THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF MATHEMATICS

BERNOULLI POLYNOMIALS AND RIEMANN ZETA FUNCTION

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SPRING 2013

A thesis
submitted in partial fulfillment
of the requirements
for baccalaureate degrees in Mathematics and Economics
with honors in Mathematics

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ABSTRACT

In this paper, we derived Bernoulli polynomials from differential operators $e^{x\, d/dx}$, which was originally considered for summation of powers. We proved that Bernoulli polynomials are the unique polynomials satisfying certain properties. This further enables us to obtain a structure theorem for summing powers of integers. As an unusual case, we proved the Riemann zeta function could be represented in terms of Bernoulli numbers. Moreover, it has an analytic continuation by applying properties of Bernoulli polynomials.

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ACKNOWLEDGEMENTS

First and foremost, I would like to show my deepest gratitude to my supervisor, Professor Moisey Guysinsky, a respectable, responsible and resourceful scholar, who has provided me with valuable guidance in every stage of the writing of this thesis. His keen and vigorous academic observation enlightens me not only in this thesis but also in my future study.

I shall extend my thanks to Professor Sergei Tabachnikov for all his kindness and help. I would also like to thank all my professors who have helped me to develop the fundamental and essential academic competence.

Last but not least, I' d like to thank my parents and all my friends, for their encouragement and support.

Chapter 1

Introduction

Can one find a general formula for the summation of power

$$\sum_{k=1}^{n} k^m$$

in terms of n and m? Followed by the aim, the idea is to consider the differential operator $e^{\frac{d}{dz}}$ act on the space of all holomorphic functions over the complex numbers. Hence, denote $T = e^{\frac{d}{dz}}$ and X for the space, we thus have

$$T\Big(f\Big)(z) \ = \ e^{\frac{d}{dz}}\Big(f\Big)(z) \qquad \text{for all } \ f \in X. \tag{0. 1}$$

Observe that

$$e^{\frac{d}{dz}}\Big(f\Big)(z) = \Big(1 + \frac{d}{dz} + \frac{1}{2!}\frac{d^2}{dz^2} + \ldots\Big)f(z) \ = \ f(z) + f'(z) + \frac{f''(z)}{2!} + \ldots \tag{0. 2}$$

which is the Taylor expansion of f at $z+1$ where each coefficient is evaluated at z. Therefore, the map

$$T : f(z) \mapsto f(z+1)$$

is a shift linear operator on X.

Now, let the holomorphic function to be z^m and denote the differential operator $D = d/dz$, we observe that the sum of powers can be written as

$$\begin{aligned}
\sum_{k=1}^{n} k^m \ &= \ \Big(T^n + T^{n-1} + \ldots + T + 1\Big)\Big(z^m\Big)(0) \\
&= \ \Big(\frac{T^{n+1} - 1}{T - 1}\Big)\Big(z^m\Big)(0) \\
&= \ \Big(T^{n+1} - 1\Big)\frac{D}{T-1}\Big(D^{-1}z^m\Big)(0)
\end{aligned}$$

where $D^{-1}\Big(z^m\Big) = \displaystyle\int_0^z t^m dt.$

Remark: The reason to insert D in the above computation is that we will be interest in the map $e^{(n+1)z} - z/e^z - 1$ instead for replacing D with z. The problem of the only pole of the function $1/e^z - 1$ has been avoided by considering the function $z/e^z - 1$ and instead mapping the function f with $D^{-1}(f)$. In X, we have a nice set of holomorphic functions with their primitives. In the real case, we shall require the function to be integrable in addition to smoothness.

The shift operator T can be generalized by considering $e^{x\frac{d}{dz}} : X \mapsto X$. Again, let the function to be $z^m \in X$ and

$$\begin{aligned} e^{x\frac{d}{dz}}\Big(z^m\Big) &= \Big(1 + x\frac{d}{dz} + \frac{x^2}{2!}\frac{d^2}{dz^2} + \dots\Big)z^m \\ &= z^m + \mathbf{C}_m^1 z^{m-1}x + \mathbf{C}_m^2 z^{m-2}x^2 + \dots + \mathbf{C}_m^m x^m \\ &= (z+x)^m \end{aligned}$$

We thus have $e^{x\frac{d}{dz}} : z^m \mapsto (z+x)^m$. The Bernoulli polynomials $B_n(x)$ for $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$ are the coefficients in the Taylor expansion of the function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \qquad (0.\ 3)$$

evaluated at $z = 0$. The Bernoulli Numbers $B_n = B_n(0)$ is thus the coefficients in the Taylor expansion of the function $z/e^z - 1$ evaluated at $z = 0$.

We are going to investigate several important properties of Bernoulli numbers and Bernoulli polynomials that lead us to conclude a general formula of the summation of powers. In particular, if we instead, consider the power to be negative, it enable us to consider the sum of an infinite sequence provided that it is convergent. This further enlighten us the application of the properties of Bernoulli polynomials to the Riemann zeta function which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) > 1$ originally. It is well known that the function has an analytic continuation to the entire complex plane where the complete zeta function is

$$\xi(s) = \frac{1}{2}(s-1)s\Big(\frac{1}{\pi^{s/2}}\Big)\Gamma\Big(\frac{s}{2}\Big)\zeta(s) \qquad (0.\ 4)$$

which satisfies the functional equation $\xi(s) = \xi(1-s)$. However, the proof of the analytic continuation appeared in almost every text books is long and relies on the functional-analytic properties of Gamma and Theta functions. In this project, we will show that Riemann zeta function can be represented by Bernoulli numbers when s is taken to be real and even. Furthermore, we will derive a new proof for the analytic continuation of the Riemann zeta function by applying the properties Bernoulli numbers and Bernoulli polynomials, which eventually help us to obtain the formula and identity of $\xi(s)$.

Chapter 2

Bernoulli Polynomial

In this chapter, we prove an important theorem about Bernoulli polynomials. We shall see that Bernoulli polynomials satisfies a number of properties in related to obtain the general formula of summation of powers. Surprisingly, it turns out that the converse is also true.

Theorem.1: Bernoulli Polynomials are the only polynomials that satisfy

(i) $B_o(x) = 1$

(ii)$B'_n(x) = nB_{n-1}(x) \quad \text{for } n \geq 1$

(iii)$\int_0^1 B_n(x)dx = 0$ for $n \geq 1$, moreover, one has

$$\int_x^{x+1} B_n(t)dt = x^n$$

Proof: First, we show that Bernoulli polynomials satisfy the three properties by proving a series of lemmata.

Lemma 2.1

$$B_n(x) = \sum_{k=0}^{n} C_n^k B_k x^{n-k}$$

Proof: The proof is easy. We simply write out the Taylor expansion of e^{xz}, multiply it with formula

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \tag{0. 5}$$

will give us the result. □

Lemma 2.2

$$B'_n(x) = nB_{n-1}(x)$$

Proof: From lemma 2.1, we have

$$B_n(x) = \sum_{k=0}^{n} C_n^k B_k x^{n-k}$$

by differentiating with respect to x on both side, we have

$$B'_n(x) = \sum_{k=0}^{n} C_n^k B_k (n-k) x^{n-1-k} = n \cdot \sum_{k=0}^{n-1} C_{n-1}^k B_k x^{n-1-k} = nB_{n-1}(x) \tag{0. 6}$$

□

Lemma 2.3

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

and for $n \geq 2$ we thus have $B_n(1) = B_n(0) = B_n$.

Proof: We recall that $e^{x\frac{d}{dz}} : z^m \mapsto (z+x)^m$. Therefore

$$mx^{m-1} = \left(e^{\frac{d}{dz}} - 1\right)\frac{\frac{d}{dz}e^{x\frac{d}{dz}}}{e^{\frac{d}{dz}} - 1}\left(z^m\right)(0)$$

$$= \left(e^{\frac{d}{dz}} - 1\right)\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}\frac{d^n}{dz^n}\left(z^m\right)(0) \quad (0.\ 7)$$

$$= \mathbf{C}_n^1 B_{n-1}(x) + \mathbf{C}_n^2 B_{n-2}(x) + ... + B_0(x)$$

From lemma 2.2, we have $B_n'(x) = nB_{n-1}(x)$, so that

$$B_n(x+1) = e^{d/dx}(B_n(x)) = B_n(x) + \mathbf{C}_n^1 B_{n-1}(x) + \mathbf{C}_n^2 B_{n-2}(x) + ... + B_0(x) \quad (0.\ 8)$$

This concludes that

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

clearly by choosing $x = 0$, we have for $n \geq 2$, $B_n(1) = B_n(0)$ and this is equal to B_n by applying lemma 2.1. □

We observe that Bernoulli polynomials satisfy the three properties (i)-(iii). Indeed, (i) is obvious from computation, (ii) is proved by lemma 3.2 and (iii) is proved from lemma 3.2 and lemma 3.3 together.

Conversely, let $F_n(x)$ satisfy the three properties. Clearly that $F_o(x) = 1 = B_o(x)$, from (iii), we must have $F_n(1) = F_n(0)$, from (ii), we first observe that

$$F_1'(x) = 1 \cdot F_o(x) = 1 \qquad F_1(x) = x + c \qquad \text{for some } c \quad (0.\ 9)$$

Further, we shall have

$$F_2(x) = x^2 + 2cx + c' \qquad \text{for some } c' \quad (0.\ 10)$$

But $F_2(1) = F_2(0)$ implies $c = -1/2$ which is saying that $F_1(x) = B_1(x)$.

Inductively, suppose $F_k(x) = B_k(x)$, then

$$F_{k+1}' = (k+1)B_k(x) = B_{k+1}'(x) \qquad F_{k+1}(x) = (k+1)\int_1^x B_k(x)dx - (k+1)B_k(1) \quad (0.\ 11)$$

We thus have

$$F_{k+1}(x) = B_{k+1}(x) + c \qquad \text{for some } c \quad (0.\ 12)$$

Hence

$$F_{k+2}'(x) = (k+2)B_{k+1}(x) + c(k+2) \quad (0.\ 13)$$

and

$$F_{k+2}(x) = B_{k+2}(x) + c(k+2)x + c' \qquad \text{for some } c' \quad (0.\ 14)$$

However, we need $0 = F_{k+2}(1) - F_{k+2}(0) = B_{k+2}(1) - B_{k+2}(0) + c(k+2)$ which implies $c = 0$ □

Chapter 3

Summation of Powers

In this section, we derive the general formula for the summation of powers. Moreover, we are going to prove an important theorem which is about the structure of the summation of powers by using Bernoulli polynomials. Certain structure is first known due to the work of Johann Faulhaber.

We define

$$S_m(n) \;=\; 1^m + 2^m + \ldots + (n-1)^m \tag{0. 15}$$

The general formula in terms of Bernoulli polynomials is provided by the following lemma.

lemma 3.1

$$(m+1)S_m(n) \;=\; B_{m+1}(n) - B_{m+1}$$

Proof: From lemma 2.3, we have

$$B_m(x+1) - B_m(x) = mx^{m-1} \qquad\qquad B_m(1) = B_m(0) = B_m \quad \text{for } m \geq 2$$

Therefore we can write

$$\begin{aligned} B_m(x+1) - B_m \;&=\; \Big[\Big(B_m(x+1) - B_m(x)\Big) \;+\; \ldots \;+\; \Big(B_m(2) - B_m(1)\Big)\Big] \\ &=\; m(x^{n-1} + x^{n-2} + \ldots + x + 1) \end{aligned} \tag{0. 16}$$

by choosing $x = n$, we thus obtain the result. □

We now proceed to our next theorem which illstrate the structure of S_m.

Theorem.2
If m is odd, then $1^m + 2^m + \ldots + n^m$ has a factor $n^2(n+1)^2$.

If m is even, then $1^m + 2^m + \ldots + n^m$ has a factor $2n+1$.

Proof: We have

$$S_m(n) = B_{m+1}(n) - B_{m+1}$$

Assume m is odd and recall the formula

$$B_n(x) \;=\; \sum_{k=0}^{n} C_n^k B_k x^{n-k}$$

hence

$$S_m(n) \;=\; \sum_{k=0}^{m} C_m^k B_k n^{m-k} \;-\; B_{m+1} \tag{0. 17}$$

Because m is odd, $B_m = 0$ for $m \geq 3$, we thus have

$$S_m(n) \;=\; n^2 \cdot \sum_{k=0}^{m-2} C_m^k B_k n^{m-k-2} \tag{0. 18}$$

This shows that n^2 is a factor of $S_m(n)$ for $m \geq 3$ odd. It is also must be true that n^2 is also a factor of

$$S_m(n+1) \;=\; 1^m + 2^m + \ldots + (n-1)^m + n^m.$$

Since $(n, n+1) = 1$,we thus have $n^2(n+1)^2$ is a factor of $S_m(n+1)$. This concludes the first part of the theorem in related to m is odd. With this in hands, we are able to show something stronger where $1^m + 2^m + \ldots + (n-1)^m + n^m$ is in fact a polynomial respect to $n^2(n+1)^2$.

We first consider

$$2ln^{2l-1} \;=\; n^l(n+1)^l \;-\; n^l(n-1)^l \;-\; P_l(n) \tag{0. 19}$$

where $P_l(n)$ is some polynomial with degree less than $2l-1$.

We proceed the proof by using a sieve method. From (0. 19), we observe that

$$2ln^{2l-1} \;=\; 2ln^{2l-1} \;+\; \Big(2\mathbf{C}_l^3 n^{2l-3} \;+\; 2\mathbf{C}_l^5 n^{2l-5} \;+\; \ldots \;+\; 2\mathbf{C}_l^{l-1} n^{2l-(2l-1)}\Big) \;-\; P_l(n) \tag{0. 20}$$

This implies

$$P_l(n) = 2\mathbf{C}_l^3 n^{2l-3} \;+\; 2\mathbf{C}_l^5 n^{2l-5} \;+\; \ldots \;+\; 2\mathbf{C}_l^{l-1} n^{2l-(2l-1)} \tag{0. 21}$$

We thus have the formula for the summation

$$2l \sum_{k=1}^{n} k^{2l-1} \;=\; n^l(n+1)^l \;-\; \Big(2\mathbf{C}_l^3 \sum_{k=1}^{n} k^{2l-3} \;+\; \ldots \;+\; 2\mathbf{C}_l^{l-1} \sum_{k=1}^{n} k\Big) \tag{0. 22}$$

Hence

$$\sum_{k=1}^{n} k^{2l-1} \;=\; \frac{n^l(n+1)^l}{2l} \;-\; \frac{1}{2l}\Bigg[2\mathbf{C}_l^3\Big(\frac{n^{l-1}(n+1)^{l-1}}{2(l-1)} \;+\; \frac{1}{2(l-1)}\Big(2\mathbf{C}_{l-1}^5 \sum_{k=1}^{n} k^{2l-5} \;+\; \ldots \;+\; 2\mathbf{C}_{l-1}^{l-2} \sum_{k=1}^{n} k\Big)\Big)$$

$$+ \Big(2\mathbf{C}_l^5 \sum_{k=1}^{n} k^{2l-5} \;+\; \ldots \;+\; 2\mathbf{C}_l^{l-1} \sum_{k=1}^{n} k\Big)\Bigg]$$

where the rest terms $\sum_{k=1}^{n} k^{2l-5}$, $\sum_{k=1}^{n} k^{2l-7}$ ... can be inductively written in terms of the powers of $n(n+1)$ with constant coefficients depend on l. We hereby denote $N = n(n+1)$ and observe that by certain sieve method, we can write the summation in terms of N's powers. Indeed, we shall obtain

$$\sum_{k=1}^{n} k^{2l-1} \;=\; \frac{N^l}{2l} \;+\; \frac{a_1 N^{l-1}}{2(l-1)} \;+\; \frac{a_2 N^{l-2}}{2(l-2)} \;+\; \ldots \;+\; \frac{a_{l-1} N}{2} \tag{0. 23}$$

for $a_j \in \mathbb{R}$ and $j = 1, 2, \ldots, l-1$.

We recall that $B_j = 0$ for $j \geq 3$ and j is odd. So, if we consider the summation as function of x, i.e $\sum (x-k)^m$, then it is rational function and both the expressions the one here and the one in terms of Bernoulli numbers have to be equal. Therefore, we should have $a_{l-1} = 0$.

We now can estimate the summation of all even powers where

$$\sum_{k=1}^{n} k^{2l} = \sum_{k=0}^{n-1} (x-k)^{2l} \Big|_{x=n} \tag{0. 24}$$

and

$$\begin{aligned} (2l+1)\textstyle\sum_{k=1}^{n} k^{2l} &= \tfrac{d}{dx}\Big(\textstyle\sum_{k=0}^{n-1}(x-k)^{2l+1}\Big)\Big|_{x=n} \\ &= \frac{2n+1}{2}\Big(N^{l+1} + a_1 N^{l} + a_2 N^{l-1} + \ldots + a_{l-1}N\Big) \end{aligned} \tag{0. 25}$$

hence

$$\sum_{k=1}^{n} k^{2l} = \frac{2n+1}{2(2l+1)}\Big(N^{l+1} + a_1 N^{l} + a_2 N^{l-1} + \ldots + a_{l-1}N\Big) \tag{0. 26}$$

where $2n+1$ is a factor of $1^m + 2^m + \ldots + (n-1)^m + n^m$ while m is even. This completes the whole proof. □

Remark.2: The proof of the structure of summation of powers also known as Faulhaber's polynomials has several versions illustrated with modern mathematical language. In compare with it, our proof here followed a nature manner with an elementary taste. However, if we draw back to the time when Faulhaber first discover the formula without even knowing Bernoulli numbers, it is surprisingly to find that the proof we did here is more close to the original idea of Faulhaber.

Chapter 4

Riemann Zeta Function

In this section, we will show that the Riemann zeta function can be represented in terms of Bernoulli numbers. Moreover, we will prove the analytic continuation of the Riemann zeta function, namely ξ which is entire and satisfies the identity $\xi(s) = \xi(1-s)$ for $s \in \mathbb{C}$ by using the properties of Bernoulli polynomials. Recall that the Riemann zeta function is defined for $\mathfrak{R}(s) > 1$,

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Our next theorem is

Theorem.3

$$2\zeta(2m) \;=\; (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}$$

Proof: There are various ways to prove the formula. The one here we proceed is by proving a series of properties of Bernoulli numbers. In fact, the proof can be much shorter.

Lemma 4.1 Bernoulli numbers satisfies the formula

$$B_n \;=\; \frac{-1}{n+1}\sum_{k=0}^{n-1} \mathbf{C}_{n+1}^{k} B_k$$

Proof: By choosing the function $f(z) = z^n$ where $z \in \mathbf{C}$, we thus have

$$\begin{aligned}\Big(1\cdot f\Big)(0) &= \Big(e^{\frac{d}{dz}}-1\Big)\Big(\tfrac{\frac{d}{dz}}{e^{\frac{d}{dz}}-1}\Big)\Big(\tfrac{1}{n+1}z^{n+1}\Big)(0)\\ &= \Big(e^{\frac{d}{dz}}-1\Big)\Big(B_0\tfrac{z^{n+1}}{n+1}+B_1z^n+n\tfrac{B_2}{2!}z^{n-1}+\ldots\ldots+\tfrac{n(n-1)\ldots3}{(n-1)!}B_{n-1}z^2+B_nz+B_{n+1}\Big)(0)\end{aligned} \tag{0. 27}$$

Therefore

$$0 \;=\; B_0+(n+1)B_1+\frac{(n+1)(n)}{2!}B_2+\ldots\ldots+\frac{(n+1)(n)}{2!}B_{n-1}+(n+1)B_n \tag{0. 28}$$

implies

$$B_n \;=\; \frac{-1}{n+1}\sum_{k=0}^{n-1} C_{n+1}^{k} B_k$$

□

Lemma 4.2
$B_n = 0$ for n is odd and $n \geq 3$.

Proof: We have

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

It is obvious to find that the first two derivatives at $z = 0$ where $B_0 = 1$ and $B_1 = \frac{-1}{2}$. We can rewrite the formula to

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n \tag{0. 29}$$

The idea is to show that the function

$$\frac{z + ze^z}{2(e^z - 1)} = \frac{z}{e^z - 1} + \frac{z}{2} = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n \tag{0. 30}$$

is even. Let $z \mapsto -z$, we thus have

$$\frac{z + ze^z}{2(e^z - 1)} \mapsto \frac{-z - ze^{-z}}{2(e^{-z} - 1)} \tag{0. 31}$$

by multiplying e^z to both above and below, we have

$$\frac{-z - ze^{-z}}{2(e^{-z} - 1)} = \frac{-ze^z - z}{2(1 - e^z)} = \frac{z + ze^z}{2(e^z - 1)} \tag{0. 32}$$

Hence, on the right hand side, all odd term Bernoulli numbers must be zero. □

Lemma 4.3

$$z \cot z = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}$$

Proof: We first observe that

$$z \cot z = \frac{z \cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})z}{e^{iz} - e^{-iz}} \tag{0. 33}$$

Next, from the formula

$$\frac{z}{e^z - 1} + \frac{z}{2} = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n$$

and lemma 2.2, we can rewrite both the left side and right side into

$$\frac{z(1 + e^z)}{2(e^z - 1)} = 1 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n} \tag{0. 34}$$

Now,by replacing z with iz, we thus have

$$\frac{iz(1 + e^{iz})}{2(e^{iz} - 1)} = 1 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (iz)^{2n} \tag{0. 35}$$

and by dividing $e^{iz/2}$ on both nominator and denominator we get

$$\frac{iz(e^{-iz/2}+e^{iz/2})}{2(e^{iz/2}-e^{-iz/2})} \;=\; 1 \;+\; \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(iz)^{2n} \tag{0. 36}$$

And now replace $z/2$ to z, we get

$$\frac{iz(e^{-iz}+e^{iz})}{(e^{iz}-e^{-iz})} \;=\; 1 \;+\; \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(i2z)^{2n} \;=\; 1 \;+\; \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}z^{2n} \tag{0. 37}$$

□

Recall that the Riemann zeta function is defined by

$$\zeta(s) \;=\; \sum_{n=1}^{\infty}\frac{1}{n^s} \qquad \text{for all } \mathrm{Re}(s) > 1$$

we now ready to prove theorem 3. For α is a complex number not equal to an integer, we calculate the Fourier series of $f(x) = \cos(\alpha x)$. where we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\alpha x)\sin(nx)dx \;=\; 0 \quad \text{for all } n\in\mathbb{Z} \tag{0. 38}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\alpha x)\cos(nx)dx \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\Big(\cos((n+\alpha)x)+\cos((n-\alpha)x)\Big)dx \;=\; \frac{-\alpha\sin(\alpha\pi)}{\pi(n^2-\alpha^2)} \tag{0. 39}$$

for all $n \in \mathbb{Z}$.

Therefore we conclude that the Fourier series of

$$\pi\cos(\alpha x) \;=\; \sum_{-\infty}^{\infty}\frac{-\alpha\sin(\alpha x)}{n^2-\alpha^2} \tag{0. 40}$$

where the equality is provided by the absolutely convergent series $1/n^2$. Then, we thus have

$$\sum_{n=1}^{\infty}\frac{1}{n^2-\alpha^2} \;=\; \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha\tan(\alpha\pi)} \tag{0. 41}$$

So, for all $z \in \mathbf{C} - \pi\mathbb{Z}$, we have

$$\cot(z) \;=\; \frac{1}{z} \;+\; 2\sum_{n=1}^{\infty}\frac{z}{z^2-n^2\pi^2}. \tag{0. 42}$$

Now, consider the function

$$f(z) = \sum_{n=1}^{\infty}\frac{1}{z^2-n^2} \tag{0. 43}$$

by writing out its Taylor expansion series and evaluate at $z = 0$, we have

$$f(z) \;=\; f(0) \;+\; f'(0)z \;+\; \frac{f''(0)}{2!}z^2 \;+\; \ldots \tag{0. 44}$$

We observe that

$$f(0) = -\zeta(2) \quad f'(0) = 0 \quad f''(0) = -\zeta(4) \quad f'''(0) = 0 \quad f^{(4)}(0) = -\zeta(6) \quad \ldots \tag{0. 45}$$

since the term z^2 appears in the denominators of $f^{(n)}(0)$, for every $f^{(odd)}$, $2z$ always appears at the nominators of $f^{(odd)}$ which implies they are all equal to zero. Hence, we have

$$f(z) \;=\; -\sum_{m=0}^{\infty} \zeta(2m) z^{2m-2} \tag{0. 46}$$

Then, a straight forward computation will lead us the following identity

$$z\cot(z) \;=\; 1 - 2\sum_{m=1}^{\infty} \frac{\zeta(2m)}{\pi^{2m}} z^{2m} \tag{0. 47}$$

together with lemma 2.3, we thus obtain

$$2\zeta(2m) \;=\; (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}$$

□

Before we proceed to prove our next theorem about the analytic continuation of ζ into the entire complex plane, we need some estimates on the Gamma function defined as

$$\Gamma(s) \;=\; \int_0^{\infty} e^{-t} t^{s-1} dt$$

define for $\Re s > 0$.

lemma 4.4: Γ has analytic continuation to the entire complex plane whose only singularities are simple poles at $0, -1, -2, \ldots$.

Proof: Integrating by parts over the finite interval gives

$$\int_{\varepsilon}^{1/\varepsilon} \frac{d}{dt}(e^{-t}t^s)dt = -\int_{\varepsilon}^{\frac{1}{\varepsilon}} e^{-t}t^s dt + s\int_{\varepsilon}^{1/\varepsilon} e^{-t}t^{s-1}dt. \tag{0. 48}$$

By letting $\varepsilon \longrightarrow 0$, we obtain the left hand side vanishes since $e^{-t}t^s \longrightarrow 0$ as t tends to 0 or ∞. Also, we have

$$\Gamma(1) \;=\; \int_0^{\infty} e^{-t}dt \;=\; 1. \tag{0. 49}$$

Hence, (4.22)-(4.23) gives us

$$\Gamma(s+1) = s\Gamma(s). \tag{0. 50}$$

To prove the function has an analytic continuation to a meromorphic function on $\mathbf{C}$ whose only singularities are simple poles at $0, -1, -2, \ldots$, it is suffices to extend Γ to each half plane $\Re(s) > -m$ for every $m \in \mathbb{N}$. For $\Re(s) > -1$, define

$$F_1(s) \;=\; \frac{\Gamma(s+1)}{s}$$

which is meromorphic with only one simple pole at $s = 0$ since $\Gamma(1) = 1$. By letting

$$F_m(s) \;=\; \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s}$$

we thus obtain the result stated above. We now ready to state and prove our next theorem.

□

Theorem 4: The Riemann zeta function has an analytic continuation to an entire function

$$\xi(s) \;=\; \frac{1}{2}(s-1)s\Big(\frac{1}{\pi^{s/2}}\Big)\Gamma\Big(\frac{s}{2}\Big)\zeta(s)$$

which satisfies the functional equation $\xi(s) = \xi(1-s)$.

Proof: 1. We first consider the following exponential summation

$$\sum_{n=1}^{\infty} e^{-nx} \;=\; \frac{1}{e^x - 1}$$

Then, integrating

$$\int_0^{\infty}\sum_{n=1}^{\infty} x^{s-1}e^{-nx}dx \;=\; \sum_{n=1}^{\infty}\int_0^{\infty}\frac{t^{s-1}}{n^s}e^{-t}dt \;=\; \zeta(s)\cdot\Gamma(s) \tag{0. 51}$$

by changing variable $x \longrightarrow \frac{t}{n}$ gives us the relevant structures for ζ in terms of Γ. Hence, for $s > 1$, one can write

$$\zeta(s) \;=\; \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx \;=\; \frac{1}{\Gamma(s)}\Big(\int_0^1 + \int_1^{\infty}\Big)\frac{x^{s-1}}{e^x-1}dx.$$

where integral over 1 through infinity defines an entire function. The integral

$$\int_0^1 \frac{x^{s-1}}{e^x-1}dx \;=\; \sum_{m=0}^{\infty}\frac{B_m}{m!(s+m-1)}$$

where B_m denotes the m^{th} Bernoulli number defined by

$$\frac{x}{e^x-1} \;=\; \sum_{m=0}^{\infty}\frac{B_m}{m!}x^m$$

further enable us to estimate the convergence property of ζ via the properties of Bernoulli numbers.

2. Since the function $\frac{z}{e^z-1}$ is holomorphic for $|z| < 2\pi$, the radius of convergence implies that we must have

$$\limsup_{m\longrightarrow\infty}\Big|\frac{B_m}{m!}\Big|^{\frac{1}{m}} \;=\; \frac{1}{2\pi} < 1.$$

Hence, by root test, we have the series converges except possibly when $s = 1, 0, -1, -2, \cdots$ where there could be a simple zero on the denominator when m is taking over these values. But, the truth is the singularity only appears at $s = 1, -1, -3, -5, \ldots$. This comes from the fact that the poles are simple and all Bernoulli numbers $B_m = 0$ for $m \geq 3$ and m is odd which is proved in lemma 4.1.

3. However, those poles at all odd negative integers will be removed when we multiply $1/\Gamma(s)$ and left with simple zeros at $0, -2, -4, \ldots$. We have proved that $\Gamma(s)$ has simple poles at zero and all negative integers. This enable us to multiply $\Gamma(s/2)$ in the formula of $\xi(s)$. Moreover, multiplying $s(s-1)$ will further remove the singularities at 1. This proves ξ is entire. Observe further that by letting $s \longrightarrow 1-s$, the function $\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$ is still entire followed by the same argument as above. With respect to the identity (4.9), the extra term $\frac{1}{2}\frac{1}{\pi^{s/2}}$ in addition appeared in the formula of $\xi(s)$ ensures the functional equation

$$\xi(1-s) \;=\; \xi(s).$$

□

An alternative proof: In fact, the analytic continuation of the Riemann zeta function can be more directly proved by applying properties of Bernoulli polynomial. In that case, the faster proof lose the information for the functional identity of $\xi(s)$.

1. Recall

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) > 1$. It is suffice to show that $\zeta(s)$ has anayltic continuation for $\Re(s) > -k$ for any $k \in \mathbb{N}$. Consider the integrals

$$\int_n^{n+1} \frac{s}{x^{s+1}}dx \;=\; \frac{1}{n^s} \;-\; \frac{1}{(n+1)^s}$$

for $n = 1, 2, \ldots$. If we instead, put a weight of $\lfloor x \rfloor$ to each of the unit interval $[n, n+1)$ for the above integrals, we have

$$\int_n^{n+1} \frac{s\lfloor x \rfloor}{x^{s+1}}dx \;=\; \frac{n}{n^s} \;-\; \frac{n}{(n+1)^s}.$$

We thus have

$$\int_1^{\infty} \frac{s\lfloor x \rfloor}{x^{s+1}}dx \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which is our Riemann zeta function. Hence, by writing $x = \lfloor x \rfloor + \{x\}$, we have obtained a new representation formula of ζ such that

$$\zeta(s) \;=\; \frac{s}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx$$

where $\displaystyle\int_1^{\infty} \frac{1}{x^s}dx \;=\; \frac{1}{s-1}$. Observe that this extend ζ to $\Re(s) > 0$.

2. In another hand, for $0 < x < 1$, We have

$$B_1(x) \;=\; x \;-\; \frac{1}{2}.$$

and the Fourier series of $B_1(x)$ can be easily computed where

$$B_1(x) \;=\; x \;-\; \frac{1}{2} \;=\; \frac{-1}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k}$$

recall that all $B_n(x)$ are analytic, their Fourier series converge.

Since we have $B'_n(x) = nB_{n-1}(x)$, we can integrate and conclude that

$$B_{2n}(x) = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^{2n}} \tag{0. 52}$$

and

$$B_{2n+1}(x) = (-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^{2n+1}} \tag{0. 53}$$

by combining these two equations, in general we thus have for $0 < x < 1$,

$$B_n(x) = -\frac{n!}{(2\pi i)^n}\sum_{k\neq 0}\frac{e^{2\pi ikx}}{k^n}. \tag{0. 54}$$

Observe that $B_n(x)$ is bounded and as $n \longrightarrow \infty$, $|B_n(x)| \longrightarrow 0$ uniformly on the unit interval.

3. Now, write $Q(x) = \{x\} - \frac{1}{2}$ which is the sawtooth function coincided with $B_1(x)$ on the unit interval. Hence that

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s\int_1^{\infty}\frac{Q(x)}{x^{s+1}}dx.$$

It is clear that we can thus construct a sequence of functions $Q_k(x)$ recursively so that

$$\int_0^1 Q_k(x)dx = 0, \qquad \frac{dQ_{k+1}}{dx} = Q_k(x), \qquad Q_0(x) = Q(x), \qquad Q_k(x+1) = Q_k(x)$$

which is periodic one and maintain the relationship with the Bernoulli polynomials

$$Q_k(x) = \frac{B_{k+1}(x)}{(k+1)!}$$

for $0 < x < 1$. Therefore, with the convergence property of B_k and hence Q_k as $k \longrightarrow \infty$, together with the recursive differential property $\frac{dQ_{k+1}}{dx} = Q_k(x)$, we can write

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s\int_1^{\infty}\left(\frac{d^k}{dx^k}Q_k(x)\right)x^{-(s+1)}dx.$$

Indeed, an k-fold integration by parts gives the analytic continuation for $\zeta(s)$ when $\Re(s) > -k$.

□

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Academic Vitae

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March 18, 2013

1 Education

- Penn State University, University Park Schreyer Honors College
- Major : Mathematics Advisor : Sergei Tabachnikov tabachni@math.psu.edu
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- Master of Art Expected Graduate Time : May 2013
- Major : Economics
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2 Advanced Study Experience

- Graduate Level reading course on Game theory supervised by Professor Kalyan Chatterjee
- Mathematics Advanced Study Semester (MASS) program Fall 2010

3 Honors

- MASS Honors List : Best Performance in Final Oral Examinations 2010
- MASS Fellowship
- Schreyer Honors College Integrated Undergraduate/Graduate Study Scholar
- Research Fellowship from Schreyer Honors College 2011
- Women in Mathematics Scholarship
- Dean's List 2008-2013

4 Social

- Active member of WIM (Women in mathematics) 2009-present
- Active member of WISE (Women in Science and Engineering) 2009-present

5 Computer Skills

- LaTex, C++, R, Mathematica, Matlab, Python.