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INHOMOGENEOUS SPACE-TIME GEOMETRIES AS AN INTERACTING QUANTUM SYSTEM

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ABSTRACT

General Relativity is the current theory of gravity and it accounts for observational data that Newtonian gravity cannot explain. There are a few drawbacks to this otherwise successful theory. The greatest of these shortcomings is that General Relativity breaks down at high energy densities, like those in the early stages of the universe. It is suspected that Quantum Mechanical effects dominate in this limit, but unfortunately there is not a complete quantum theory of gravity yet. This thesis investigates the behavior of a perturbed space-time from the perspective of Loop Quantum Cosmology, a proposed quantum theory of gravity modified for applications to cosmology.

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1. INTRODUCTION

1.1 General Relativity

Einstein published his General Theory of Relativity in 1916. After publishing his Special Theory of Relativity, Einstein investigated a theory of gravity that would be consistent with Special Relativity. It is worth noting here that Newtonian Gravity is independent of time, implying changes in gravitation between two objects arbitrarily far away happen instantaneously. This is in contradiction with the speed limit imposed by Special Relativity and indicates the need for a new theory of gravity. The resulting theory is a completely different view of gravity than Newton's force law. General Relativity is a geometric description of gravity; it describes particles traveling along geodesics in curved space-time as opposed to a force deflecting particle trajectories.

A radical new idea like General Relativity requires significant empirical data behind it. In this aspect, General Relativity has been a resounding success. There are three particularly convincing results that support General Relativity: gravitational lensing, gravitational waves, and a precession in Mercury's orbit.

Gravitational lensing is a new concept introduced by General Relativity. Recall that General Relativity describes gravity in terms of curved space-time as opposed to Newtonian Gravity which describes gravity as a force between massive particles. The implication of this is Newton's picture demands that gravity does not deflect electromagnetic radiation whereas Einstein's picture does demand that gravity "deflect" electromagnetic radiation (since the waves propagate through curved space-time). Arthur Eddington first observed this effect in 1919 during a solar eclipse.[1] A star that was obscured by the sun was visible, its apparent position beside the sun.

Gravitational radiation is another new concept introduced by General Relativity. Unlike gravitational lensing, gravitational radiation is a phenomenon that has yet to be observed directly. However, there is very strong indirect evidence suggesting that gravitational radiation is a genuine physical process. General Relativity predicts that binary star systems lose energy through the emission of gravitational radiation and this loss in orbital energy causes the orbit itself to shrink and the orbital period to decrease. The quintessential observation of this effect is the orbital decay of the Hulse-Taylor pulsar binary, discovered in 1975. The measured orbital decay matches the General Relativity prediction within (0.13±0.21) percent.[2]

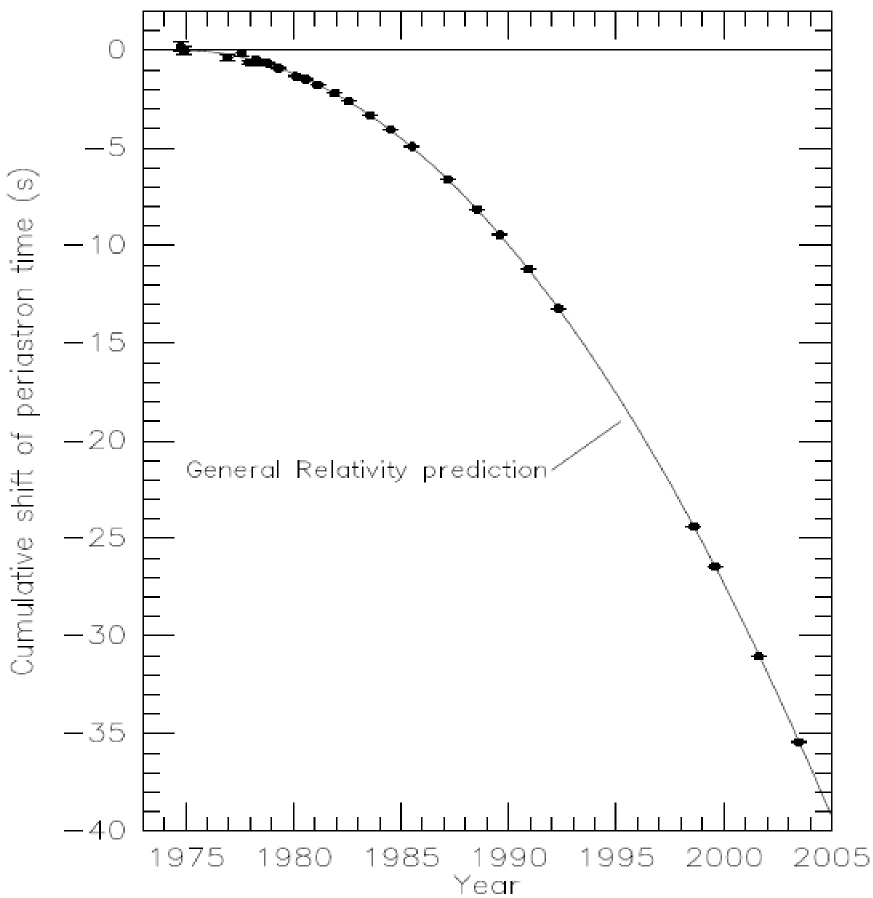

Figure 1.1: Orbital decay of PSR B1913+16. The data points indicate the observed change in the epoch of the periastron with date while the parabola illustrates the theoretically expected change in epoch for a system emitting gravitational radiation[2]

For all of the successes of Newtonian Gravity, it had a few known shortcomings. One of these was an unexplained precession in Mercury's orbit of 43 arcseconds per century. This fact was known since 1859. The prevailing theories at the time involved the effects of yet undetected planets. While this hypothetical object was given a name, it was never found. Einstein's General Relativity however predicted a correction term of 42.98 arcseconds per century in Mercury's orbit, accounting for the deviation from Newtonian gravity.[1]

In spite of its success, General Relativity is still not the last word on gravity. As elegant as Einstein's theory is, it is plagued by some very condemning problems. One major trouble is General Relativity is incompatible with Quantum Mechanics. If this issue was not difficult enough, a more fundamental problem arises from within the theory itself. Many solutions of Einstein's Equation contain singularities that cannot be removed by means of changing coordinates, implying that these infinities are somehow real physical phenomena. Worse still, it turns out these singularities are unavoidable. Hawking and Penrose published a paper in 1970 proving that any solution of Einstein's Equation corresponding to realistic conditions demanded the existence of a "physical" singularity in the past or the future.[3] All of this together suggests that General Relativity is incomplete, and a new theory of gravity is necessary.

1.2 Loop Quantum Gravity

General Relativity, for all of its successes, is incomplete, so a new theory of gravity is necessary. One attempt at a more complete theory of gravity is Loop Quantum Gravity. This is a relatively new theory postulated in 1990 by Carlo Rovelli and Lee Smolin. Loop Quantum Gravity attempts to extend the Ashtekar variables formulation of General Relativity into the quantum regime. Martin Bojowald introduced a finite, symmetry-reduced model of Loop Quantum Gravity that is especially useful for cosmology, called Loop Quantum Cosmology, in 1999. Loop Quantum Cosmology has been used to analyze the early universe, structure formation, and even offers a resolution to the problem of singularities in the classical theory of gravity.

A key property of Loop Quantum Gravity that is that space-time is that geometry is quantized; space-time itself is composed of "atoms." This feature plays a major role in Loop Quantum Cosmology. Particularly, quantized geometry is at the core of Loop Quantum Cosmology's resolution to the singularity problem. This thesis investigates the property of quantized geometry in cosmology.

1.3 Motivation

Loop Quantum Gravity is a promising new theory, but there is currently no empirical evidence to support it. It is possible that the quantized geometry may lead to some kind of testable, qualitative effect. This project aims to construct a nonlinear, Schrödinger-like equation for these volume chunks with the goal of comparing this equation to that of a Bose-Einstein condensate. The hope is that this equation will suggest a testable property of quantized geometry.

2. Classical Foundations

2.1 Overview

This project begins in classical General Relativity before shifting to quantum gravity and then into cosmology. As this thesis attempts to venture into how quantum effects could affect both structure formation and singularities in the early universe, it makes sense to use a cosmological model as a starting point.

The study begins with a simple model of universe that is isotropic, homogeneous, and space has a time-dependent factor. The system analyzed in this thesis remains isotropic, but it does not remain homogeneous. In the context of space-time, homogeneity demands an absence of any collections of matter. To remedy this, a first order, scalar perturbation is introduced to the system.

Once the system represents a time-dependent universe where matter can collect, that system changes formulation. The ultimate goal of this project is a wave-function that represents the behavior of discrete chunks of space, analogous to a non-linear Schrödinger equation. The Schrödinger equation is constructed in terms of a Hamiltonian, so the constructed space-time is reconstructed as a Hamiltonian.

2.2 Perturbation

This project begins with the familiar Friedmann-Robertson-Walker space-time from basic cosmology. Cosmology is at the core of this project, therefore the time evolution of this universe is of great importance. However, the Friedmann-Robertson-Walker solution is a vacuum solution; it doesn't admit clumps of matter to form. This is not a particularly interesting system to study (shown below).

$$g_{ab} = \begin{pmatrix} \text{-1} & 0 & 0 & 0 \\ 0 & a^2(\tau) & 0 & 0 \\ 0 & 0 & a^2(\tau) & 0 \\ 0 & 0 & 0 & a^2(\tau) \end{pmatrix} \tag{2.1}$$

Following the procedure in [4], a scalar perturbation with longitudinal gauge is added to the system to allow clumps of matter to form. The perturbed space-time metric becomes

$$g_{ab} = \begin{pmatrix} -\left(1+\frac{2L}{a^2}\right) & 0 & 0 & 0 \\ 0 & a^2+2L & 0 & 0 \\ 0 & 0 & a^2+2L & 0 \\ 0 & 0 & 0 & a^2+2L \end{pmatrix} \tag{2.2}$$

where a is the usual scale factor associated with the Friedmann-Robertson-Walker metric, and $L = L(\tau, x, y, z)$ is the perturbation variable (see Appendix A for more details).

2.3 Hamiltonian Formulation

The next stage of this process is to construct the classical Hamiltonian of this perturbed space-time. Specifically, this project employs the Arnowitt, Deser and Misner, or ADM, formalism for building a gravitational Hamiltonian. Once the perturbed space-time is in its Hamiltonian formulation, the process of discretizing volume and changing classical values into quantum operators becomes much easier. The Hamiltonian is constructed following the procedure in [4], using the space-time metric, the induced metric, and curvature tensors built from the induced metric. The expression found is then integrated over a surface of constant time, effectively decoupling space and time from each other. This becomes very helpful later on.

$$16\pi H_G = \int_{\Sigma_t} dy^3 [N\left(K^{ab}K_{ab} - K^2 -^3 R\right) - 2N_a D_b\left(K^{ab} - Kh^{ab}\right)]\sqrt{h} \tag{2.3}$$

In the equation above, h_{ab} is the induced metric, K_{ab} is the induced extrinsic curvature tensor, ${}^3R$ is the induced Ricci scalar, N is the lapse function, and D_b is a covariant derivative. This thesis uses the convention of Greek indices running from zero to four and Roman indices running from one to three. The induced metric, the lapse function, and N_a are found through direct comparison as follows. For this gauge choice, the line element takes the general form of

$$ds^2 = -N^2 d\tau^2 + h_{ab}\left(dx^a + N^a d\tau\right)\left(dx^b + N^b d\tau\right) \tag{2.4}$$

As the perturbed space-time metric is known, it is easy to find that the associated line element is

$$ds^2 = -\left(1 + \frac{2L}{a^2}\right) d\tau^2 + \left(a^2 + 2L\right)\left(dx^2 + dy^2 + dz^2\right) \tag{2.5}$$

The absence of terms of the form $f(\tau)dx^a d\tau$ suggests that $N^a = 0$. This immediately simplifies the Hamiltonian density to its first term. What is left is to read off the expressions for the lapse function and the induced metric.

$$N^2 = \left(1 + \frac{2L}{a^2}\right) \tag{2.6}$$

$$h_{ab} = \begin{pmatrix} (a^2 + 2L) & 0 & 0 \\ 0 & (a^2 + 2L) & 0 \\ 0 & 0 & (a^2 + 2L) \end{pmatrix} \tag{2.7}$$

The components of the Hamiltonian density still absent are the induced intrinsic curvature tensor and its associated scalar, and the induced metric Ricci scalar. These calculations are more complicated than just comparing line elements, beginning with the extrinsic curvature tensor.

It helps again that N_a is actually the zero vector. The semicolons represent the covariant derivative, but as they are applied to the zero vector, they are not particularly relevant. The next stage is constructing the extrinsic curvature tensor.

The handy equation for this below is found in [5].

$$K_{ij} = \frac{1}{2N}\left(\dot{h_{ij}} - N_{j;i} - N_{i;j}\right) = \frac{1}{2N}\left(\dot{h_{ij}}\right) = \frac{\dot{h_{ij}}}{2\sqrt{1+\frac{2L}{a^2}}} = \frac{a\dot{a}+\dot{L}}{\sqrt{1+\frac{2L}{a^2}}}\delta_{ij} \tag{2.8}$$

The extrinsic curvature tensor K appears in the Hamiltonian density as a contraction with itself and the square of its trace. The contraction is made easier by the fact that all of the tensors involved are diagonal. It is then possible to use the induced metric h_{ab} to raise indices and then contract this new tensor with the original K_{ab}. Following this, take the trace of K_{ab} and square the result.

$$K^{ab} = h^{ac}h^{bd}K_{cd} = h^{a1}h^{b1}K_{11} + h^{a2}h^{b2}K_{22} + h^{a3}h^{b3}K_{33} \tag{2.9}$$

$$K^{ab}K_{ab} = K^{11}K_{11} + K^{22}K_{22} + K^{33}K_{33} = 3\left(\frac{1}{a^2+2L}\right)^2\left(\frac{a\dot{a}+\dot{L}}{\sqrt{1+\frac{2L}{a^2}}}\right)^2 \tag{2.10}$$

$$K^2 = \left(h^{ij}K_{ij}\right)^2 = 9\left(\frac{1}{a^2+2L}\right)^2\left(\frac{a\dot{a}+\dot{L}}{\sqrt{1+\frac{2L}{a^2}}}\right)^2 \tag{2.11}$$

The calculation of the induced Ricci scalar is slightly more involved. This process begins with calculating the Christoffel symbols from the induced metric. Heavily relying on symmetry conditions, it becomes apparent that the only non-zero Christoffel symbols are those of the form Γ^i_{ii}, $\Gamma^i_{ij} = \Gamma^i_{ji}$, and Γ^i_{jj} where $i \neq j$. The argument is as follows.

$$\Gamma^a_{bc} = \frac{h^{ad}}{2}\left(\frac{\partial h_{db}}{\partial x^c} + \frac{\partial h_{dc}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right) \tag{2.12}$$

The fact that the induced space-time metric is diagonal allows the immediate simplification of this expression to

$$\Gamma^a_{bc} = \frac{h^{aa}}{2}\left(\frac{\partial h_{ab}}{\partial x^c} + \frac{\partial h_{ac}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^a}\right) \tag{2.13}$$

It immediately follows from the fact that h_{ab} is diagonal that for a Christoffel symbol to be nonzero, either $a = b$, $a = c$, or $b = c$, inclusively. This narrows the collection of Christoffel symbols down from sixty-four to fifteen. A few quick calculations show that

$$\Gamma^i_{ii} = \frac{\partial_i L(\tau, x, y, z)}{a^2(\tau) + 2L(\tau, x, y, z)} \tag{2.14}$$

$$\Gamma^i_{ji} = \Gamma^i_{ij} = \frac{\partial_j L(\tau, x, y, z)}{a^2(\tau) + 2L(\tau, x, y, z)} \tag{2.15}$$

$$\Gamma^i_{jj} = -\frac{\partial_i L(\tau, x, y, z)}{a^2(\tau) + 2L(\tau, x, y, z)} \tag{2.16}$$

The last step in collecting the pieces of the Hamiltonian density is assembling the induced Ricci scalar. From the definition of the Ricci tensor,

$$R_{ab} = \frac{\partial \Gamma^c_{ab}}{\partial x^c} - \frac{\partial \Gamma^c_{ac}}{\partial x^b} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc} \tag{2.17}$$

For these Christoffel symbols, the full induced Ricci tensor ${}^3R_{ab}$ becomes, entry-wise,

$$R_{ii} = \frac{L_j^2 - 2(\partial_j^2 L)L + L_k^2 - 2(\partial_k^2 L)L + 4L^2 - 4(\partial_x^2 L)L - a^2(2\partial_i^2 L + \partial_j^2 L + \partial_k^2 L)}{(a^2 + 2L)^2} \tag{2.18}$$

$$R_{ij} = R_{ji} = \frac{3L_j L_i - (a^2 + sL)\partial_i \partial_j L}{(a^2 + 2L)^2} \tag{2.19}$$

However, it is not the Ricci tensor itself that is of immediate importance for the Hamiltonian density. This calculation, while a touch tedious, is not difficult.

$${}^3R = h^{ab} R_{ab} = h^{ab} \left[\frac{\partial \Gamma^c_{ab}}{\partial x^c} - \frac{\partial \Gamma^c_{ac}}{\partial x^b} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc} \right] \tag{2.20}$$

$${}^3R = \frac{2\left(3(\frac{\partial L}{\partial x})^2 - 4L\frac{\partial L^2}{\partial^2 x} + 3(\frac{\partial L}{\partial y})^2 - 4L\frac{\partial L^2}{\partial^2 y} + 3(\frac{\partial L}{\partial z})^2 - 4L\frac{\partial L^2}{\partial^2 z}\right)}{(a^2 + 2L)^3} \tag{2.21}$$

All of the necessary components of the Hamiltonian density are assembled; all that remains is to build the full expression. Recall that

$$16\pi H_G = \int_{\Sigma_t} dy^3 \left[N\left(K^{ab}K_{ab} - K^2 -^3 R\right) - 2N_a D_b\left(K^{ab} - Kh^{ab}\right)\right]\sqrt{h} \quad (2.22)$$

Also recall that $N_a = 0$, making the expression of interest

$$16\pi H_G = \int_{\Sigma_t} dy^3 \left[N\left(K^{ab}K_{ab} - K^2 -^3 R\right)\right]\sqrt{h} = \int_{\Sigma_t} \mathcal{H}_{\mathcal{G}} dy^3 \quad (2.23)$$

Substituting everything in, the exact expression for the Hamiltonian density becomes

$$\mathcal{H}_{\mathcal{G}} = \sqrt{1+\frac{2L}{a^2}}\left(-6\left(\frac{1}{a^2+2L}\right)^2\left(\frac{a\dot{a}+\dot{L}}{\sqrt{1+\frac{2L}{a^2}}}\right)^2 - \frac{2\sum_{i=1}^{3}\left(3(\frac{\partial L}{\partial x^i})^2 - 4L\frac{\partial^2 L}{\partial x^{i2}}\right)}{(a^2+2L)^3}\right)\sqrt{h} \quad (2.24)$$

where h is the determinant of the induced metric,

$$\sqrt{h} = (a^2+2L)^{3/2} \quad (2.25)$$

Before progressing further into the project itself, it is worth doing a quick evaluation of this Hamiltonian density. Since this object is still firmly in the realm of classical General Relativity, it is easy to check that this Hamiltonian density reduces to the Hamiltonian density of the standard Friedmann-Robertson-Walker space-time when the perturbation variable L goes to zero. Repeating all of these calculations for the Friedmann-Robertson-Walker space-time gives $K^{ab}K_{ab} = \frac{3\dot{a}^2}{a^2}$, $K^2 = \frac{9\dot{a}^2}{a^2}$, ${}^3R = 0$, $N = 1$, and $N_a = 0$. Using these, the Hamiltonian density of this background metric is

$$16\pi H_{G_{FRW}} = \int_{\Sigma_t} dy^3 \frac{-6\dot{a}^2}{a^2} a^3 = \int_{\Sigma_t} -6a\dot{a}^2 dy^3 \quad (2.26)$$

$$\lim_{L\to 0} \mathcal{H}_{\mathcal{G}} = \frac{-6}{a^4}(a\dot{a})^2 a^3 dy^3 = -6a\dot{a}^2 dy^3 \quad (2.27)$$

After computing the Hamiltonian density of the perturbed space-time and checking that it does indeed reduce to that of the background Friedmann-Robertson-Walker space-time as the perturbation variable goes to zero, the only thing left to do in the framework of classical gravity is finding the the appropriate momentum tensor of this system. A quick limit shows that this momentum is consistent with its background metric.

$$\pi^{ab} = \sqrt{h}\left(K^{ab} - K^c_c h^{ab}\right) = \frac{-2}{\sqrt{1+\frac{2L}{a^2}}}\frac{a\dot{a}+\dot{L}}{\sqrt{a^2+2L}}\delta^{ab} = -2\left(\dot{a} + a\partial_\tau\left[\frac{L}{a^2}\right]\right)\delta^{ab} \tag{2.28}$$

$$\lim_{L\to 0}\pi^{ab} = -2\frac{a\dot{a}}{a} = -2\dot{a} \tag{2.29}$$

3. Departure from the Classical

3.1 Overview

Everything up until this point is an exact calculation made in the framework of classical General relativity. It is at this point that both exactness and classical gravity are left behind.

First, neither the Hamiltonian density nor the momentum are in a helpful form. This is where the approximations begin. The Hamiltonian density is expanded using power series, and all terms not second order in the perturbation variable are excluded. The effects of interest to this project would all occur in second order, therefore no important behavior is lost this way.

The second step of this chapter is the critical discretization condition. It is here that space is cut up into discrete chunks: the primary property that this project aims to analyze. It is worth noting here that this step discretizes space only and that time is left as it is.

The final step of this chapter is quantization: the process of changing classical variables into quantum mechanical operators. Making the assumption that discrete volume chunks compose space is not part of General Relativity, but it is still a classical model in the sense that it is not a quantum theory. This shift from classical values to quantum operators within the Hamiltonian construction mirrors the development of the Schroödinger equation.

3.2 Approximation

The Hamiltonian density found in Chapter 2 is exactly the Hamiltonian density of the constructed space-time. However, not only is this object cumbersome, but all interesting effects of the system occur in second order of the perturbation variable. It is at this point in the project that calculations cease to be exact. The regime of interest is, again, terms that are in second order with the perturbation variable and its derivatives. The natural course of action is to use power series expansion in the perturbation variable to make these terms more obvious and to exclude all uninteresting terms.

The process of approximation begins by examining each expression that appears in a denominator or in a radical within the Hamiltonian density. These expressions are not conducive to the steps that follow, so they are replaced by their respective power series approximations. Recall from Chapter 2 that the Hamiltonian density of this system is

$$\mathcal{H}_{\mathcal{G}} = \sqrt{1+\frac{2L}{a^2}}\left(-6\left(\frac{1}{a^2+2L}\right)^2\left(\frac{a\dot{a}+\dot{L}}{\sqrt{1+\frac{2L}{a^2}}}\right)^2 - \frac{2\sum_{i=1}^{3}\left(3(\frac{\partial L}{\partial x^i})^2 - 4L\frac{\partial^2 L}{\partial x^{i2}}\right)}{(a^2+2L)^3}\right)\sqrt{h} \tag{3.1}$$

where h is the determinant of the induced metric,

$$\sqrt{h} = (a^2+2L)^{3/2} \tag{3.2}$$

The perturbation variable L is small by assumption, so it follows to replace the problematic expressions within these equations with their respective Taylor Series in L centered at zero.

There are four expressions to expand. Beginning with the left of the Hamiltonian density and moving towards the right, the expressions to expand are:

$$\sqrt{1+\frac{2L}{a^2}} = 1 + \frac{1}{a^2} - \frac{L^2}{2a^4} + O(L^3) \tag{3.3}$$

$$\frac{1}{(a^2+2L)^2}\frac{1}{1+\frac{2L}{a^2}} = \frac{1}{a^4} - \frac{6L}{a^6} + \frac{24L^2}{a^8} + O(L^3) \tag{3.4}$$

$$\frac{1}{(a^2+2L)^3} = \frac{1}{a^6} - \frac{6L}{a^8} + \frac{24L^2}{a^{10}} + O(L^3) \tag{3.5}$$

$$\sqrt{h} = (a^2+2L)^{3/2} = a^3 + 3aL + \frac{3L^2}{2a} + O(L^3) \tag{3.6}$$

Making the appropriate substitutions and simplifying produces the three terms:

$$\mathcal{H}_{\mathcal{G}} = A + B + C \tag{3.7}$$

$$A = \frac{-2\Sigma_{i=1}^{k=3}(-4\partial_i^2 L + 3(\partial_i L)^2 + 3a^2(a\dot{a} + \dot{L})^2)}{a^3} \tag{3.8}$$

$$B = \frac{4\Sigma_{i=1}^{k=3}(-4\partial_i^2 L + 3(\partial_i L)^2 + 3a^2(a\dot{a} + \dot{L})^2)L}{a^5} \tag{3.9}$$

$$C = -\frac{8\Sigma_{i=1}^{k=3}(-4\partial_i^2 L + 3(\partial_i L)^2 + 3a^2(a\dot{a} + \dot{L})^2)L^2}{a^7} \tag{3.10}$$

Removing all all terms that are not of order two in the perturbation variable and its derivatives yields

$$A = \frac{-6\Sigma_{i=1}^{k=3}((\partial_i L)^2 + a^4\dot{a}^2 + a^2\dot{L}^2)}{a^3} \tag{3.11}$$

$$B = \frac{4\Sigma_{i=1}^{k=3}(-4L\partial_i^2 L + 3a^3\dot{a}L\dot{L})}{a^5} \tag{3.12}$$

$$C = -\frac{24a^4\dot{a}^2 L^2}{a^7} \tag{3.13}$$

Combining all three terms produces

$$\mathcal{H}_{\mathcal{G}} \approx -6\left(a\dot{a}^2 + \frac{\dot{L}^2 - 4(\dot{a}/a)\dot{L}L + 4(\dot{a}/a)^2L^2}{a} + \frac{1}{a^3}\sum_{s=1}^{3}\left(\left(\frac{\partial L}{\partial x^s}\right)^2 - \frac{4}{3}L\left(\frac{\partial^2 L}{\partial x^{s^2}}\right)\right)\right) \tag{3.14}$$

It is with some foresight that the Hamiltonian density goes through one more simplification here. The discretization process goes more smoothly as well as making the expression a little more compact. The expression will be rewritten in terms of L/a^2 instead of L by itself. Not only does this streamline the discretization in the next section, it also makes the expression more compact. Notice that

$$\dot{L}^2 - 4(\dot{a}/a)\dot{L}L + 4(\dot{a}/a)^2L^2 = (\partial_\tau(L/a^2))^2 \tag{3.15}$$

This makes the approximated Hamiltonian density

$$\mathcal{H}_{\mathcal{G}} \approx 6a^3 \left(\left(\frac{\dot{a}}{a}\right)^2 + \left(\partial_\tau \left(\frac{L}{a^2}\right)\right)^2 + \frac{7}{3a^2} \sum_{s=1}^{3} \left(\frac{\partial(L/a^2)}{\partial x^s}\right)^2 \right) \tag{3.16}$$

This concludes the approximations and the time spent using the framework of General Relativity.

3.3 Discretization

In General Relativity, space-time is considered as continuous (except at singularities). For this project, space is broken up into discrete chunks or patches. Time, however, is left as it is. Affecting this change may have proved difficult in working with a space-time metric, but this is an advantage of using the Hamiltonian formulation. Recall that the expression for the Hamiltonian decouples space and time by integrating over all space on a surface of constant time. Moreover, the form of the gravitation Hamiltonian will directly assist in defining these volume patches.

First, assume that there are a finite number of volume patches, contributing to a finite total volume and define these chunks of space as $V_{i,j,k}$, where i, j, k are natural numbers used to count patches in x, y, z respectively. Using these patches, it is possible to define a spatial slice Σ as a collection of these spatial chunks.

$$\Sigma = \bigcup_{i,j,k=1}^{\mathcal{N}^{1/3}} \mathcal{V}_{i,j,k} \tag{3.17}$$

Equation (3.17) represents a union of $\mathcal{N}$ spatial patches $\mathcal{V}_{i,j,k}$, namely $\mathcal{N}^{1/3}$ patches in each direction. It is important to say here that $\mathcal{N} = \mathcal{N}_1\mathcal{N}_2\mathcal{N}_3$ is equivalent to saying that there are an equal number of chunks in each direction. This can be assumed since the system is known to be isotropic. Before continuing any further, it should be noted that this is a simplified picture where $\mathcal{N}$ is constant. In a more general construction $\mathcal{N}$ would be a function of time, but for this thesis, $\mathcal{N}$ is held fixed.

The next major assumption is that all of these spatial patches are of the same coordinate volume, namely

$$\int_{\mathcal{V}_{i,j,k}} d^3y = l_0^3 = V_0/\mathcal{N} \tag{3.18}$$

where V_0 is the total coordinate volume of the spatial slice Σ. To calculate the actual geometric volume $V_{i,j,k}$ of an individual space chunk, all that is needed is the induced space-time metric h_{ab} and the standard definition of volume.

$$V_{i,j,k} = \int_{\mathcal{V}_{i,j,k}} d^3y\sqrt{h} = a^3 \int_{\mathcal{V}_{i,j,k}} d^3y(1+2L/a^2)^{3/2} \tag{3.19}$$

The expression is not particularly pleasant, so another approximation is made, once again using power series expansion and that the total volume $V = a^3V_0 = a^3l_0^3\mathcal{N}$.

$$V_{i,j,k} \approx a^3 l_0^3 + 3a \int_{\mathcal{V}_{i,j,k}} d^3yL \approx \frac{V}{\mathcal{N}} + 3aL(x_{i,j,k})l_0^3 \tag{3.20}$$

This last approximation is critical. Using this approximation, it is possible to solve for the perturbation variable L in terms of the fluctuations in volume of the individual patches.

$$L(x_{i,j,k}) = \frac{V_{i,j,k} - V/\mathcal{N}}{3al_0^3} \tag{3.21}$$

The term $V/\mathcal{N}$ is effectively the average volume of any single patch, making the perturbation variable L a scaled difference between the actual volume of a chunk of space and the average volume of a chunk of space.

With volume discretized and a discrete expression for perturbation variable L, it is now possible to rewrite the Hamiltonian of this system as a function of volume variables and their associated momenta. This happens using the momentum tensor π^{ab} found in Chapter 2, the new expression for L, and the fact that both the induced metric h_{ab} and π^{ab} can be split into background and inhomogeneous components.

The results of all of this are the two momenta conjugate to V and $V_{i,j,k}$.

$$\Pi_V = -\frac{4}{3}\frac{\dot{V}}{V} \tag{3.22}$$

$$\Pi_{V_{i,j,k}} = -\frac{4}{3}\partial_\tau\left(\frac{\mathcal{N}V_{i,j,k}}{V}\right) \tag{3.23}$$

Making the observations that $\dot{a}/a = -4\pi G\Pi_V$ and that $\partial_\tau(L/a^2) = -4\pi G\Pi_{V_{i,j,k}}$ makes it easy to change variables in the approximated Hamiltonian density. From here, two things are required to construct a full Hamiltonian. The first is to change the expression from an integral of the Hamiltonian density over the spatial slice to a sum over the discrete volume patches. The second is to change spatial derivatives of L to finite differences. In order,

$$\mathcal{H}_\mathcal{G} \approx 6a^3\left(\left(\frac{\dot{a}}{a}\right)^2 + \left(\partial_\tau\left(\frac{L}{a^2}\right)\right)^2 + \frac{7}{3a^2}\sum_{s=1}^{3}\left(\frac{\partial(L/a^2)}{\partial x^s}\right)^2\right) \tag{3.24}$$

$$H_G \approx -6\pi GV\left(\Pi_V^2 + \frac{1}{\mathcal{N}}\sum_{i,j,k}\Pi_{V_{i,j,k}} + ...\right) \tag{3.25}$$

where the ellipsis represents the terms containing spatial derivatives of (L/a^2). These terms change from continuous derivatives to finite differences between adjacent patches,

$$\frac{\partial}{\partial x^s}\frac{L(x_{i,j,k})}{a^2} \rightarrow \frac{V_{(i,j,k)+\tilde{s}} - V_{(i,j,k)-\tilde{s}}}{6l_0(V/\mathcal{N})} \tag{3.26}$$

where $\tilde{s}$ indicates the unit vector in the s-direction. There are a few important things to note here. The first is that this is construction is not the only way to handle the fact that spatial derivatives are no longer possible. The manner in which these terms are discretized is a choice. Since one of the goals of this project is to compare the resultant non-linear, Schrödinger-like equation to that of a Bose-Einstein condensate, the discretization method selected is that which most closely resembles interactions of many-body systems dependent on the distances between particles. The other important note here is that V and $V_{i,j,k}$ are not independent from each other. By their definitions, $\sum_{i,j,k} V_{i,j,k} = V$, but, in order to focus on the self-interaction of the system, this fact is largely ignored.

3.4 Quantization

All efforts up until this point is independent work by the author. At this point, the quantization, the research group as a whole then took this perturbed, discretized Hamiltonian and shifted it from the classical to the quantum regime. Namely, the classical variables become quantum operators following the quantization used in Loop Quantum Gravity. The individual patches of volume $V_{i,j,k}$ and their associated expansions are isotropic and can be quantized as follows. The patch volume simply becomes a multiplicative operator. Quantizing the associated momentum is not quite as easy. In Loop Quantum Gravity, wavefunctions $\psi_{V_{i,j,k}}$ are elements of $\ell^2(\mathbb{R})$ so it is actually easier to quantize exponentials of momenta. Define the operator

$$e^{i\widehat{\delta_{i,j,k}\Pi_{i,j,k}}/\hbar}\psi_{V_{i,j,k}} = \psi_{V_{i,j,k}+\delta_{i,j,k}} \tag{3.27}$$

where $\delta_{i,j,k}$ are real numbers fixed by quantization choices. Usually to get a standalone momentum operator, one would take a derivative of this operator with respect to $\delta_{i,j,k}$, but unfortunately this is impossible here. The action of this operator on elements of $\ell^2(\mathbb{R})$ is discontinuous in $\delta_{i,j,k}$ so no such derivative can exist.[6]

The driving force behind all that follows from this point is the discrete nature of the inhomogeneity in this project's construction. Assuming that this inhomogeneity is small, it is safe to also assume that the dynamics of the individual volume patches are virtually independent from each other. This way it is possible to treat the overall state as a product of states, namely, $\Psi(V_1, V_2, ...) = \psi_1(V_1)\psi_1(V_2)...$ where the ψ_i is a wavefunction of an individual patch. Negligible inhomogeneity demands that difference equations govern the development of the individual ψ_i wavefunctions. For the scenario of small, but not negligible, inhomogeneity considered here interaction terms between the individual wave functions appear. Fortunately, perturbation theory handles these terms well.

There is one more approximation to be had at the expense of the small inhomogeneity. At the level of quantum geometry this fact also implies that the individual wavefunctions ψ_i are all actually similar to each other.[6] It is therefore possible to approximate this system as a product of identical wavefunctions in different values, namely $\Psi(V_1, V_2, ...) = \psi(V_1)\psi(V_2)...$ where each ψ is the same. It is in this final approximation that the connection between this system and the Bose-Einstein condensate becomes clear.

This form of product states it is now possible to map this many-body problem to a one-body problem in a specific potential, creating a non-linear wave equation. For a system like a Bose-Einstein condensate where the many ψ are known to be equal and assuming interactions behave like point the associated Hamiltonian is

$$\hat{H} = \sum_{i=1}^{n} \left(\frac{1}{2m} \hat{p}_i^2 + V(\hat{x}_i) \right) + \frac{\alpha}{2} \sum_{i \neq j} \delta(\hat{x}_i - \hat{x}_j) \tag{3.28}$$

with n particles of mass m in individual potentials $V(x_i)$. This Hamiltonian leads to the Gross-Pitaevski equation, the nonlinear Schrödinger-like equation with an additional term in the potential scaling with $|\psi(x)|^2$ shown below. The trick to computing this equation comes from taking the expectation value of the above Hamiltonian operator.

$$i\hbar \frac{\partial \psi_n}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + \frac{1}{2}(n-1)\alpha |\psi(x)|^2 \psi \tag{3.29}$$

3.5 The Nonlinearity

The previous sections establish a universe that evolves with time, contains matter clumps, and has discrete "quanta" of spatial volume. The dynamics of these volume patches are encoded in a classical Hamiltonian, which is pushed into the quantum regime. The result of this is a quantum mechanical Hamiltonian operator. The last stage of this project is to produce a nonlinear Schrödinger-like equation from this Hamiltonian operator, following a similar procedure to the derivation of Gross-Pitaevski equation.

The key point of interest in producing this equation is the resulting nonlinear term. Unfortunately the nonlinearity is not as simple as in the Gross-Pitaevski equation. What eventually falls out are polynomials, not delta functions, and the overall equation is not only nonlinear but it is also nonlocal. However, a similar procedure of taking an expectation value of the Hamiltonian operator still produces a useful result, with a slight modification. The minor modification is take the expectation value of the interaction component of the Hamiltonian in a product state. Consider the arbitrary V_1 and V_2 interacting through the potential $W_{int}(V_1, V_2) = \alpha(V_1 - V_2)^2/V^2$, where V is again the total volume. Translating this into operators and rearranging gives the expectation value

$$\langle \hat{W}_{int} \rangle_\Psi = \frac{\alpha}{V^2} \int dV_1 dV_2 |\psi(V_1)|^2 |\psi(V_2)|^2 (V_1 - V_2)^2 \tag{3.30}$$

$$\langle \hat{W}_{int} \rangle_\Psi = \frac{\alpha}{V^2} \int |\psi(V_1)|^2 dV_1 \int d(\delta V) |\psi(V_1 + \delta V)|^2 (V_1 + \delta V 2)^2 \tag{3.31}$$

where δV is defined as $V_2 - V_1$.

The integration looks unpleasant, but there is another approximation that simplifies the process considerably. Namely, the second integration happens independently of the first. Assume that ψ has a sharp peak around the expectation value of V_1: i.e. when

$$\int d(\delta V) |\psi(V_1 + \delta V)|^2 (V_1 + \delta V 2)^2 = (\Delta V)^2 \tag{3.32}$$

This means the V_1 values that make the second integration equal to the quantum fluctuations of V in the associated state dominate $\langle \hat{W}_{int} \rangle_\Psi$. All of this together demands the wave equation of this construction is of the form below.[6]

$$i\hbar \frac{\partial \psi_n}{\partial t} = \psi_{n+1} - 2\left(1 - \frac{1}{2}\alpha \frac{(\Delta n)^2_\psi}{n}\right)\psi_n + \psi_{n-1} \tag{3.33}$$

$$(\Delta n)^2_\psi = \sum_n (n - \langle n \rangle_\psi)^2 \, |\psi_n|^2 \tag{3.34}$$

Notice now that the nonlocality of this wave equation is now clearly visible. The coefficient $(\Delta n)^2_\psi$ depends on all wavefunctions simultaneously.

This wave equation is valid only if ψ has a sharp peak around the expectation value of V_1. Considering systems without a sharp peak or higher order approximations requires a derivative expansion in ψ, namely

$$|\psi(V_1 + \delta V)|^2 = |\psi(\langle V \rangle) + \delta V + (V_1 - \langle V \rangle))|^2 \tag{3.35}$$

Expanding this it is possible to rewrite $\langle \hat{W}_{int} \rangle_\Psi$ as

$$\langle \hat{W}_{int} \rangle_\Psi = \int dV_1 |\psi(V_1)|^2 W_{nonlin}(V_1) \tag{3.36}$$

$$W_{nonlin}(V) = \sum_{j=0}^{\infty} \frac{1}{j!} (\Delta V)^2_{\rho(j)} (V - \langle V \rangle)^j \tag{3.37}$$

where $(\Delta V)^2_{\rho(j)}$ is the fluctuation in V computed with the distribution $\rho^{(j)}$, which is just the j-th derivative of $\rho(V) = |\psi(V)|^2$.[6] These derivatives are not necessarily normalized or even positive, so probability distributions are not actually present. However, the values of these derivatives are still well-defined parametrizations.[6]

4. In Summary

4.1 Interpretations

The nonlinearity inherent to the system studied in this thesis creates immediate problems for studying physics. The system does not have the unitarity property in standard quantum mechanics; namely, wavefunctions that satisfy the nonlinear wave equation do not have preserved inner products with other states. In particular, this makes studying the Hamiltonian operator especially difficult. There is some hope in that while inner products of different states $\langle\phi|\psi\rangle$ are not preserved, norms of individual states $\langle\psi|\psi\rangle$ are preserved. Moreover, the original many-body quantum system constructed in this thesis does have the unitarity property, meaning that this issue is the result of approximations and simplifications used to go from a many-body-system to a one-body system.[6] The method that shifts the many-body problem to a single-body problem is actually a roundabout method of computing the expectation value of the Hamiltonian, re-writing that in terms of a single wavefunction, and calling the additional term an extra potential depending on that single wavefunction.

The physics of this system does indeed have the unitarity property.[6] Also, the preservation of the norm of a given wavefunction suggests that the probabilistic interpretation still holds. The useful idea here is that the Gross-Pitaevski equation here is not the fundamental wave equation of the system, but it does model interesting dynamic effects, such as interaction of superposed states.[6] Even though this particular equation loses unitarity in its construction, things like overlap of superposed states or the distance between different distributions are calculable without unitarity. Fortunately, an equation like this is very useful in quantum cosmology because the exact state of quantum space is unknown and likely to remain that way for some time. The difference equation

$$i\hbar\frac{\partial\psi_n}{\partial t} = \psi_{n+1} - 2\left(1 - \frac{1}{2}\alpha\frac{(\Delta n)^2_\psi}{n}\right)\psi_n + \psi_{n-1} \tag{4.1}$$

analogous to Gross-Pitaevski equation is reasonable.

4.2 Conclusions

This thesis constructs a novel inhomogeneous quantum cosmology model for the purpose of discovering the key features of quantum space-time. This model begins as the Friedmann-Robertson-Walker space-time of classical cosmology, becomes inhomogeneous via a perturbation, and discretizes by chopping spatial volume into patches. This model becomes a many-body problem in quantum mechanical operators and then reduces to a single-body problem through a number of assumptions based on the fact the inhomogeneity is small and ideas from condensed matter physics.

There are a few key differences between the constructed model at that of a condensed matter model. The first difference is that interactions in the quantum cosmology model do not take place in actual space. The volume patches interact based on their geometries instead of their physical distance from each other; the Hamiltonian largely depends on the deviations of volume patch geometries from their average. The second major difference between the two models is the nonlinear potential. The Gross-Pitaevski equation has a delta function as its interaction potential: very effective for point-like interactions. The quantum cosmology model constructed here has a polynomial potential as its nonlinear term, which arises from expanding the Hamiltonian in the geometries of the volume patches. The particular form of this nonlinearity also forces the model to be nonlocal. The final major difference between these two models lies in their associated many-body Hamiltonians. This is a well-known object for condensed matter system, but a consistent Hamiltonian in quantum gravity for an inhomogeneous system is not well known.

Nonlinear wave equations, like the one developed in this thesis, are particularly interesting because they predict effects that do not require extremely high densities. It is possible that the nonlinear term could be large because the combined effects of all patches could large even if the contribution of each individual patch appears insignificant. This is, in part, due to the fact that the fluctuations are positive by construction and therefore do not cancel each other.

The project at the heart of this thesis is an exploratory effort to expose essential behavior of an inhomogeneous universe with discretized volume. It is quite possible to extend this project to space-times with vector and tensor perturbations as well as scalar perturbations, hopefully producing more complicated nonlinear wave equation that better represents the dynamics of a quantum inhomogeneous space-time.

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APPENDIX A
Perturbation Procedure

The procedure for perturbing the Friedmann-Robertson-Walker space-time from classic cosmology is fairly straightforward. This procedure, in its full detail, is found in [4]. The basic idea of the perturbation is that the theory of General Relativity is invariant under coordinate transformations. Define the coordinate transformation ψ, and let g be some space-time metric. Because the theory is invariant, both the original g and the coordinate transformed g have to describe the same geometry. The best way to make use of this fact is to do an infinitesimal coordinate transformation. Since the background (unperturbed) space-time metric is held fixed, it can be shown that the coordinate transformation is the identity map to first order.

$$g \to g + \frac{1}{a^2} L_X g \tag{A.1}$$

The above equation shows the basic form of such a transformation given the selected background metric. The L_X is the Lie derivative along the arbitrary vector $X = (T, L_i)$. It should be noted here that this is the form of a scalar perturbation. This limits the scope of the overall project, but this is an exploratory work. Since this project produced an effective wave equation, a logical extension would be to include vector and tensor perturbations to from a more accurate effective wave equation. The space-time metric that follows from this construction is rather messy and full of coordinate dependent terms.

$$L_X g_{\alpha\beta} dx^\alpha dx^\beta = -2\dot{T} d\tau^2 + 2\left(\frac{\partial T}{\partial x^i} + \dot{L}_i - 2\frac{\dot{a}}{a} L_i\right) d\tau dx^i + 2\left(L_{i;j} + L_{j,i} - a\dot{a}T\delta_{ij}\right) dx^i dx^j \tag{A.2}$$

As neither messy nor coordinate dependent is desirable, it is necessary to change variables, make a gauge choice, and simplify. The change of variables follows from the inverse Fourier Transform. This is not a difficult calculation, but it is very long and tedious. The line element in new variables will simply be quoted.

$$L_X g_{\alpha\beta} dx^\alpha dx^\beta = -2(L + \frac{1}{a^2}\dot{T})d\tau^2 + 2\left(L + \frac{H_T}{3} - \frac{\dot{a}}{a^3}T\right)\delta_{ij} dx^i dx^j \tag{A.3}$$

In the above equation, L is the perturbation variable used throughout the main body of this thesis. Selecting the longitudinal gauge causes both T and H_T are both zero. Combining this simplified perturbation term with the background metric produces the perturbed space-time used in this project, namely,

$$g_{ab} = \begin{pmatrix} -\left(1 + \frac{2L}{a^2}\right) & 0 & 0 & 0 \\ 0 & a^2 + 2L & 0 & 0 \\ 0 & 0 & a^2 + 2L & 0 \\ 0 & 0 & 0 & a^2 + 2L \end{pmatrix} \tag{A.4}$$

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- M. Bojowald, A. L. Chinchilli, D. Simpson, C. C. Dantas, and M. Jaffe, “Nonlinear (loop) quantum cosmology,” Physical Review D 2012 86, 124027