = x(yz)$.

3. Identity: there is an element e in G such that $eg = g = ge$ for all g in G.

4. Inverses: for every element x in G there is an element y in G such that $xy = e = yx$.

A *representation* of a group G on a vector space V is a group homomorphism $\rho : G \to GL(V)$ such that for all g_1, g_2 in G,

$$\rho(g_1)\rho(g_2) = \rho(g_1 g_2).$$

A representation is *irreducible* if there is no proper subspace W of V such that W is invariant under ρ.

Passing to the representation allows us to make the group G more concrete, and allows for simpler computations. Since we can represent elements of G as linear transformations, aka matrices, we can take a complex calculation involving the group operation and turn it into a matrix calculation.

Groups and their representations occur all over the place; for example, we can form groups from the rotations or symmetries of an object. In representation theory, the focus shifts from G to the category of all representations of G. In fact, a group G is determined by its irreducible representations.

In this paper, we're interested in Lie groups of matrices. The two groups we're interested in are the special unitary groups $SU(2)$ and $SU(3)$. A matrix U is *unitary* if $U^* = U^{-1}$, where U^* denotes the conjugate transpose.

Definition 1.1. The Lie group $SU(2)$ is the group of 2×2 unitary matrices with determinant 1. Similarly, the Lie group $SU(3)$ is the group of 3×3 unitary matrices with determinant 1.

Definition 1.2. The Lie group $SL(2, \mathbb{C})$, is the 2×2 special linear group. It contains all complex 2×2 matrices with determinant 1. Likewise, $SL(3, \mathbb{C})$ contains all complex 3×3 matrices with determinant 1.

A *Lie algebra* $\mathfrak{g}$ is a vector space with a skew-symmetric bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

i.e.,

$$\begin{aligned}
[X, Y] &= -[Y, X] \\
[aX, bY] &= ab[X, Y] \\
[X + W, Y] &= [X, Y] + [W, Y] \\
[X, Y + Z] &= [X, Y] + [X, Z]
\end{aligned}$$

for all $W, X, Y, Z \in \mathfrak{g}, a, b \in \mathbb{C}$, such that

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

(the Jacobi identity).

In the case of matrix groups, representations of a matrix group G correspond to representations of a Lie algebra $\mathfrak{g}$. Matrix groups and their corresponding Lie algebras are related by the exponential map, as follows:

The *Lie algebra* $\mathfrak{g}$ of a matrix group G is given by

$$\mathfrak{g} = \{X |\, exp(tX) \in G \ \ \forall\, t \in \mathbb{R}\}$$

Definition 1.3. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ contains all of the complex 2×2 matrices which have trace equal to zero. Likewise, $\mathfrak{sl}(3, \mathbb{C})$ contains all complex 3×3 matrices with trace zero.

Definition 1.4. The Lie algebra $\mathfrak{su}(2)$ contains all of the real 2×2 skew-adjoint matrices with trace zero.

Representations of $\mathfrak{sl}(2, \mathbb{C})$ correspond to representations of $SU(2)$, not to representations of $SL(2, \mathbb{C})$ as one might think. Why?

First, representations of $SU(2)$ correspond to representations of $\mathfrak{su}(2)$. Representations of $su(2)$ are real-linear maps which map an algebra into the algebra of linear operators on some complex vector space. $su(2)$ and $\mathfrak{sl}(2,\mathbb{C})$ are related, since

$$\mathfrak{sl}(2,\mathbb{C}) = su(2) + isu(2).$$

It's a fact that every real-linear representation of $su(2)$ extends uniquely to a complex-linear representation of $\mathfrak{sl}(2,\mathbb{C})$. We can also go in the opposite direction to get a unique representation of $su(2)$ from a representation of $\mathfrak{sl}(2,\mathbb{C})$. This shows that representations of $SU(2)$ correspond to complex-linear representations of $\mathfrak{sl}(2,\mathbb{C})$. We can say the same thing for $SU(3)$ and $\mathfrak{sl}(3,\mathbb{C})$.

This is what we'll study: the problem of finding all the irreducible representations of the Lie algebras $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$.

2 Representation Theory of $sl(2, \mathbb{C})$

In this section, we'll show how one can classify the finite-dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ by highest weight. We begin with the standard, or adjoint, representation, whose weights are called roots, and use this to build the other irreducibles.

Definition 2.1. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra of all complex 2×2 matrices which have trace equal to zero. A natural basis for $\mathfrak{sl}(2, \mathbb{C})$ is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where

$$\begin{aligned} [H, E] &= 2E \\ [H, F] &= -2F \\ [E, F] &= H \end{aligned}$$

Definition 2.2. A *representation* of a Lie algebra $\mathcal{A}$ on a vector space V is a vector space homomorphism

$$\varphi : \mathcal{A} \to \mathfrak{gl}(V) = \mathrm{End}(V).$$

A representation of a lie algebra must also preserve the bracket relations

$$\varphi[X, Y] = [\varphi[X], \varphi[Y]]$$

for all $X, Y \in \mathfrak{g}$.

Definition 2.3. We call a representation *irreducible* if there is no proper subspace W of V such that W is invariant under φ.

The most obvious representation is the trivial representation, which maps all elements to zero (on a one-dimensional space). The simplest non-trivial representation is the *adjoint* representation, in which $\mathfrak{sl}(2, \mathbb{C})$ acts on itself as an $\mathfrak{sl}(2, \mathbb{C})$-module via the commutator:

$$\begin{aligned} \mathrm{ad} : \mathfrak{sl}(2,\mathbb{C}) &\to \mathrm{End}(\mathfrak{sl}(2,\mathbb{C})) \\ X &\mapsto \mathrm{ad}_X \end{aligned}$$

where for all $X, Y \in \mathfrak{g}$,

$$\mathrm{ad}_X(Y) = [X, Y].$$

Sometimes a representation is denoted simply by the vector space itself, so to avoid confusion it can be helpful to see the connection

$$\text{modules of } \mathfrak{g} \leftrightarrow \text{representation of } \mathfrak{g}.$$

(one can always be constructed from the other). Given a lie algebra representation

$$\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$$

the vector space V can be made into a $\mathfrak{g}$-module via

$$X \cdot v = \varphi(X)v$$

for all $X \in \mathfrak{g}, v \in V$. (This turns an element of $\mathfrak{sl}(2,\mathbb{C})$ into an invertible linear transformation.) Conversely, if V is a $\mathfrak{g}$-module, then we can make V a representation of $\mathfrak{g}$ by defining

$$\begin{aligned} \varphi : \mathfrak{g} &\to \mathfrak{gl}(V) \\ X &\mapsto \varphi(X) \\ \varphi(X) : &v \mapsto X(v) \end{aligned}$$

Definition 2.4. A lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h}$ is abelian and every $H \in \mathfrak{h}$ is semisimple, and $\mathfrak{h}$ is maximal with respect to these properties. It is essentially a maximal subalgebra of $\mathfrak{g}$ on which the Lie bracket is zero (the maximal abelian subalgebra of $\mathfrak{g}$).

For $\mathfrak{sl}(2,\mathbb{C})$, the Cartan subalgebra contains only multiples of one element:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

since any diagonalizable element of $\mathfrak{sl}(2,\mathbb{C})$ is in the span of H.

Given a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$,

$$\varphi : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V),$$

we can use a 'unitarian trick': let $\{v_\alpha\}$ be a basis for V, then

$$\varphi(H)v_\alpha = \lambda_\alpha v_\alpha$$

for some eigenvalue λ_α. Thus the action of H on V is diagonalizable. The eigenvalues λ_α are called the *weights* of the representation.

Definition 2.5. A *weight* for a Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}(V)$ is linear map $\lambda : \mathfrak{h} \to \mathbb{C}$ such that

$$V_\lambda = \{v \in V \mid H(v) = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$$

is a nonzero subspace of V.

Definition 2.6. The above vector space V_λ is called the *weight space* of V with weight λ, and is non-zero if and only if V contains a common, or simultaneous, eigenvalue for the elements of $\mathfrak{h}$.

Definition 2.7. In the special case when φ is the adjoint representation, the weights are called *roots* and the weight space is called the *root space*.

When we study the weight spaces of $\mathfrak{sl}(2,\mathbb{C})$, we are really studying the weight spaces of H. The problem is now to understand how H acts on V.

Again, let ad be the Lie algebra representation acting on $\mathfrak{sl}(2,\mathbb{C})$ by

$$\mathrm{ad}(H)(v) = [H, v].$$

To understand the representation, we need to understand what it does on the basis elements $\{E, F, H\}$:

$$\begin{aligned}
\mathrm{ad}(H)(E) &= [H, E] = 2E \\
\mathrm{ad}(H)(F) &= [H, E] = -2F \\
\mathrm{ad}(H)(H) &= [H, E] = 0
\end{aligned}$$

so we have roots $\lambda = -2, 0, 2$ and we can decompose $\mathfrak{sl}(2, \mathbb{C})$ into a direct sum of root spaces (the "root-space" decomposition):

$$\mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We denote these subspaces by V_{-2}, V_0, V_2 respectively, so we have the decomposition

$$\mathfrak{sl}(2, \mathbb{C}) \cong V_{-2} \oplus V_0 \oplus V_2.$$

Now let V_π be an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. We know that the action of H is diagonalizable, and

$$\pi(H)(v_\alpha) = \alpha v_\alpha,$$

where $v_\alpha \in V_\alpha$, which gives the following decomposition:

Lemma 2.8. *Let V_π be an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. Then*

$$V_\pi = H \oplus \left(\bigoplus_\alpha V_\alpha \right)$$

where each α is a weight of the representation.

Now we want to know how E and F act on each V_α; in particular, how H acts on $E(v)$ and $F(v)$ where $v \in V_\alpha$. This will tell us how E and F act on our decomposition since it is broken up by the action of H. Specifically, for $v \in V$,

$$\begin{aligned} H(E(v)) &= E(H(v)) + [H, E](v) = E(H(v)) + (\mathrm{ad}(H)(E))(v) \\ &= E((\alpha)v) + 2E(v) \\ &= (\alpha + 2)E(v). \end{aligned}$$

So

$$v \in V_\alpha \Rightarrow E(v) \in V_{\alpha+2},$$

which means that if v is an eigenvector of H with eigenvalue α, then $E(v)$ is an eigenvector of H with eigenvalue $\alpha + 2$. Similarly,

$$v \in V_\alpha \Rightarrow F(v) \in V_{\alpha-2}.$$

This tells us that

$$E(V_\alpha) \subset V_{\alpha+2}$$
$$F(V_\alpha) \subset V_{\alpha-2}$$

For every α occuring in the decomposition. By irreducibility, each α in the direct sum decomposition must be congruent to every other one mod 2, and the α's occur as $\ldots, \alpha-2, \alpha, \alpha+2, \ldots$:

$$\begin{tikzcd} \cdots V_{n-4} \arrow[loop below, "H"] \arrow[r, shift right, "E"'] & V_{n-2} \arrow[loop below, "H"] \arrow[l, shift right, "F"'] \arrow[r, shift right, "E"'] & V_n \arrow[loop below, "H"] \arrow[l, shift right, "F"'] \end{tikzcd}$$

Say the "highest" weight is n (as in the above diagram), which exists since any finite set of weights can be ordered so that there is a highest one. If we take $v \in V_n$, then v has eigenvalue n, and $E(v) = 0$ since otherwise $E(v)$ would have eigenvalue $n+2$, but V_n is the highest weight space.

We can't go any higher by applying E, so now we will apply F successively to $v \in V_n$. This forms the set

$$\{v, F(v), F^2(v), \ldots, F^k(v)\}$$

where $F^{k+1}(v) = 0$. This process must end at some value of k since V is finite dimensional and each V_α is one dimensional.

Claim 2.9. *The representation V is spanned by the set of vectors $\{v, F(v), F^2(v), \ldots, F^k(v)\}$.*

Proof. To show this we need to show that the span is invariant under the action of $\mathfrak{sl}(2, \mathbb{C})$ (clearly the vectors already span a subspace. Then, since V is irreducible, the span will be exactly V.)

We know F carries $F^m(v)$ to $F^{m+1}(v)$, Also $F^m(v)$ is an eigenvector for H since

$$\begin{aligned} H(F^m(v)) &= F^m(H(v)) + [H, F^m](v) \\ &= nF^m(v) - 2mF^m(v) \\ &= (n-2m)F^m(v), \end{aligned}$$

so the action of H is invariant on the span of the the set $\{v, F(v), F^2(v), \ldots, F^k(v)\}$. Now we just need to check that the action of E preserves the span:

$$\begin{aligned}
E(v) &= 0 \in \operatorname{span}(\{v, F(v), F^2(v), \ldots, F^k(v)\} \\
E(F(v)) &= F(E(v)) + [E, F](v) \\
&= F(0) + H(v) \\
&= nv \\
E(F^2(v)) &= (EF)(F(v)) \\
&= F(EF(v)) + [EF, F](v) \\
&= F(nv) + E[F, F](v) + [E, F]F(v) \\
&= nF(v) + 0 + H(F(v)) \\
&= nF(v) + (n-2)F(v) \\
&= (2n-2)F(v)
\end{aligned}$$

and in general

$$\begin{aligned}
E(F^m(v)) &= (n + (n-2) + (n-4) + \cdots + (n-2m+2))F^{m-1}(v) \\
&= m(n-m+1)F^{m-1}(v),
\end{aligned}$$

so the action of E preserves the span. ■

We have both an upper and lower bound on the V_α's, so $F^{k+1}(v) = 0$ for some k. If m is the smallest power such that $F^m(v) = 0$, then

$$\begin{aligned}
0 &= E(F^m(v)) \\
&= m(n-m+1)F^{m-1}(v)
\end{aligned}$$

and $F^{m-1}(v) \neq 0$, so

$$n - m + 1 = 0,$$

where n is the highest weight. So $n = m - 1$ and $n \in \mathbb{Z}^+$, since m was a positive integer. We get a decomposition

$$\{0\} = V_{n-2m} \xleftarrow{F} V_{-n} = V_{n-2(m-1)} \xleftarrow{F} \cdots \xleftarrow{F} V_{n-4} \xleftarrow{F} V_{n-2} \xleftarrow{F} \underset{\underset{H}{\circlearrowleft}}{V_n} = V_{m-1}$$

So we see that $\alpha_{\max} = -\alpha_{\min}$.

Given our highest weight n, we see that we get $n+1 = m$ many V_α's in our decomposition, with eigenvalues $n, n-2, \ldots, -n+2, -n$. So for each $n \in \mathbb{Z}^+$ there is a unique representation $V^{(n)}$ which is the $(n+1)$-dimensional representation with eigenvalues (i.e., weights) $n, n-2, \ldots, -n+2, -n$.

The representation is determined by the highest weight because once we know our highest weight we are able to determine the α's in the decomposition

$$V = \bigoplus V_\alpha,$$

and can then describe the entire representation.

This is related to the adjoint representation because the action of E and F on our representation V is determined by the action of $\mathrm{ad}(H)$ on E and F. For example,

$$H(E(v) = E(H(v)) + \mathrm{ad}(H)(E)(v).$$

This is why the values for α are a string of numbers congruent modulo 2: 2 and -2 are the roots of E and F respectively, and E and F act by adding or subtracting 2 from the weights of our representation.

We can now determine what our "new" $\{E, F, H\}$ are in the representation V with highest weight n.

H should act on each vector only by multiplying by an eigenvalue (the eigenvector remains unchanged):

$$H = \begin{pmatrix} n & 0 & \cdots & \cdots & 0 \\ 0 & n-2 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & -n+2 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}$$

Here the first column is given by $H(v_n)$, the second by $H(v_{n-2})$, and eventually the last by $H(v_{-n})$.

The action of E and F are given by

$$E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & & n-1 & 0 \\ 0 & 0 & \cdots & & 0 & n \\ 0 & 0 & \cdots & & 0 & 0 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ n & 0 & \cdots & 0 & 0 & 0 \\ 0 & n-1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & & 2 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

(Note: the value of the nonzero entries of E and F are different depending on text, but they always occur in E just above the diagonal so that E moves vectors from V_α to $V_{\alpha+2}$, while the nonzero entries in F always occur just below the diagonal because F moves vectors from V_α to $V_{\alpha-2}$.)

Examples 2.10. We saw above that we can think of the elements of a representation as matrices which act on a vector space V. For $\mathfrak{sl}(2,\mathbb{C})$, we can also think of its elements being represented as operators which act on the vector space of homogeneous polynomials of degree n in two variables.

1. The trivial representation is just $V = \mathbb{C}$, where all $c \in \mathbb{C}$ are sent to themselves.

2. The standard two-dimensional representation is $V = \mathbb{C}^2$. It's called the standard representation because this is the representation $\mathfrak{sl}(2,\mathbb{C})$ 'comes with' - it's the vector space on which $\mathfrak{sl}(2,\mathbb{C})$ acts naturally. If we let x and y be the standard basis vectors in $\mathbb{C}^2$, then $H(x) = x$ and $H(y) = -y$. So we get a decomposition

$$V = \mathbb{C}x \oplus \mathbb{C}y = V_{-1} \oplus V_1$$

   Another way we can picture this representation is to think of V as the space of homogeneous polynomials of degree two in x and y. We can identify E as $x\frac{d}{dy}$, F as $y\frac{d}{dx}$, and $H = [E, F]$ as $x\frac{d}{dx} - y\frac{d}{dy}$. Then we still get all the relations we would like.

3. The adjoint representation

   Definition 2.11. Inside of $V \otimes V$, we can also look at the span of all elements of the form $v_1 \otimes v_2 + v_2 \otimes v_1$. This is a subrepresentation of $V \otimes V$, given by

$$\{V \otimes V\}/\{v_1 \otimes v_2 + v_2 \otimes v_1 = 0\}$$

   This space is called the *symmetric square*, denoted Sym^2V.

   The adjoint representation is $W = Sym^2V = Sym^2(\mathbb{C}^2)$. From our analysis above, we saw that a basis consisted of the matrices E, F, and H, and that the W decomposes into three one-dimensional spaces:

$$W = W_{-2} \oplus W_0 \oplus W_2$$

Again, we can also picture this representation as the space of homogeneous polynomials of degree three in x and y. We identify E, F, and H in the same way we did above, and have a basis $\{x^2, xy, y^2\}$ of Sym^2V. We again get all of the relations we would like; for example,

$$\begin{aligned} H(x^2) &= 2x^2 \\ H(xy) &= 0 \\ H(y^2) &= -2y^2 \end{aligned}$$

So, we can also think of the decomposition of W as

$$W = \mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$$

4. More generally, the (n+1)-dimensional representation is $W = Sym^n(\mathbb{C}^2)$, with basis $\{x^n, x^{n-1}y, \ldots, y^n\}$. We can write

$$\begin{aligned} W &= \mathbb{C}x^n \oplus \mathbb{C}x^{n-1}y \oplus \ldots \oplus \mathbb{C}y^n \\ &= W_{-n} \oplus W_{-n+2} \oplus \ldots \oplus W_n \end{aligned}$$

since

$$\begin{aligned} H(x^{n-k}y^k) &= H(x^{n-k})y^k + x^{n-k}H(y^k) \\ &= (n-k)x^{n-k}y^k + x^{n-k}(-ky^k) \\ &= (n-2k)x^{n-k}y^k \end{aligned}$$

Since we can get such an irreducible representation for any n, we can see that

Lemma 2.12. *Any irreducible representation of* $\mathfrak{sl}(2,\mathbb{C})$ *is a symmetric power of the standard representation* $V = \mathbb{C}^2$.

3 Representation Theory of $sl(3, \mathbb{C})$

Now that we've analyzed the irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$, the next natural step is to look at the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$. We will proceed by analogy to our study of $\mathfrak{sl}(2, \mathbb{C})$.

For $\mathfrak{sl}(2, \mathbb{C})$, we decomposed an irreducible representation V of $\mathfrak{sl}(2, \mathbb{C})$ into a direct sum of eigenspaces based on the action of H, where H was the unique element of the Cartan subalgebra (up to scalars). Each $v \in V_\alpha$ was an eigenvector of H with eigenvalue α, a scalar.

The major difference for $\mathfrak{sl}(3, \mathbb{C})$ is that we have more than one element in our Cartan subalgebra - it's now a two-dimensional subspace $\mathfrak{h}$. This means that if we want to decompose an irreducible representation V of $\mathfrak{sl}(3, \mathbb{C})$ into a direct sum of eigenspaces based on the action of $\mathfrak{h}$, the weights of the decomposition need to be eigenvalues for each $H \subset \mathfrak{h}$, i.e. for a given weight α, every $v \in V_\alpha$ is an eigenvector for every $H \in \mathfrak{h}$. This means that for any eigenvector $v \in V_\alpha$ there is a function $\alpha : H \to \mathbb{C}$ such that for all $H \in \mathfrak{h}$,

$$H(v) = \alpha(H)v$$

where α is a complex-valued function depending linearly on H.

Definition 3.1. Given a complex vector space V, the *dual space* $V^* = Hom_{\mathbb{C}}(V, \mathbb{C})$ is the set of linear functionals on V.

Our statement above, combined with this definition, means that we want the eigenvalues for $\mathfrak{sl}(3, \mathbb{C})$ to be elements of $\mathfrak{h}^*$ instead of just scalars. This will make our study more complex.

$\mathfrak{h}$ is the subspace of commuting diagonalizable matrices of $\mathfrak{sl}(3, \mathbb{C})$, so all of the $H \in \mathfrak{h}$ are simultaneously diagonalizable. This says that if v is an eigenvector for an element $H_0 \in \mathfrak{h}$, then it is an eigenvector for every $H \in \mathfrak{h}$, just with different eigenvalues. This gives us an idea of why the following lemma is true (although the proof is not so obvious).

Lemma 3.2. *Any finite-dimensional representation V of $\mathfrak{sl}(3, \mathbb{C})$ has a decomposition*

$$V = \oplus V_\alpha,$$

where each α is an eigenvalue of $\mathfrak{h}$.

Definition 3.3. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(3, \mathbb{C})$ is defined as

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \middle| a_1 + a_2 + a_3 = 0 \right\}$$

Notation 3.4. Here we'll choose a basis for $\mathfrak{h}$. A natural choice of basis is the set of matrices H_{12} and H_{23}, where

$$H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We'll also define another matrix H_{13}:

$$H_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Note that $H_{12} + H_{23} = H_{13}$.

From this, we can describe $\mathfrak{h}^*$ explicitly, in terms of generators and relations.

Notation 3.5. There are three natural functionals on $\mathfrak{h}$, namely L_1, L_2, and L_3, where

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i$$

Now we can write $\mathfrak{h}^*$ as follows:

Definition 3.6. The dual of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{h}^*$, is identified with

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3 = 0)$$

For $\mathfrak{sl}(2, \mathbb{C})$, we looked first at the adjoint representation, and found a basis $\{X, Y, H\}$ of matrices where X and Y were eigenvectors for the action of ad_H on $\mathfrak{sl}(2, \mathbb{C})$. This allowed us to decompose the representation into a direct sum of weight spaces.

We want to similarly find a basis of $\mathfrak{sl}(3, \mathbb{C})/\mathfrak{h}$ where each basis element is an eigenvector for the action of $\mathrm{ad}_\mathfrak{h}$ on $\mathfrak{sl}(3, \mathbb{C})$. Once we do this, we'll have a decomposition

$$\mathfrak{sl}(3, \mathbb{C}) = \mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha),$$

where the $\alpha \in \mathfrak{h}^*$ are eigenvalues of $\mathfrak{h}$, i.e.

$$[H, X] = \alpha(H)X$$

for all $X \in \mathfrak{g}_\alpha$ and for all $H \in \mathfrak{h}$.

Notation 3.7. The following matrices constitute a basis of $\mathfrak{sl}(3, \mathbb{C})/\mathfrak{h}$: the six matrices E_{ij} ($i \neq j$) where the ijth entry is 1 and the other entries are all zero.

To see that these matrices are eigenvectors of $\mathrm{ad}_\mathfrak{h}$, we'll check for one of them, say E_{12}:

$$[H, E_{12}] = HE_{12} - E_{12}H = a_1E_{12} - a_2E_{12} = (a_1 - a_2)E_{12}$$

and in general, we get that

$$[H, E_{ij}] = (a_i - a_j)E_{ij} = (L_i - L_j)(H)E_{ij}$$

for each E_{ij}. From this, we can see that the matrices E_{ij} are eigenvectors of $\mathfrak{h}$ with corresponding eigenvalues $L_i - L_j$. Note that $[E_{ij}, E_{ji}] = H_{ij}$.

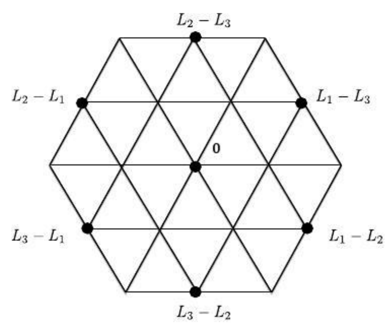
Notation 3.8. The matrices E_{ij}, along with H_{12} and H_{23}, constitute a basis of $\mathfrak{sl}(3, \mathbb{C})$. The Lie bracket relations are as follows:

$$[H_{ij}, E_{ij}] = 2E_{ij}$$
$$[H_{ij}, E_{ji}] = -2E_{ji}$$
$$[E_{ij}, E_{ji}] = H_{ij}$$

For the case of $\mathfrak{sl}(2,\mathbb{C})$, we only had two operators X and Y, our raising and lowering operators, which moved weight vectors to different weight spaces. Because of this, we were able to think of the weight spaces as lying on a line, with X and Y moving weight vectors up and down the line to neighboring weight spaces. X and Y are opposites in a sense - they act on the same line in opposite directions, and $X(Y(v))$ just sends $v \in V_\alpha$ to αv.

For $\mathfrak{sl}(3,\mathbb{C})$, we now have six such operators, so we will get a more complicated diagram, which will be in two dimensions. However, we similarly have three pairs of corresponding operators E_{ij} and E_{ji} which act on a line in opposite directions. It's not a surprise then that a representation of $\mathfrak{sl}(3,\mathbb{C})$ will have its weight spaces occuring on a triangular lattice, with points on the lattice differing by linear combinations of the L_i.

The diagram for the adjoint representation is as follows:



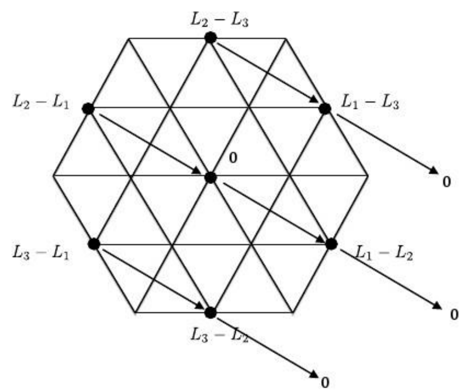
We know how $\mathfrak{h}$ acts on each of the weight spaces V_α, since every vector $v \in V_\alpha$ is an eigenvector of $\mathfrak{h}$ with eigenvalue α. We'd like to know how the other elements act, and we can find this out via an explicit calculation. Namely, let X be an element of $\mathfrak{g}_\alpha$ and Y be an element of $\mathfrak{g}_\beta$, where α and β are roots occuring in the decomposition of V. Then for any $H \in \mathfrak{h}$,

$$\begin{aligned}
[H,[X,Y]] &= [X,[H,Y]] + [[H,X],Y] \\
&= [X,\beta(H)Y] + [\alpha(H)X,Y] \\
&= (\alpha(H)+\beta(H))[X,Y].
\end{aligned}$$

This tells us that

Lemma 3.9. *Let V be the adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$, and let $X \in \mathfrak{g}_\alpha$. Given an eigenvector Y with weight β, $[X,Y] = ad(X)(Y)$ is again an eigenvector with weight $\alpha+\beta$.*

So $\mathrm{ad}(\mathfrak{g}_\alpha)$ sends each $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$, which means that each $\mathfrak{g}_\alpha$ acts by translation on the weight spaces by adding α to the current weight. For example, we can picture the action of $\mathfrak{g}_{L_1-L_2}$ as follows:



Now, let's look at an arbitrary representation $V = \bigoplus V_\alpha$ of $\mathfrak{sl}(3,\mathbb{C})$. To see how $\mathfrak{sl}(3,\mathbb{C})$ acts on V, we'll compute how the elements of $\mathfrak{h}$ act on images of the $v \in V_\alpha$. Let $X \in \mathfrak{g}_\alpha$ (where α is a root of the adjoint representation), $v \in V_\beta$, and $H \in \mathfrak{h}$. Then

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H,X](v) \\ &= X(\beta(H)v) + (\alpha(H)X)(v) \\ &= (\alpha(H) + \beta(H))X(v) \end{aligned}$$

So if v is an eigenvector for $\mathfrak{h}$ with weight β, then $X(v)$ is again an eigenvector of $\mathfrak{h}$ with weight $\alpha+\beta$. Since the roots α are of the form $L_i - L_j$, we see that:

Lemma 3.10. *The eigenvalues α occuring in an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ differ from one another by integral linear combinations of the vectors $L_i - L_j \in \mathfrak{h}^*$.*

This means that all of the weights of an irreducible representation lie in the same translate of the lattice generated by the $L_i - L_j$.

Definition 3.11. The lattice generated by the $L_i - L_j$, denoted here as $\Lambda_R \in \mathfrak{h}^*$, is called the *root lattice*.

This is just like in the case of $\mathfrak{sl}(2, \mathbb{C})$ - in an arbitrary irreducible representation, the weights differed by multiples of ± 2, which are the roots of the adjoint representation. From this we got a diagram of the weight spaces lying in a translate of $2\mathbb{Z}$.

For $\mathfrak{sl}(2, \mathbb{C})$, our next step was to take a vector v from the extremal weight space V_α. We found that E killed v, so we repeatedly applied F to generate the representation in terms of our highest weight vector v.

We'd like to do the same thing for $\mathfrak{sl}(3, \mathbb{C})$. At first glance, choosing an extremal weight space seems difficult, since we're not just on a line and we can move in more than one direction. However, there is a nice way to choose a highest weight space analogously to our method for $\mathfrak{sl}(2, \mathbb{C})$, and in fact our choice doesn't affect the end result.

What we want to do is find a vector $v \in V_\alpha$ which is killed by half of our operators E_{ij}. These three operators will be like our $E \in \mathfrak{sl}(2, \mathbb{C})$, and will be 'raising' operators. Then we will use the other three operators to generate the representation from this highest weight vector v, as we did before with $F \in \mathfrak{sl}(2, \mathbb{C})$. These operators will be our 'lowering' operators. To do this, we will first need to choose a linear functional l such that $l(\alpha) > 0$ for half of the roots $L_i - L_j$, so that our highest weight vector will be killed by half of the E_{ij}.

Notation 3.12. We'll choose to have the E_{ij} where $i < j$ as the raising operators (although any choice of three E_{ij} would give us the same representation). Define our linear functional $l : \mathfrak{h}^* \to \mathbb{C}$ as

$$l(a_1 L_1 + a_2 L_2 + a_3 L_3) = a a_1 + b a_2 + c a_3$$

where $a + b + c = 0$ and $a > b > c$. Then $l(L_i - L_j) > 0$ for $i < j$, and the corresponding E_{ij} generate the positive root spaces; likewise, $l(L_i - L_j) < 0$ for $i > j$, and these E_{ij} generate the negative root spaces.

From this choice, we see that

Lemma 3.13. *Let V be an irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$. Then there exists a $v \in V$ such that*

1. *$v \in V_\alpha$ for some α*
2. *v is killed by E_{12}, E_{13}, and E_{23}*

Such a vector v is called a *highest weight vector*.

Now we'll show that we can generate an irreducible representation by repeatedly applying the three lowering operators E_{ij} to a highest weight vector v.

Claim 3.14. *Let V be an irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$ and $v \in V$ a highest weight vector. Then V is generated by the images of v under successive application of the three operators E_{21}, E_{31}, and E_{32}.*

Proof. We'll approach this proof in the same way that we approached Claim 2.9 for $\mathfrak{sl}(2, \mathbb{C})$: we'll show that the subspace W of V that is spanned by the images of v under the repeated application of the operators E_{21}, E_{32}, and E_{31} is preserved by all of $\mathfrak{sl}(3, \mathbb{C})$. Obviously these three operators preserve W; so do elements of $\mathfrak{h}$, since they keep weight vectors in the same weight space. So we just need to check that the action of the three raising operators E_{12}, E_{23}, and E_{13} preserves W. It's enough to check this for only the first two, since $[E_{12}, E_{23}] = E_{13}$.

Let v be a highest weight vector with weight α. Then

$$\begin{aligned}
E_{ij}v &= 0 \quad \forall\, i < j \\
E_{12}(E_{21}(v)) &= (E_{21}(E_{12}(v)) + [E_{12}, E_{21}](v) \\
&= \alpha([E_{12}, E_{21}])v \\
E_{23}(E_{21}(v)) &= (E_{21}(E_{23}(v)) + [E_{23}, E_{21}](v) \\
&= 0
\end{aligned}$$

We can similarly compute that the image of $E_{32}(v)$ under these two operators remains in W. We can now make a general calculation, using induction, that will show that any vector in W is preserved by $\mathfrak{sl}(3, \mathbb{C})$. To do this, let w_n be any word of length n or less in the letters E_{21} and E_{32}, and let W_n be the vector space spanned by the vectors $w_n(v)$. Note that W then is the union of the spaces W_n. We'll show that E_{12} and E_{23} carry W_n into W_{n-1}. First, we can rewrite w_n as either $E_{21}(w_{n-1})$ or $E_{32}(w_{n-1})$, and (say) that $w_{n-1}(v)$ has eigenvalue β. Then using the first expression of w_n, we compute that

$$\begin{aligned}
E_{12}(w_n(v)) &= E_{12}(E_{21}(w_{n-1}(v))) \\
&= E_{21}(E_{12}(w_{n-1}(v))) + [E_{12}, E_{21}](w_{n-1}(v)) \\
&\in E_{21}(W_{n-2}) + \beta([E_{12}, E_{21}])w_{n-1}(v) \\
&\in W_{n-1} \\
E_{23}(w_n(v)) &= E_{23}(E_{21}(w_{n-1}(v))) \\
&= E_{21}(E_{23}(w_{n-1}(v))) + [E_{23}, E_{21}](w_{n-1}(v)) \\
&\in E_{21}(W_{n-2}) \\
&\in W_{n-1}
\end{aligned}$$

We can do the same thing for the second expression of w_n, which finishes the proof. ■

This claim has three consequences:

1. None of the images of $v \in V_\alpha$ under these three operators will end up back in V_α, since the weight of $E_{ji}(v)$ is equal to $\alpha - (L_j - L_i)$, which cannot equal α. From this we see that $dim V_\alpha = 1$. Thus v is the only unique element of V_α, up to scalars.

2. All the eigenvalues of the representation lie in a third of the plane with a corner at α. This $\frac{1}{3}$-plane is cut out by the lines containing the eigenspaces of the vectors $(E_{21})^k(v)$ and $(E_{32})^k(v)$. The representation can't lie in the other two-thirds of the plane because the three raising operators will kill any vectors on these two border lines.

3. Each of these border weight spaces $V_{\alpha+k(L_2-L_1)}$ and $V_{\alpha+k(L_3-L_2)}$ are at most one-dimensional, since they are each spanned by only the vector $(E_{21})^k(v)$ and $(E_{32})^k(v)$, respectively. There is no other way to reach these eigenspaces using only the three lowering operators E_{ij}.

From the proof of the claim above, we can also see that

Proposition 3.15. *If V is any (not necessarily irreducible) representation of $\mathfrak{sl}(3,\mathbb{C})$ and $v \in V$ is a highest weight vector, then the subrepresentation W of V generated by the images of v under successive applications of the operators E_{21}, E_{31}, and E_{32} is irreducible.*

Proof. From Lemma 3.13 and Claim 3.14, we know that W is a subrepresentation of V (since it is preserved by all of $\mathfrak{sl}(3,\mathbb{C})$) and that W_α is one-dimensional, where α is the weight of v. If W were not irreducible, then we would be able to decompose W into a direct sum of two subrepresentations, say $W = Y \bigoplus Z$. The projection of an element of W to the sum of an element of Y and an element of Z commutes with the action of $\mathfrak{h}$, so we have that $W_\alpha = Y_\alpha \bigoplus Z_\alpha$. W_α is one-dimensional, so one of the two spaces in the direct sum must be zero. Then v is in either Y or Z, which means that W is either Y or Z. ■

This proposition gives us that

Corollary 3.16. *Any irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ has a unique highest weight vector, up to scalars. More generally, for each irreducible subrepresentation W of V, we get a linear subspace which has dimension equal to the number of times W appears in the direct sum decomposition of V; this subspace corresponds to the highest weight of W. The union of these linear subspaces corresponds to the set of highest weight vectors of V.*

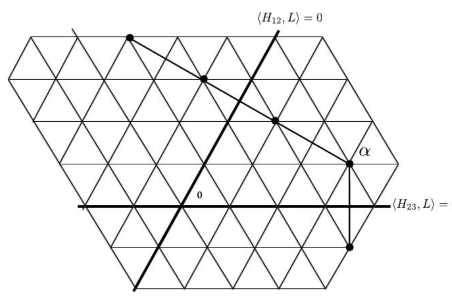
To understand how the rest of the representation unfolds, we'll first look at the border vectors of our $\frac{1}{3}$-plane. The vectors $(E_{21})^k(v)$ form one side of this border, and the corresponding weight spaces $\mathfrak{g}_{\alpha+k(L_2-L_1)}$ are all isomorphic to $\mathbb{C}$ (because they each contain only one unique vector) until we get to the first nonzero weight space.

How many nonzero weight spaces are there along this line? We can figure this out quite simply by first realizing that the elements E_{12}, E_{21}, and H_{12} span a subalgebra of $\mathfrak{sl}(3,\mathbb{C})$ which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, via an isomorphism which sends these three elements to E, F, and H, respectively. We can see this in the picture because the these three elements of $\mathfrak{sl}(3,\mathbb{C})$ all act along a line; namely, a translate of the line $L_2 - L_1$. Call this subalgebra s_{12}.

Since this subalgebra only moves vectors in the direction of $L_2 - L_1$ (and in the opposite direction), the subspace $W = \bigoplus \mathfrak{g}_{\alpha+k(L_2-L_1)}$ of V is preserved by the action of s_{12}. This means that W is a representation of $s_{12} \cong \mathfrak{sl}(2,\mathbb{C})$. From our analysis of $\mathfrak{sl}(2,\mathbb{C})$, we know that the weights arising from the action of H_{12} on our subspace W must be integral and symmetric with respect to zero. In this case, symmetric with respect to zero means symmetric across the line of reflection $\langle H_{12}, L\rangle = 0$. (This line is equivalent to the line aL_3, since H_{12} acts trivially on L_3.)

In general, the elements E_{ij}, E_{ji}, and H_{ij} span a subalgebra $s_{ij} \cong \mathfrak{sl}(2,\mathbb{C})$ of $\mathfrak{sl}(3,\mathbb{C})$; these three elements are mapped to E, F, and H, respectively.

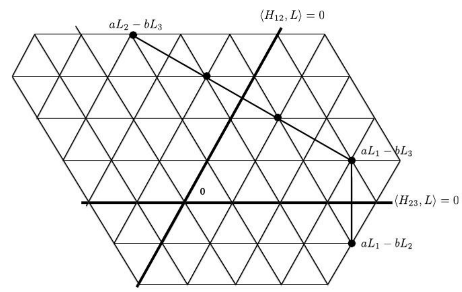
We can do the same thing with the border vectors $(E_{32})^k(v)$, getting a representation of $\mathfrak{sl}(2,\mathbb{C})$ which is preserved under reflection in the line $\langle H_{23}, L\rangle = 0$. Our picture so far looks like this:



We know that α is the weight of our highest weight vector v, where $\alpha = aL_1 - bL_3$ $(a, b > 0)$. Then $\alpha(H_{12}) = a$, $\alpha(H_{23}) = b$, and $\alpha(H_{13}) = a+b$.

Thinking about the vectors $(E_{12})^k(v)$ as belonging to a representation of $\mathfrak{sl}(2,\mathbb{C})$, we have that a is the heighest weight of this subrepresentation, based on the action of H_{12}. So there are (a+1) nonzero weight spaces on this line, symmetric with respect to the line $< H_{12}, L >= 0$, corresponding to weights $a, a-2, ..., -a+2, -a$ (based on the action of H_{12}). Each time we apply E_{21} to v we add $L_2 - L_1$ to α, and we take a steps, so the weight of the last weight space in this line is $aL_2 - bL_3$.

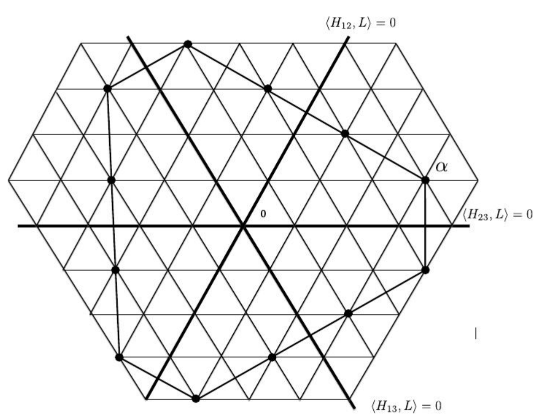
We can do the same thing for the vectors $(E_{32})^k(v)$, and find out that this line contains $(b+1)$ weight spaces, since $(E_{32})^b(v)$ is the last non-zero vector. Each time E_{32} is applied we add $L_3 - L_2$ to α, so the last weight space has weight $aL_1 - bL_2$. We can relabel our picture as follows:



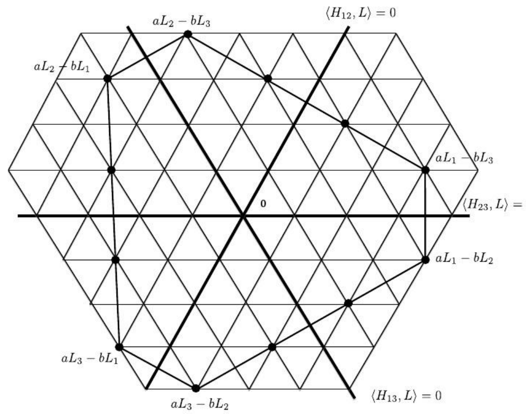
Let's go back to the last weight space on the string of vectors $E_{21}^k(v)$, which has weight $aL_2 - bL_3 := \beta$. If $w \in V_\beta$, then we know of course that $E_{21}w = 0$, but also (from the picture) that $E_{23}w = 0$ and $E_{13}w = 0$. Thus w would be a highest weight vector if we had chosen our linear functional l differently, since w is killed by half of the E_{ij}'s. We can use the same analysis of the previous two strings; this time we are applying E_{31} repeatedly to w, and get another representation of $\mathfrak{sl}(2,\mathbb{C})$. $(aL_2 - bL_3)H_{13} = b$, so this string allows for b applications of E_{31}. The final weight space has weight $aL_2 - bL_1$.

We can do the same thing with this weight space $V_{aL_1-bL_2}$; we'll get an a-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, with final weight $aL_3 - bL_2$.

Repeating this step once more with both the weight spaces $V_{aL_2-bL_1}$ and $V_{aL_3-bL_2}$, we'll get two more representationss of $\mathfrak{sl}(2,\mathbb{C})$ which meet at the same weight space, $V_{aL_3-bL_1}$. This completes the hexagonal border that has been formed. The diagram at this point is as follows:



And labeling the picture based on our expression of α in terms of the L_i, the diagram looks as follows:



Definition 3.17. The *dihedral group* D_n is the group of symmetries of the regular n-sided polygon. It has $2n$ elements - n rotations and n reflections. It can be generated by the n reflections.

Definition 3.18. The *symmetric group* S_n is the group of all permutations on n letters.

We generated this border by reflecting across the three lines $\langle H_{ij}, L\rangle = 0$. Our representation then is related to the dihedral group of a triangle, which is the same as the group S_3. So, we can say that the six corner weight spaces of the hexagon (which all could have been the highest weight space, depending on our choice of linear functional) are the images of V_α under the group S_3.

We know that the weights of any representation of $\mathfrak{sl}(2,\mathbb{C})$ must be integers. So, the weights along this hexagonal border must evaluate to integer values when we apply them to elements of $\mathfrak{h}$. This means that all of the weight functionals must be integral linear combinations of the L_i.

Proposition 3.19. All the eigenvalues of any irreducible finite-dimensional representation of $\mathfrak{sl}(3,\mathbb{C})$ must lie in the lattice generated by the L_i, and be congruent modulo the lattice $\Lambda_R \in \mathfrak{h}^*$ (the lattice generated by the $L_i - L_j$).

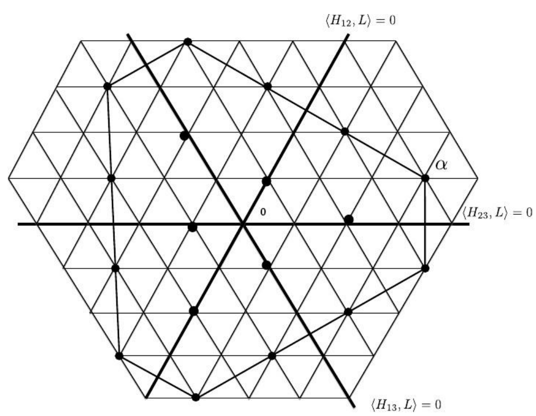
Definition 3.20. The lattice generated by the L_i is called the *weight lattice*.

Note that this is true for $\mathfrak{sl}(2,\mathbb{C})$ as well. In that case, any irreducible finite-dimensional representation lies in the lattice $\mathbb{Z}$ and is congruent modulo the lattice $2\mathbb{Z}$.

The subspace $W = \bigoplus \mathfrak{g}_{\alpha+k(L_1-L_2)}$ was preserved by the action of the subalgebra s_{12}. We can say the same thing about any subspace which lies on a line parallel to $L_2 - L_1$. In fact, for any $\beta \in \mathfrak{h}^*$ which is a weight of V, the direct sum

$$W = \bigoplus \mathfrak{g}_{\beta+k(L_i-L_j)}$$

will be a representation of $s_{ij} \cong \mathfrak{sl}(2,\mathbb{C})$ (but not necessarily irreducible). This means that for each such string of weight spaces, the values of k that appear form a consecutive string of integers. We now can get the rest of the representation by filling the inside of this hexagon with the weight spaces which lie on the same translate of Λ_R as the weight spaces along the border. The complete diagram of V is as follows:



Proposition 3.21. Let V be any irreducible, finite-dimensional representation of $\mathfrak{sl}(3,\mathbb{C})$. Then for some α which is a weight in the weight lattice, the set of eigenvalues occuring in V is exactly the set of linear functionals congruent to α modulo the lattice Λ_R and lying in the hexagon whose vertices are the images of α under the group generated by reflections in the lines $\langle H_{ij}, L\rangle = 0$.

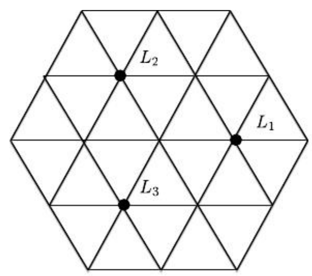
In the next section, we will get a classification of the irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$ by highest weight.

4 Examples and Description of the Irreducible Representations of $\mathfrak{sl}(3, \mathbb{C})$

Examples 4.1. Here, we'll first look at some examples of representations of $\mathfrak{sl}(3, \mathbb{C})$.

1. The standard representation, with $V \cong \mathbb{C}^3$

   $\mathfrak{sl}(3, \mathbb{C})$ acts naturally on $\mathbb{C}^3$ by matrix multiplication. The weight vectors are the standard basis vectors e_1, e_2, and e_3, with weights L_1, L_2, and L_3, respectively. The diagram for V is given by

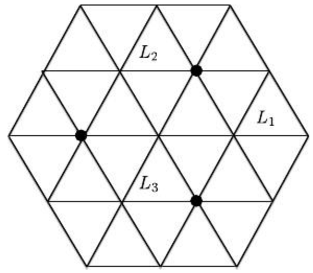
   

   Note: For all subsequent representations, V will continue to be the standard representation.

2. The dual representation V^*

   Definition 4.2. If $\pi : \mathfrak{g} \to End(V)$ is a representation of a Lie algebra $\mathfrak{g}$ on a vector space V, then we can define a represenation π^* of $\mathfrak{g}$ on V^* as follows:
   $$(\pi^*(X)\phi)v = \phi(-\pi(X)^T v)$$
   for all $v \in V$, $X \in \mathfrak{g}$, and $\phi \in V^*$.

   This is a representation, since the action of π^* respects the commutator. The weights of the dual V^* of V are the negatives of the weights of V. Then the diagram of V^* is as follows:



3. The exterior square $\Lambda^2 V$ and the general exterior product $\Lambda^n V$

   Definition 4.3. Let π be a representation of a Lie algebra $\mathfrak{g}$ on a vector space V, and ρ be a representation of $\mathfrak{g}$ on a vector space W. Then we can define a representation $\pi \otimes \rho$ of $\mathfrak{g}$ on $V \otimes W$ as follows:

   $$((\pi \otimes \rho)(X))(v \otimes w) = \pi(X) \otimes w + v \otimes \rho(X)w$$

   for all $X \in \mathfrak{g}$, $v \in V$ and $w \in W$.

   From this definition, given a representation π of $\mathfrak{g}$ on V, we can define a representation $\pi \otimes \pi$ on $V \otimes V$. Inside of $V \otimes V$, we can look at the span of all elements of the form $v_1 \otimes v_2 - v_2 \otimes v_1$. This is a subrepresentation of $V \otimes V$; we can write it as

   $$\{V \otimes V\}/\{v_1 \otimes v_2 - v_2 \otimes v_1 = 0\}$$

   Definition 4.4. The subrepresentation $\{V \otimes V\}/\{v_1 \otimes v_2 - v_2 \otimes v_1 = 0\}$ of $V \otimes V$ is called the *exterior square*, denoted $\Lambda^2 V$.

   Say that v_1 and v_2 are eigenvectors for the action of $\pi(H)$ with eigenvalues α and β, respectively. We can compute that $v_1 \otimes v_2 - v_2 \otimes v_1$ is an eigenvector for the action of $(\pi \otimes \pi)(H)$, with eigenvalue $\alpha + \beta$. This means that for our choice of V as the standard representation of $\mathfrak{sl}(3, \mathbb{C})$, the eigenvalues for $\Lambda^2 V$ are the pairwise sums of the distinct weights of V. From this, we can see that the diagram of $\Lambda^2 V$ will be the same as the diagram for V^*. Likewise, the diagram for $\Lambda^2 V^*$ is the same as the diagram for V.

   The general exterior product (also called the wedge product), $\Lambda^n V$, is also a representation of $\mathfrak{g}$, with weights which are n-fold sums of the distinct weights of V.

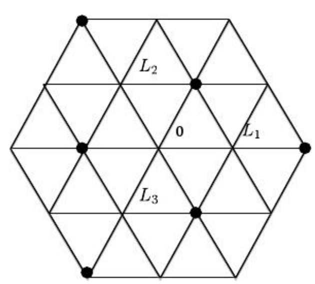
4. The symmetric square Sym^2V and the general symmetric product Sym^2V

Inside of $V \otimes V$, we can also look at the span of all elements of the form $v_1 \otimes v_2 + v_2 \otimes v_1$. This is a subrepresentation of $V \otimes V$, given by

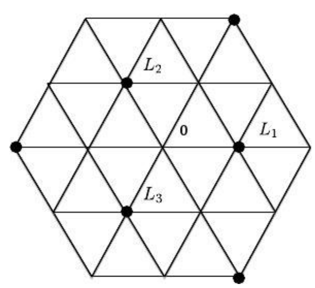
$$\{V \otimes V\}/\{v_1 \otimes v_2 + v_2 \otimes v_1 = 0\}$$

Definition 4.5. The subrepresentation $\{V \otimes V\}/\{v_1 \otimes v_2 - v_2 \otimes v_1 = 0\}$ of $V \otimes V$ is called the *symmetric square*, denoted Sym^2V.

We can compute that $v_1 \otimes v_2 + v_2 \otimes v_1$ is an eigenvector for the action of $(\pi \otimes \pi)(H)$, with eigenvalue $\alpha + \beta$. This means that for our choice of V as the standard representation of $\mathfrak{sl}(3, \mathbb{C})$, the eigenvalues for Sym^2V are the pairwise sums of the weights of V (we get all sums of the weights, not just the sum of distinct weights, because in this case we don't get the zero vector when v_1 and v_2 are the same vector). From this, we can see that the diagram of Sym^2V is given by



Similarly, the diagram for Sym^2V^* is as follows:

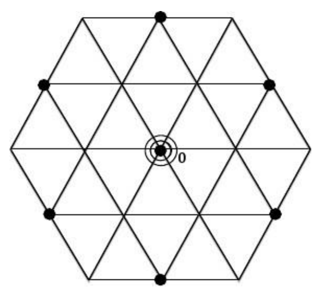


They are both irreducible, since neither set of weights is the union of the weights of two other representations of $\mathfrak{sl}(3, \mathbb{C})$.

In general, the symmetric product $Sym^n V$ is a representation of $\mathfrak{g}$ with weights which are n-fold sums of the weights of V. $Sym^n V$ and $Sym^n V^*$ are all irreducible, and they are exactly the representations with triangular diagrams instead of hexagonal ones.

5. The tensor product $V \otimes V^*$

   As we can see from the above examples using tensor products, the weights for this representation will be the sums of the weights $\{L_i\}$ of V with the weights $\{-L_j\}$ of V^*. The weights then will just be the linear functionals $L_i - L_j$, and 0. Each $L_i - L_j$ will occur once, and 0 will occur three times, once for each i. The diagram is as follows:

   We can see from the picture that this representation is not irreducible - the diagram is that of the adjoint representation, but with an extra two dimensions at the zero weight space. We can see this algebraically as well: there is a linear map from $V \otimes V^*$ to $\mathbb{C}$, given just by plugging $v \in V$ into a functional $\pi \in V^*$. This maps $v \otimes \pi$ to $\pi(v)$, and is a map of $\mathfrak{sl}(3, \mathbb{C})$-modules. If we identify $V \otimes V^*$ with the space $Hom(V, V)$, then this map is just the trace. Then the kernel of this map is the space of traceless matrices, which is just the adjoint representation.

Now we can describe how to get the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$, just like how we could say that for $\mathfrak{sl}(2, \mathbb{C})$, there is an $(n+1)$-dimensional representation for any $n \in \mathbb{N}$. First, we need

Lemma 4.6. *If the representations V and W have highest weight vectors v and w with weights α and β, respectively, then the vector $v \otimes w \in V \otimes W$ is a highest weight vector of weight $\alpha + \beta$.*

We saw this when we looked at $\Lambda^n V$ and $Sym^n V$. This lemma helps us to prove that

Theorem 4.7. *For any pair of natural numbers a, b there exists a unique irreducible, finite-dimensional representation of $\mathfrak{sl}(3, \mathbb{C})$ with highest weight $aL_1 - bL_3$.*

Proof. Existence: $Sym^n V$ and $Sym^n V^*$ are always irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$. Then $Sym^a V \otimes Sym^b V^*$ will contain an irreducible subrepresentation with highest weight $aL_1 - bL_3$.

Uniqueness: Suppose we have two irreducible representations V and W, both with highest weight α; let v and w be the highest weight vectors of V and W, respectively. Then (v, w) is a highest weight vector of the representation $V \oplus W$, again with highest weight α. Let $U \in V \oplus W$ be the irreducible representation generated by this highest weight vector. The projections maps $\pi_1 : U \to V$ and $\pi_2 : U \to W$ must be isomorphisms, since they are nonzero maps between irreducible representations. Then $\pi_1^{-1} \circ \pi_2 : V \to W$ is an isomorphism. ■

References

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