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# Cursed Equilibrium Refinements in Sequential Auction Games

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# Abstract

In this thesis, we apply the concept of a *cursed equilibrium* to a sequential auction environment. In a cursed equilibrium, players fail to take into consideration the relationship between their opponents' strategies and types with probability $\chi$. In auctions, bidders who exhibit this behavior are said to suffer from the winner's curse: the price they pay upon winning the auction exceeds the true value of the object. As an extension of Eyster and Rabin's original paper, we derive symmetric equilibria strategies for a two-round sequential auction as well as the corresponding price series for any fixed level of cursedness. In addition, we posit a plausible connection between the winner's curse and the downward price anomaly in sequential common-value auctions.

# Contents

# 1 Introduction

An auction is a method of efficiently allocating objects to individuals who value those objects the most. In most auction formats, those individuals who wish to acquire the object for sale submit sealed bids, from which the seller chooses the winner. Usually, but not always, the winner is the individual who submits the highest bid, paying a price equal to his bid to the seller. This is commonly known as a first-price auction. A well known alternative is the Vickrey auction (also known as the second-price auction), where the highest bidder wins but instead pays a price equal to the second-highest bid. In real world markets, there exist a variety of auction formats. While some auctions rely on selling goods simultaneously–such as government securities–others split the auction into rounds of bidding, with a single object sold in each round. Objects from houses, wine, paintings, flowers, and even satellite transponders are sold by sequential auctions. When objects are sold in sequence,as opposed to simultaneously, bidders have multiple chances to win, giving an opportunity for bidders to re-evaluate their strategy following each round.

In an auction, the worth of the objects for sale has either one of two possible states (or sometimes a combination of the two), called *private* or *common* values, respectively. Objects with a common-value possess some intrinsic worth equal to all bidders. Consider, for example, an oil field. The oil field has a specific monetary value directly related to the amount of petroleum potentially available to extract. Bidders bidding on an oil field do not know the $definite$ value of the field, since they do not know the exact amount of petroleum underground. This leads bidders to make an approximated "guess" in the form of a sealed bid, dependent on some previously available public or private information. For example, a firm may hire an oil field service company to estimate the potential amount of petroleum. This private information, often called a *signal*, is strategically considered in submitting a bid on the object. Observed frequently in real-world auctions, bidders whose private signals partially reveal the common-value of the object often bid more than the equilibrium theory predicts (see Krishna, 2002).

When the winning bid exceeds not only the equilibrium bid but the object's true value, the bidder is said to exhibit the phenomenon of the "winner's curse". The winner's curse is a psychological bias widely observed in experimental literature (see Kagel and Levin, 1986 and Klemperer, 1998). In contrast to laboratory experiments, theory dictates that bidders never bid over the true value of the object in an equilibrium, no matter how many bidders there are. The Bayesian Nash equilibrium (BNE) corrects for this possibility in the form of "bid shading," or bidding a fraction of one's estimate of the object's value. One existing explanation for the winner's curse is that bidders typically fail to fully appreciate the correlation between other bidders' signals, and the probability of winning the auction. If other bidders have "low" signals, a bidder with a comparatively "high" signal who fails to temper his bid leads him to believe that the object is worth more than it truly is. Winning the auction with a high signal reveals something about the object; specifically that other bidders took some negative information about the object into consideration that the winning bidder did not.

In this thesis, we extend the existing work on the "cursed equilibrium" concept created by Eyster and Rabin (2005) that captures this intuition for simultaneous auctions to the environment of sequential auctions. This concept allows bidders to *partially* appreciate the correlation between other bidders' signals and their actions with positive probability that their beliefs regarding this correlation are incorrect. Their $\chi$-virtual game corresponds to a parametric model of the BNE. Our paper's novel contribution is establishing a possible relationship between the winner's curse and the widely-observed 'downward price anomaly' in sequential auctions.

We begin the following section (Section two) with a literature review, briefly outlining pertinent contributions to the analysis of sequential auctions as well as an introduction to the cursed equilibrium concept.

In Section three, we outline the auction environment for first-price and second-price auctions and draw some assumptions from the model.

In Section four, we formally derive symmetric cursed equilibria for the two auction formats. We show that the cursed equilibrium leads to an ambiguous effect on the seller's revenue, dependent on the number of competing bidders as well as the distribution of values. In an example, we show that a fully-cursed auction can produce either higher or lower expected revenues than the Bayesian auction equivalent.

In Section six, we study the equilibrium price series for both formats. We show that for the symmetric bidders case, the price series at equilibrium is at best a martingale for a fixed level of cursedness. A (discrete-time) martingale is a stochastic process $X_1, X_2$, etc. that satisfies $\mathbb{E}[X_{t+1}|X_1, \ldots, X_t] = X_t$ for any time $t$.

In Section seven, we refine the cursed equilibrium concept in an example with asymmetrically cursed bidders. We show that the price series of the auction is no longer bound as a non-decreasing function. Thus for some instances of asymmetric cursedness, the expected prices are downward drifting, tying the winner's curse to the downward price anomaly in our conclusion.

We end the thesis (Section eight) with a brief discussion and reflect on extensions and possible future directions of our research.

## 2 Literature Review

Auctions have been used as a market mechanism since the time of the ancient Babylonians. As a famous historical example, the entire Roman Empire was sold via an ascending auction in 193 A.D. By comparison, the study of auctions in economic literature is relatively recent. Vickrey's seminal paper (1962) was the first to truly study auctions as games. Though early contributors like Ortega-Reichert (1968) and Wilson (1969) made significant headway in augmenting the discipline, Milgrom and Weber's paper (1982) signals the beginning of the modern era of the theory of auctions. See Klemperer (1999) and Krishna (2002) for a thorough introduction to the foundation of the theory.

By comparison to the simultaneous auction environment, the study of sequential auctions is rather sparse and generally still considered incomplete. In 2000, Milgrom and Weber published a follow-up to their early work that defines Nash equilibrium conditions in sequential auctions under the assumption of affiliated values. Unlike Weber's private signal result, the expected price path no longer follows a martingale (e.g., random walk), but a submartingale (e.g., upward drifting). At this point, there is a bifurcation between the empirical data and theoretical results.

Many empirical papers on sequentially sold homogeneous objects find that the price paid for the object declines as the auction progresses. Ashenfelter's (1989) early contribution to the study of wine auctions is among the first of many to bring attention to this result. This peculiarity has since been dubbed by economists as the "downward price anomaly," or alternatively, the "afternoon effect". Van den Berg et al. (2000) find evidence for the downward price anomaly in sequential Dutch (first-price) auctions of roses in large flower auctions.

McAfee and Vincent (1993) offer an elegant solution to the problem of downward drifting price series' in the form of risk averse bidders. However, their result depends on risk aversion *increasing* with wealth, and so it remains doubtful that this offers a plausible explanation for the observed anomaly. Bernhardt and Scoones (1994) explain the decline in prices by assuming the objects are heterogeneous. In a different manor, Jeitschko (1999) considers a multi-unit demand model. The downward price anomaly has inspired many competing theories, though there is no clear-cut explanation for its frequent occurrence in the data.

In addition to the relevant auction literature, the basis of this thesis draws heavily on Eyster and Rabin's work (2005) on an equilibrium concept in games with private information, the cursed equilibrium. In that paper, they apply the concept to models of bilateral trade, signaling games, and common-value auctions. In the latter, cursed equilibrium elegantly captures the phenomenon of the winner's curse. Their parameterized equilibrium inbeds the Bayesian Nash Equilibrium (BNE) as an extreme case, in which they show a range of 'cursedness' values fitting empirical data better than general equilibrium analysis suggests. In the discussion of their paper, Eyster and Rabin comment on the possibility of extending their equilibrium concept to sequential games. In the remaining portion of this thesis, we extend the cursed equilibrium into a two-stage auction environment.

# 3 The Environment

In this section, we incorporate the cursed effect in a two-stage model for both first-price and second-price auctions. Assume there are 2 identical objects to be auctioned individually in sequence, and $N$ symmetric bidders with a single-unit demand. Bidder $i$ observes the realization $x_i$ of a private signal $X_i$, a random variable with support on $[\check{x}, \hat{x}]$. We assume that the signals are independently and identically distributed with a probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$. As a usual convention in the literature, we define an order statistic $Y_j$, as the $j$-th highest signal of the $N-1$ non-winning bidders, $j \geq 1$. Let $\mathbf{x}$ be a vector of all bidders signals and $\mathbf{x}_{-i}$ be the vector of all signals except $i$. Furthermore, the object for auction has a common-value $V$, dependent on the vector $\mathbf{x}$. Most of the literature on common-values assumes either $V = \sum x_i$ or $V = \frac{1}{N} \sum x_i$, the sum of the signals or the average of the signals, respectively.

All aspects of the environment are assumed to be known by bidder $i$ with the exception of the realization of $\mathbf{x}_{-i}$. We assume that bidders are risk neutral–that is, they are profit-maximizing agents. Therefore, in round $t$ a perfectly rational bidder is willing to bid up to his own value $v_t(x_i, \mathbf{x}_{-i})$, as determined by the player-specific perception of the object's common-value. This value is altered by the extent to which a player is "cursed," defined as the probability $\chi$ that a bidder believes other bidders play the average action regardless of the realized value of their private signal. Consequently, the cursed bidder attaches the correct (equilibrium) strategy of all other bidders with probability $(1 - \chi)$.

In this framework, we view auctions as a sequential game played among symmetric bidders. In each round, bidder $i$ submits a $\chi$-cursed bid $b_i$ to a seller, where $b_i$ is chosen from a strategy function $\beta_i^{\chi} : [\check{x}, \hat{x}] \rightarrow \mathbb{R}^+$ common to all bidders. At the end of the first round of bidding, the highest bid wins, and his bid is made public to the $N-1$ bidders that remain to bid in the second round. This bid announcement behaves as a signal, from which the remaining bidders correctly infer the realization $y_1$ of the previous winner's signal. From here, we independently examine the two most common formats: first-price sealed-bid, and second-price sealed-bid auctions. In the first-price environment, the bidder with the highest bid wins the object and pays his bid, $b_i$. We define a bidder's payoff $\pi_i$ for a chosen bid of $b_i$ as:

$$\pi_i = \begin{cases} v_t(x_i, \mathbf{x}_{-i}) - b_i & \text{if } b_i > \max_{j \neq i}(b_j) \\ 0 & \text{if } b_i \leq \max_{j \neq i}(b_j) \end{cases}$$

In the second-price equivalent, the bidder who submits the highest bid wins and pays a price equal to the second-highest bid, $\max_{j \neq i}(b_j)$. A bidder's payoff $\pi_i$ for a chosen bid of $b_i \in \beta_i^\chi$ is:

$$\pi_i = \begin{cases} v_t(x_i, \mathbf{x}_{-i}) - \max_{j \neq i}(b_j) & \text{if } b_i > \max_{j \neq i}(b_j) \\ 0 & \text{if } b_i \leq \max_{j \neq i}(b_j) \end{cases}$$

From the cursed effect, in either format bidder $i$'s $\chi$-cursed profit conditional on winning the auction with a bid of $b_i$ and paying a price of $p_i$ is

$$\pi_i = (1 - \chi) v_t(x_i, \mathbf{x}_{-i}) + \chi C(x_i) - p_i$$

where $p_i = b_i$ for the first-price auction and $p_i = \max_{j \neq i}(b_j)$ in the second-price equivalent. This expression is simply the $\chi$-weighted average of the object's true value to $i$ and the value conditioned on the realization of $X_i$. There are two vital functions in our analysis: the value of the object $v_t(x_i, \mathbf{x}_{-i})$, and what we refer to as the cursed function $C(x_i) = \mathbb{E}[V | X_i = x_i]$, equal to the expected value of the object conditioned only on the realized value of $i$'s signal $x_i$. As an addition to the concepts developed prior, we feel that it is necessary to outline some basic assumptions regarding the state of the auction environment. Hence:

**A1.** The function $v_t(x_i, \mathbf{x}_{-i})$ is increasing in $x_i$, twice differentiable, and symmetric for all $N$ bidders.

**A2.** The expression $f_{Y_t}(\cdot|x_i)F_{Y_t}(\cdot|x_i)^{-1}$ is increasing in $x_i$. Here, $F(\cdot|x_i)$ is the distribution of $Y_j$ conditional on $X_i = x_i$ and $f(\cdot|x_i)$ denotes the conditional density function associated with the distribution.

**A3.** The distribution of $Y_t$ given the realization of $X_1$, $Y_1$ and given $Y_1 \leq X_1$ is stochastically increasing in $x_i$.

**A4.** We assume $X_1 \succ Y_1 \succ \ldots \succ Y_{t-1}$, i.e. the signal of the first winner stochastically dominates the signal of the second winner, which in turn dominates that of the next highest bidder, and so forth.

**A5.** Finally, we have $f(x|y)f(x'|y') > f(x'|y)f(x|y') \qquad \forall x > x', y > y'$

The fifth assumption is commonly referred to as the strict monotone likelihood ratio property (SMLRP). We use the SMLRP in our derivation of the expected price series at cursed equilibrium. The following section is dedicated to incorporating the cursed equilibrium into the general symmetric model first derived by Milgrom and Weber (1982). We develop symmetric cursed equilibrium conditions for both sequential first-price and second-price common-value auctions and offer a complete example that clearly illustrates cursed bidding behavior.

# 4 Symmetric Cursed Equilibria

We now derive the symmetric cursed equilibria for the two common-value auction formats. The sequential cursed equilibrium concept not only has roots in psychology, but is easily adaptable to the current theory. We provide some discussion on the difference in bidding behavior for increasing values of $\chi$.

Eyster and Rabin's (2005) definition of the cursed equilibrium for simultaneous auctions allows players to have partially incorrect beliefs about the correlation between other players' signals and their bids. In a $\chi$-cursed equilibrium, each bidder optimally responds to a belief that with probability $\chi$ his opponents' bids do not depend on their signals, and with probability $(1-\chi)$ their bids do depend on their signals. Hence, the valuation of the object to any bidder is the $\chi$-weighted average of the object's true value and the expectation of its value conditional only on that bidder's signal. Our work extends Eyster and Rabin's model to account for a common bidder's dilemma; each bidder must account for the possibility of winning the object not only in the first round, but in the $t-1$ future rounds that follow. The equilibrium strategy, consisting of $t$ bids, leads to a less aggressive first-round strategy, i.e. bidders shade their bid below what they would have in a simultaneous auction to account for the possibility of a future payoff in following rounds.

**Lemma 4.1.** For all $\chi \in [0,1]$, the $\chi$-weighted average of the value of the object to those in the second and final round of the auction is $V^{\chi} = v_2(x;x,y_1) + \chi[C(x) - v_2(x;x,y_1)]$

*Proof.* The result follows from the definition of a $\chi$-cursed equilibrium. A bidder $i$'s expectation of the common value of the object $V$ conditional on $i$'s signal $X_i = x$ and the highest of the remaining bidders' signals being $x$ is just $v_2(x;x,y_1)$. His valuation of the object conditional on only his signal $X_i$ is dependent on the realization of the signal, $x$ and the expected value of the random variable $X_i$. The $\chi$-weighted average of the two valuations is

$$
\begin{aligned}
V^{\chi} &= \mathbb{E}\left\{(1-\chi)V + \chi\mathbb{E}[V|X_i = x]|X_i = x, Y_1 = y_1, Y_2 = x\right\} \\
&= (1-\chi)v_2(x; x, y_1) + \chi C(x) \\
&= v_2(x; x, y_1) + \chi[C(x) - v_2(x; x, y_1)] \qquad (1)
\end{aligned}
$$

□

In a cursed equilibrium, the $\chi$-weighted average of these two valuations is used in place of the Bayesian valuation to reflect the ignorance of the correlation between signals and bids. Note that for $\chi = 0$ the cursed value is equivalent to the fully rational value, and for $\chi = 1$, the value is drawn strictly from the cursed function $C(x)$. Bidders who act as if winning the auction conveys no information about the true value of the object are called *fully cursed*. The proofs of the two main theorems introduced in this section follow succinctly from arguments by Milgrom and Weber (2000) as well as Mezzetti et al. (2004), and so will not be discussed for purposes of clarity.

## 4.1 First-Price Auctions

With the definition of $V^{\chi}$ in hand, consider a sequential first-price auction with the winning bid announced *ex post*. Since the highest bidder wins the first-price auction and pays the amount equal to his own bid, this format coincides with the general price announcement model developed by Milgrom and Weber (2000). Let $S^* = (\beta_1^{\chi}, \beta_2^{\chi})$ be a symmetric equilibrium strategy of a two-round auction; $\beta_1^{\chi}(x)$ and $\beta_2^{\chi}(x; y_1)$ denote the $\chi$-cursed bid made by the bidder with signal $x$ and winning bid announcement $y_1$ in the first and second stage, respectively.

**Theorem 1.** Let $V^{\chi}$ be the cursed valuation of the object to each bidder and suppose that $y_1$ is the signal corresponding to the winning bid in the first round. A symmetric cursed equilibrium profile $S^*=(\beta_1^{\chi}, \beta_2^{\chi})$ in the sequential first-price auction is given by the solutions to the pair of differential equations

$$\beta_2^{\chi'}(x;y_1) = [V^{\chi} - \beta_2^{\chi}(x;x,y_1)]\frac{f_{Y_2}(x|x;y_1)}{F_{Y_2}(x|x;y_1)} \tag{2}$$

$$\beta_1^{\chi'}(x) = [\beta_2^{\chi}(x;x) - \beta_1^{\chi}(x)]\frac{f_{Y_1}(x|x)}{F_{Y_1}(x|x)} \tag{3}$$

The principal implication of the cursed equilibrium is that the winner's curse arises naturally from a sufficiently large pool of bidders. As the number of bidders $N$ gets large, the probability $P$ that the bidder with the highest signal submits a bid in excess of the true value of the object tends to 1. In first-price auctions, the symmetric equilibrium is complex: bidders must attempt to maximize the probability of winning while simultaneously minimizing the chance of overpaying. Milgrom and Weber (2000) prove that fully rational bidders always correct for the possibility of overpaying in equilibrium, conditioning their bid on their random signal $X_i$ being the highest, and equal to that of the next highest signal $Y_1$. However, for any non-zero $\chi$, those with high signals bid higher than predicted by the Bayesian equilibrium, increasing the probability that the winning bidder suffers the winner's curse. By comparison to the first-price auction, analysis of the second-price auction offers a less intuitive, but more tractable solution.

## 4.2 Second-Price Auctions

In a second-price auction, the highest bidder wins the auction and pays a price equal to the second highest bid. In the independent private values (IPV) model, a well-known result is that bidding one's private value $x$ is a weakly dominant strategy for any bidder. However, under the assumption of affiliate values a bidder bids the value of the object conditioned on his signal $X_1 = x$ being the highest signal and equal to the highest of the remaining bidders' signal, $Y_1$. At cursed equilibrium, bidders bid the $\chi$-weighted average, so that $\beta(x) = (1-\chi)v(x,x) + \chi C(x)$. In sequential auctions, bidders remaining in the final round of the auction still have an incentive to bid the $\chi$-weighted average of their values; however, as is the case for first-price auctions, bidders shade their bid below their cursed value in previous rounds.

**Theorem 2.** Let $V^\chi$ be the cursed value of the object to each bidder and suppose that $y_1$ is the signal corresponding to the winning bid in the first round. A symmetric cursed equilibrium strategy profile $S^*=(\beta_1^\chi, \beta_2^\chi)$ in the sequential second-price auction is given by

$$\beta_2^\chi(x; y_1) = \mathbb{E}[V^\chi | X_1 = x, Y_1 = y_1, Y_2 = x] \tag{4}$$

$$\beta_1^\chi(x) = \mathbb{E}[\beta_2^\chi(y_2; x) | X_1 = x, Y_1 = x] \tag{5}$$

First, observe that in the second round it is a symmetric equilibrium for bidders to bid their cursed value $V^\chi = (1-\chi)v_2(x; x, y_1) + \chi C(x)$. For $\chi = 0$, it is an equilibrium for bidders to bid their true value $v_2(x; x, y_1)$, conditioned on the previous winning bid $y_1$ and the realized value of the next highest signal $Y_2 = x$, consistent with the results in Milgrom and Weber (2000). For $\chi = 1$, bidders at fully cursed equilibrium bid $C(x)$.

In the first round, a bidder knows that if he could win in the first round with a bid of $b(x)$ he would also win in the second round, and pay a lower price. Therefore, in equilibrium he bids the price he expects to pay in the second round, conditioned on the realized value of his signal being the highest and equal to the highest of $N-1$ remaining bidders. The price he expects to pay is equal to the bid of the opponent with the second highest signal $Y_2$, since the winning bid in the second round with a signal $Y_1$ wins and pays $\beta_2(Y_2; x)$.

The SMLRP implies that (4) is increasing in $x$, while (5) is increasing in $x$ and $y_1$. Thus, affiliation leads $\chi$-cursed bidders to bid at least as high as their first round bid in the following round. The process of acquiring "better" information i.e., other players' signals, drives the bidding upward. We now provide a simple example of the $\chi$-cursed equilibrium in a sequential first-price auction.

# 5 Example

In this example, we illustrate the differences between perfectly rational ($\chi = 0$) and fully cursed ($\chi = 1$) symmetric equilibrium bidding strategies in a two-round first-price auction. To set up the model, first suppose that there are three bidders. The objects are sold sequentially in two rounds of bidding. For simplicity, we study bidders with strictly single-unit demand, so that after a round of bidding the bidder with the highest submitted bid wins the object and leaves the auction, leaving the remaining bidders to bid in subsequent rounds. Each bidder $i$ observes a private signal $X_i = S_i + \epsilon$, where $S$, and $\epsilon$ are uniformly and independently distributed on [0,1] for all bidders. That is, bidder 1 receives the signal $X_1 = S_1 + \epsilon$, bidder 2 receives the signal $X_2 = S_2 + \epsilon$, and bidder 3 receives the signal $X_3 = S_3 + \epsilon$. The object for sale has a common value,

$$V = \frac{1}{3}\sum_{i=1}^{3} X_i \tag{6}$$

All bidders choose a bid from a symmetric strategy function $\beta_t^{\chi}(\cdot)$. As usual in auction literature, if $X_1$ is the highest bid, we define $Y_1$ as the highest of $N-1$ bidders, $Y_2$ as the second highest, and so on. Let $f_{Y_1}(\cdot)$ and $f_{Y_2}(\cdot)$ be the probability density functions (with some arguments) of $Y_1$ and $Y_2$, and let $F_{Y_1}(\cdot)$ and $F_{Y_2}(\cdot)$ be the corresponding cumulative distribution functions.

To develop some intuition regarding the example to follow, without loss of generality consider the case where $X_1 \succ X_2 \succ X_3$. A fully rational bidder will always bid some positive fraction $k < 1$ of his signal, so the range of $\beta_i^{\chi}$ is in the closed interval $[0, X_i]$. For partially cursed bidders, this assumption does not hold. The two losing bidders simultaneously observe the winning signal $y_1$ and get another chance to win the object in the second round. We know that $Y_1 \succ Y_2$, and so knowing the realization $y_1$ of $Y_1$ in the second auction causes bidder 2 and 3 to reassess their bidding strategy. Theoretically, bidders 2 and 3 will bid at a higher fraction of their own signals, $X_2$ and $X_3$, respectively.

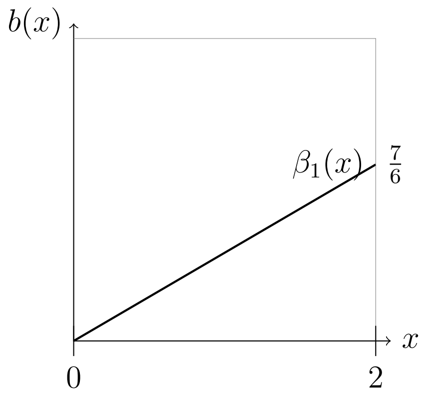


Figure 1: First period equilibrium bidding strategy, $\chi$=0

## 5.1 Fully Rational Bidders

Developing a set of BNE strategies requires a backward inductive study of the model. When $\chi = 0$, the $\chi$-cursed value $V^{\chi} = (1-\chi)v_t(\cdot) + \chi C(x) = v_t(\cdot)$. Simplifying the cursed equilibria strategies defined in the previous section, the BNE profile simplifies to the set of differential equations

$$\beta_2'(x; y_1) = [(v_2(x; x, y_1) - \beta_2(x; x, y_1)] \frac{f_{Y_2}(x|x; y_1)}{F_{Y_2}(x|x; y_1)} \tag{7}$$

$$\beta_1'(x) = [\beta_2(x; x, y_1) - \beta_1(x; x)] \frac{f_{Y_1}(x|x)}{F_{Y_1}(x|x)} \tag{8}$$

with the boundary condition $\beta_t'(\check{x}) = v_t(\check{x}; \check{x}) = 0$. To determine the equilibrium bidding strategy, we need to explicitly solve for $f_{Y_2}(x|x; y_1)$, the density function of the third highest bidder conditional on $X_1 = x, Y_1 = y_1$. This requires some further analysis into methods of Bayesian inference. It can be verified that

$$\frac{f_{Y_2}(x|x; y_1)}{F_{Y_2}(x|x; y_1)} = \frac{2}{x} \tag{9}$$

For a complete derivation of (9), see Appendix A. Now using the argument above, the differential equation defining the symmetric equilibrium with the boundary condition

$\beta_2(0; y_1) = 0$ in the second round of bidding is

$$\beta_2'(x; y_1) = \left( \frac{2x + y_1}{3} - \beta_2(x; y_1) \right) \frac{2}{x} \tag{10}$$

which has a unique solution

$$\beta_2(x; y_1) = \frac{4x + 3y_1}{9} \tag{11}$$

The symmetric equilibrium for the first round of bidding is determined in a similar manner. However, in the first round, bidders have no information other than their own private signal. It follows that

$$\frac{f_{Y_1}(x|x)}{F_{Y_1}(x|x)} = \frac{3}{x} \tag{12}$$

so the symmetric equilibrium strategy in the first round satisfies the differential equation

$$\beta_1'(x) = \left( \frac{7x}{9} - \beta_1(x; x) \right) \frac{3}{x} \tag{13}$$

with boundary condition $\beta_1(0) = 0$. This yields a symmetric equilibrium of

$$\beta_1(x) = \frac{7}{12} x \tag{14}$$

Thus, each bidder has a Bayesian Nash strategy profile,

$$S^*(x) = \left( \frac{7}{12} x, \frac{4}{9} x + \frac{1}{3} y_1 \right) \tag{15}$$

Under our assumptions, the expected revenues in the fully rational auction are

$$\begin{aligned}
\mathbb{E}[\mathrm{R}^{\mathrm{I}}] &= \mathbb{E}\left[\max\left\{\frac{7}{12}X_1, \frac{7}{12}X_2, \frac{7}{12}X_3\right\}\right] \\
&= \frac{7}{12}\mathbb{E}[\max\{S_1, S_2, S_3\}] + \frac{7}{12}\mathbb{E}[\epsilon] \\
&= \frac{35}{48}
\end{aligned}$$

for the first round of bidding, and

$$\begin{aligned}
\mathbb{E}[\mathrm{R}^{\mathrm{II}}] &= \mathbb{E}\left[\max\left\{\frac{4}{9}X_1 + \frac{1}{3}Y_1, \frac{4}{9}X_2 + \frac{1}{3}Y_1\right\}\right] \\
&= \frac{4}{9}\mathbb{E}[\max\{S_1, S_2\}] + \frac{4}{9}\mathbb{E}[\epsilon] + \frac{1}{3}\mathbb{E}[Y_1] \\
&= \frac{49}{54}
\end{aligned}$$

for the second.

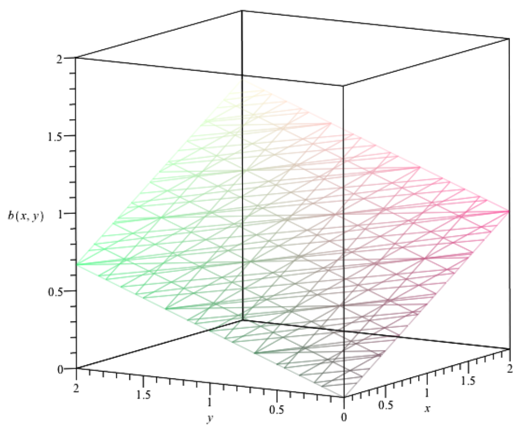


Figure 2: Second period equilibrium bidding strategy, $\chi$=0

## 5.2 Fully Cursed Bidders

We now turn our attention to studying the equilibrium bidding strategies of fully cursed bidders; i.e., when $\chi = 1 \forall i \in N$. In the fully cursed environment, bidders act as if they were participating in an auction with private (but affiliated) values. Assume two items are sold sequentially to three bidders with private signals distributed as before on the closed interval $[0, 2]$. With respect to the second period equilibrium strategy of an arbitrary winning bidder, the differential equation can be written as

$$\beta_2^{\chi\prime}(x; y_1) = [C(x) - \beta_2^{\chi}(x; y_1)] \frac{f_{Y_2}(x|x)}{F_{Y_2}(x|x)} \tag{16}$$

where

$$C(x) = \mathbb{E}[V^{\chi}|X_1 = x] = \frac{1}{3}x + \frac{(n-1)}{3}\mathbb{E}[X_i] = \frac{1}{3}x + \frac{2}{3} \tag{17}$$

From the assumption of a uniform distribution with support on $[0, 2]$ we get

$$\frac{f_{Y_2}(x|x)}{F_{Y_2}(x|x)} = \frac{2}{x} \tag{18}$$

so

$$\beta_2^{\chi\prime}(x) = \left(\frac{x+2}{3} - \beta_2^{\chi}(x)\right)\frac{2}{x} \tag{19}$$

which gives a solution to the second period strategy

$$\beta_2^{\chi}(x) = \frac{2}{9}x + \frac{2}{3} \tag{20}$$

It is interesting to note that in equilibrium, the second period strategy is completely *independent* of the previous price announcement $y_1$. That is, fully cursed bidders fail to consider any correlation between the previous winning bid and the value of the object for which they are bidding. Working backward, the first period equilibrium strategy is the solution to the differential equation

$$\beta_1^{\chi\prime}(x) = [\beta_2^{\chi}(x) - \beta_1^{\chi}(x)]\frac{f_{Y_1}(x|x)}{F_{Y_1}(x|x)} \tag{21}$$

with the boundary condition $\beta_1^{\chi}(0) = 0$. Since bidders assume their signal distributions are symmetric, the expression can be rewritten as

$$\beta_1^{\chi\prime}(x) = \left(\frac{2x+6}{9} - \beta_1^{\chi}(x)\right)\frac{3}{x} \tag{22}$$

Solving the differential equation yields the solution

$$\beta_1^{\chi}(x) = \frac{1}{6}x + \frac{2}{3} \tag{23}$$

Thus, the fully cursed symmetric equilibrium profile follows directly:

$$S^*(x) = \left(\frac{1}{6}x + \frac{2}{3}, \frac{2}{9}x + \frac{2}{3}\right) \tag{24}$$

The expected revenues in the fully cursed environment are

$$\begin{aligned}
\mathbb{E}[\mathrm{R}^{\mathrm{I}}] &= \mathbb{E}\left[\max\left\{\frac{1}{6}X_1, \frac{1}{6}X_2, \frac{1}{6}X_3\right\}\right] + \frac{2}{3} \\
&= \frac{1}{6}\mathbb{E}[\max\{S_1, S_2, S_3\}] + \frac{1}{6}\mathbb{E}[\epsilon] + \frac{2}{3} \\
&= \frac{7}{8}
\end{aligned}$$

for the first round of bidding, and

$$\begin{aligned}
\mathbb{E}[\mathrm{R}^{\mathrm{II}}] &= \mathbb{E}\left[\max\left\{\frac{2}{3}X_1, \frac{2}{3}X_2\right\}\right] \\
&= \frac{2}{9}\mathbb{E}[\max\{S_1, S_2\}] + \frac{2}{9}\mathbb{E}[\epsilon] + \frac{2}{3} \\
&= \frac{8}{9}
\end{aligned}$$

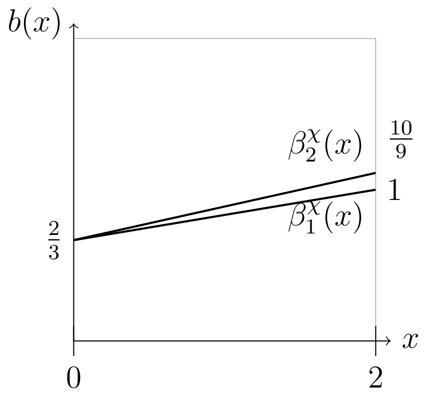


Figure 3: Equilibrium bidding strategies, $\chi$=1

for the second.

### 5.3 Notes

Though we make some relatively simplistic assumptions for the purpose of tractability, the consequences of the level of cursedness in these two examples leads to an interesting result. In the first round of bidding, the fully cursed auction generates higher revenues than the fully rational auction. However, second round revenues are higher in the fully rational auction. As shown in Figure 3, cursedness raises the intercept and lowers the slope of the equilibrium bidding function. This suggests that there is an implicit correlation between the aggressiveness of bidders in subsequent rounds and the revelation of information, specifically learning previous winning bids. Fully cursed bidders are less likely to bid aggressively because they do not take the information conveyed in the previous winning bid into consideration in the second round. The effect of the level of cursedness on seller's revenue is therefore ambiguous. In the following section, we derive the $\chi$-cursed equilibrium price series for symmetric bidders in an attempt to formalize this intuition.

# 6 Equilibrium Price Path

**Proposition 1.** Suppose that $\chi$-cursed bidders follow the symmetric bidding strategies $\beta_1^{\chi}(\cdot)$ and $\beta_2^{\chi}(\cdot)$. Then $\forall \chi \in [0,1)$, the equilibrium price series follows a submartingale, $\mathbb{E}[P_2|P_1] > P_1$.

In this section, we provide a proof of the price series proposition above. The proposition states that for any $\chi \in [0,1)$ the expected price series of both sequential first-price as well as second-price auctions are upward drifting. The assumption of affiliation causes the expected price to rise in subsequent rounds, since bid announcements carry valuable information to the remaining bidders. We use a revenue ranking lemma to show that the proposition holds for the second-price auction as well.

## 6.1 First-Price Auctions

A bidder $i$ with a signal $X_i = x$ wins the Bayesian auction in round two. Assuming bidders follow symmetric strategies, it must be that $x < y_1$, or else $i$ would have won in the previous round of bidding. From the definition of a first-price auction, the winning bidder in round two pays the price $\mathrm{P}_2$ equal to the winning bid, $\beta_2^{\chi}(\cdot)$. This implies that the price path is equivalent to

$$\begin{aligned}\mathbb{E}[P_2|P_1] &= \mathbb{E}[P_2|\beta_1^{\chi}(x)] \\ &= \mathbb{E}[\beta_2^{\chi}(x;y_1)|\beta_1^{\chi}(x)]\end{aligned}$$

Now, let

$$\frac{\frac{\mathrm{d}}{\mathrm{d}x}[F'_{Y_1}(\cdot)\beta_1^{\chi}(x)]}{f_{Y_1}(\cdot)} = \delta > 0$$

Through manipulation of the first-order differential equation for $\beta_2^{\chi}(x;y_1)$, we obtain

$$
\begin{aligned}
\beta_1'(x) &= [\beta_2^\chi(x;x) - \beta_1^\chi(x)]\frac{f_{Y_1}(\cdot)}{F_{Y_1}(\cdot)} \\
f_{Y_1}(\cdot)\beta_2^\chi(x;x) &= \frac{\mathrm{d}}{\mathrm{d}x}[F_{Y_1}'(\cdot)\beta_1^\chi(x)] + f_{Y_1}(\cdot)\beta_1^\chi(x) \\
\beta_2^\chi(x;x) &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}[F_{Y_1}'(\cdot)\beta_1^\chi(x)]}{f_{Y_1}(\cdot)} + \beta_1^\chi(x) \\
\beta_2^\chi(x;x) &= \delta + \beta_1^\chi(x)
\end{aligned}
$$

From the strict monotone likelihood ratio property, the function $\beta_2^\chi(\cdot)$ must be increasing in all its arguments. Since we know the winning signal $y_1$ is strictly greater than $x$,

$$
\mathbb{E}[\beta_2^\chi(x;y_1)|\beta_1^\chi(x)] > \mathbb{E}[\beta_2^\chi(x;x)|\beta_1^\chi(x)]
$$

and we have shown previously that for $\delta > 0$,

$$
\beta_2^\chi(x;x) > \beta_1^\chi(x)
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}[P_2|P_1] &= \mathbb{E}[\beta_2^\chi(x;y_1)|\beta_1^\chi(x)] \\
&> \mathbb{E}[\beta_2^\chi(x;x)|\beta_1^\chi(x)] \\
&> \beta_1^\chi(x) \\
&= P_1
\end{aligned}
$$

Thus, we have shown that $\mathbb{E}[P_2|P_1] > P_1$. The inequality holds from the assumption that $\beta_1^\chi(x)$ is an increasing function, coupled with the fact that $Y_2 \leq x$. The upward drift of prices is a well known result in auction theory proven first for risk neutral bidders by Milgrom and Weber (2000). When $\chi = 1$, bidders have private (but interdependent) values of the object. Furthermore, the price series with private values is

$$
\begin{aligned}
\mathbb{E}[P_2|P_1] &= \mathbb{E}[P_2|\beta_1^{\chi}(x)] \\
&= \mathbb{E}[\beta_2^{\chi}(Y_2; x, y_1)|Y_2 \leq x \leq y_1] \\
&= \beta_1^{\chi}(x) \\
&= P_1
\end{aligned}
$$

So we have $\mathbb{E}[P_2|P_1] = P_1$. It follows that for any $\chi \in [0, 1]$, the price path $P_t$ is a strictly non-decreasing function of $\beta_1^{\chi}(x)$.

## 6.2 Second-Price Auctions

Now consider the equivalent second-price auction format. In a second-price auction, the winning bidder with signal $x$ pays an amount equal to $\beta_1^{\chi}(Y_1)$, where $Y_1$ is the second-highest signal of the remaining bidders. The winning bid announcement in the first round is the same as the first-price auction, given by $\beta_1^{\chi}(x)$. It follows that the price in the second round, $P_2 = \beta_2^{\chi}(Y_2; X_1)$, where $Y_2 = x_3$ is the third-highest signal.

**Lemma 6.1.** In the sequential symmetric model with affiliated signals and common values, the first-price (FP) and second-price (SP) auctions can be ranked in terms of expected revenues, $\mathbb{E}[R^{SP}] > \mathbb{E}[R^{FP}]$.

It follows directly that the second-price auction price series also follows a submartingale (see Krishna, 2002). This concludes the proof of the proposition. Though these proofs hold for any fixed $\chi$, they rely on the assumption of symmetric bidding functions. The addition of asymmetrically $\chi$-cursed bidders may in fact lead to *lower* revenues in subsequent rounds. For instance, bidders with a higher $\chi$ value are more likely to bid above $V$. As these bidders win, those with lower (or zero) $\chi$ values are left, and though they bid more aggressively relative to their own bids in prior periods, they may still be less than the bids made by those with high $\chi$ values in previous rounds. In this case, under certain cursed conditions the equilibrium price path would *not* be submartingale but rather supermartingale, a tendency in common-value auctions with strong empirical evidence (see McAfee and Vincent, 1993).

# 7 Asymmetric $\chi$ Values

Next, we consider the implications of asymmetries in the level of cursedness among bidders. This has somewhat interesting effects on the expected price series at equilibrium. We approach asymmetry with a single model. The environment contains two types of bidders: fully cursed bidders, and fully rational bidders.

Consider a two-round auction between two fully cursed bidders and a single rational bidder. Let the true common-value of the object $V = \sum x_i$. For bidders $i = 1, 2, 3$ suppose that the signals $X_i$ are interdependent and distributed (as before) on $[0, 2]$. To differentiate between rational and cursed bidders again we denote only the cursed bids with a superscript $\chi$. The rational bidder, further referred to as bidder 3, realizes the correct distribution of the common value as the sum of the independent signals, and so his valuation is

$$v_3(x_1, x_2, x_3) = x_1 + x_2 + x_3 \tag{25}$$

However, both cursed bidders fail to value the object at its true common-value, instead opting to bid strictly from the cursed function, $C(x_i)$. Specifically,

$$V^\chi = C(x_i) = \mathbb{E}[V|X_i = x_i] = x_i + (N-1)\mathbb{E}[X_i] = x_i + (2)(1) = x_i + 2$$

Therefore, bidders 1 and 2 value the object at

$$v_1(x_1) = x_1 + 2$$
$$v_2(x_2) = x_2 + 2$$

and (at equilibrium) in round one bid symmetrically,

$$\beta_1^\chi(x_i) = k \cdot x_i + 2 \qquad k \leq 1 \tag{26}$$

where $k$ is a constant. In the first stage of bidding, we know that bidder 3 will, at most, bid his expected value conditional on his signal being the highest of his opponents, eliminating the possibility of the winner's curse. Thus in the first round,

$$\underset{x_3 \in \mathbb{R}}{\operatorname{argmax}}(\beta_1(x_3)) = 3 \cdot x_3 \tag{27}$$

is 3's maximum possible bid. Considering $\beta_1(x_3) = 3 \cdot x_3$ and $\beta_1^{\chi}(x_1) = k \cdot x_1 + 2$, there are two possible cases. If $\beta(X_3) > k \cdot \max\{x_1, x_2\} + 2$, Bidder 3 wins the first round and pays the bid of the second highest bidder, say in the case that $x_2 > x_3$, $k \cdot x_2 + 2$. The two cursed bidders then bid $V^{\chi}$ in the next round and the bidder with the higher signal pays that of the lower. On the other hand, if $k \cdot \max\{x_1, x_2\} + 2 > \beta(X_3)$ a cursed bidder pays some price and bidder 3 bids against the remaining cursed bidder. Consider the second round choice made by bidder 3. He knows that bidder 2 will, of course, bid his private value in the final round. Therefore, bidder 3's expected payoff with a bid of $b$ from the second round is

$$\begin{aligned}
\Pi_3(b, x_3; y_1) &= \int_0^b [v(x_3, y_1, x_2) - C(x_2)]\mathrm{d}x_2 \\
&= \int_0^b [(x_3 + y_1 + x_2) - (x_2 + 2)]\mathrm{d}x_2 \\
&= \int_0^b (x_3 + y_1 - 2)\mathrm{d}x_2 \\
&= b \cdot x_3 + b \cdot y_1 - 2 \cdot b
\end{aligned}$$

Maximizing $\Pi_3(\cdot)$ with respect to $b$ shows that if bidder 2 follows the strategy $\beta_2^{\chi}(x_2) = x_2 + 2$, it is optimal for bidder 3 to bid

$$\beta_2(x_3; y_1) = \begin{cases} x_3 + y_1 & \text{if } x_3 + y_1 \geq 2 \\ 0 & \text{if } x_3 + y_1 < 2 \end{cases}$$

Bidder 3 will only submit a non-zero bid if his signal combined with the signal of the previous winner is greater than 2, the minimum price he expects to pay conditional on winning in the second round. To verify this, consider the case where $\beta_2^{\chi}(\cdot) = \beta_2(\cdot)$, or $x_3+y_1 = x_2+2$. The payoff to bidder 3 upon winning is $v_2(x_1, y_1, x_3)-x_2+2 = y_1+x_3-2$. Since $y_1 + x_3 \geq 2$, bidder 3's payoff conditional on winning is strictly non-negative. Furthermore, it is trivial to show that $\mathbb{E}[\beta_2(X_3)] \leq \mathbb{E}[\beta_2(X_2; X_1)]$, so on average, bidder 2 will win the second round of the auction, and pay $\beta_2(x_3; y_1)$. However, since bidder 1 previously won, and paid a price $P_1 = k \cdot x_2 + 2 \geq 2$, it follows that $\mathbb{E}[P_2|P_1] \leq P_1$. Note that while we do not solve explicitly for the first round strategies, they are bound by the intercept from $C(x)$.

To supplement our example, we consider the impact asymmetric $\chi$ values has on the auction's competitiveness. A sequence of $N$ auctions is called *competitive* if the expected price in the Bayesian Nash equilibrium converges to the value of the object. Whenever $\chi > 0$, Eyster and Rabin (2005) show that when $N$ is sufficiently large, all $N$ bidders suffer the winner's curse in the symmetric model. In the asymmetric model, the probability that any uniquely $\chi$-cursed bidder has a signal greater than $\mathbb{E}[X_i]$ increases with the number of $\chi$-cursed bidders. As the auction progresses, the proportion of fully cursed bidders to partially cursed bidders decreases, and thus the exent of the winner's curse varies for remaining bidders with unique $\chi$s. Though our example exhibits decreasing prices, with more rounds (and thus more bidders) the expected price path becomes ambiguous.

# 8 Discussion

In this paper, we have derived symmetric cursed equilibria for sequential auctions with winning bid announcements. From extending the equilibrium concept from simultaneous to sequential auctions, we have shown the cursed equilibrium to be a robust parametric representation of a Nash equilibrium. When the first round bids are announced, symmetric $\chi$-cursed bidders update their bidding strategies to reflect information acquisition. We show the submartingale price series first derived by Milgrom and Weber (2000) for symmetric bidders with affiliated values holds for any $\chi \in [0, 1)$. Consequently, $\chi$ may increase or decrease the seller's revenue in any sequence of auctions, but $\mathbb{E}[R^I] < \mathbb{E}[R^{II}]$ for any fixed $\chi$. As a result, the effect of $\chi$ on a seller's revenue is ambiguous. We have also shown how even slight asymmetries among bidders' $\chi$ values may lead to downward drifting prices in equilibrium. Fitting $\chi$ values to empirical sequential auction data may yield more interesting results in this direction.

Furthermore, our model may be strengthened in various ways. For instance, allowing $\chi$ to follow some probability distribution across bidders may be a more realistic adaptation to sequential games than assuming binary $\chi$ values for the entire bidder population. Though this would appear to be a natural extension of our work, the possibility of proving the existence of such an equilibrium solution does not seem particularly tractable. Eyster and Rabin (2005) note that another possible refinement of a cursed equilibrium in repeated games could be allowing for a more robust definition of cursedness. If we consider the action of observing winning bids, it is perhaps more realistic to assume that even fully cursed bidders would rationally update their strategies. In this sense, bidders could become more rational as they gain more information in sequential games, i.e. $\chi$ decreases as a function of $t$. However, this idea partially undermines the essence of a cursed equilibrium since for $\chi \in (0, 1)$ bidders partially update their bidding strategies as they gain better information anyway and only fully cursed bidders fail to do so across rounds.

In conclusion, we believe that the cursed equilibrium offers an intuitive and portable adaptation to the Nash equilibrium and may help in bridging the gap between the theory of auctions and phenomena observed in empirical studies.

Appendices

# A

# Joint Densities

## A.1 Two Bidders

For clarity, let the signals of bidder $Y_1$ and $Y_2$ be denoted by $Y$ and $Z$, respectively. To get the joint distribution $f(y,z)$, note that for any fixed $\epsilon$, $Y$ and $Z$ are uniformly and independently distributed over $[\epsilon, \epsilon+1]$. Therefore,

$$f_{YZ|\epsilon}(y,z|\epsilon) = \begin{cases} 1 & \text{if } (y,z) \in [\epsilon,\epsilon+1] \times [\epsilon,\epsilon+1] \\ 0 & \text{otherwise} \end{cases}$$

The joint distribution of $Y$,$Z$, and $\epsilon$ over $\mathbb{R}^3$ is easily shown to be

$$f_{YZ\epsilon}(y,z,\epsilon) = \begin{cases} 1 & \text{if } \epsilon \leq \min(y,z) \leq \max(y,z) \leq \epsilon+1 \\ 0 & \text{otherwise} \end{cases}$$

**Case 1.** Let $z \leq y$. We now have

$$f(y,z,\epsilon) = \begin{cases} 1 & \text{if } \epsilon \leq z \leq y \leq \epsilon+1 \\ 0 & \text{otherwise} \end{cases}$$

**Sub-case A.** Let $z \leq y \leq 1$. In this case,

$$f(y,z,\epsilon) = \begin{cases} 1 & \text{if } \epsilon \leq z \\ 0 & \text{otherwise} \end{cases}$$

To obtain our joint density $f(y,z)$ from $f(y,z,\epsilon)$, we solve the indefinite integral

$$f(y,z) = \int_{-\infty}^{\infty} f(y,z,\epsilon)\,\mathrm{d}\epsilon$$

with respect to the bounds of $f(y,z,\epsilon)$. For $z \leq y \leq 1$,

$$f(y,z) = \int_0^1 f(y,z,\epsilon)\mathrm{d}\epsilon = \int_0^z \mathrm{d}\epsilon = z$$

**Sub-case B.** Let $1 \leq z \leq y$. In this case,

$$f(y,z,\epsilon) = \begin{cases} 1 & \text{if } y-1 \leq \epsilon \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus if $1 \leq z \leq y$,

$$f(y,z) = \int_0^1 f(y,z,\epsilon)\mathrm{d}\epsilon = \int_{y-1}^1 \mathrm{d}\epsilon = 2 - y$$

**Sub-case C.** Let $z \leq 1 \leq y$ and $y - 1 \leq z$. If $y - 1$ is strictly greater than $z$, there is no $\epsilon$ such that $y - 1 \leq \epsilon \leq z$. So in this case,

$$f(y,z,\epsilon) = \begin{cases} 1 & \text{if } y-1 \leq \epsilon \leq z \\ 0 & \text{otherwise} \end{cases}$$

So for $z \leq 1 \leq y$ and $y - 1 \leq z$,

$$f(y,z) = \int_0^1 f(y,z,\epsilon)\mathrm{d}\epsilon = \int_{y-1}^z \mathrm{d}\epsilon = 1 + z - y$$

Combining the cases above when $z \leq y$,

$$f_{YZ}(y,z) = \begin{cases} z & \text{if } z \leq y \leq 1 \\ 2-y & \text{if } 1 \leq z \leq y \\ 1+z-y & \text{if } z \leq 1 \leq y \leq z+1 \\ 0 & \text{otherwise} \end{cases}$$

The case when $y \leq z$ is determined symmetrically. Calculating the conditional densities requires some probability manipulation. Let

$$f_{ZY}(z|y) = \frac{f_{ZY}(y, z)}{f_Y(y)} \tag{28}$$

be the probability density function of $y$ conditional on $x$ being the realization of the variable $X$. Since $X_i = S_i + \epsilon$, the probability density function of $X_i$ over the interval $[0, \theta]$ is

$$f_Y(y; \theta) = \begin{cases} \frac{2\theta - y}{\theta^2} & \text{if } \theta \leq y \\ \frac{y}{\theta^2} & \text{if } y \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

Now, for $z \leq y \leq 1$ (the other cases are symmetric) the conditional distribution is

$$f(z|y; \theta) = \frac{z\theta^2}{y}$$

and for all $y \in [0, \theta]$, the cumulative distribution function $F(y|y; \theta)$ is

$$\begin{aligned} F(y|y; \theta) &= \int_0^y f(z|y)\mathrm{d}z \\ &= \frac{z^2\theta^2}{2z}\Big|_0^y \\ &= \frac{z\theta^2}{2} \end{aligned}$$

Thus,

$$\frac{f_{Y_2}(x|x; y_1)}{F_{Y_2}(x|x; y_1)} = \frac{2}{x} \tag{29}$$

## A.2 Three Bidders

Let the signals of bidders $X_1$, $Y_1$ and $Y_2$ be denoted by $X$, $Y$, and $Z$. To get the joint distribution $f(x, y, z)$, note that for any fixed $\epsilon$, $X$, $Y$, and $Z$ are uniformly and independently distributed over $[\epsilon, \epsilon+1]$.

**Case 1.** Let $z \leq y \leq x$. We now have

$$f(y,z,x,\epsilon) = \begin{cases} 1 & \text{if } \epsilon \leq z \leq y \leq x \leq \epsilon + 1 \\ 0 & \text{otherwise} \end{cases}$$

**Sub-case A.** Let $z \leq y \leq x \leq 1$. In this case,

$$f(x,y,z,\epsilon) = \begin{cases} 1 & \text{if } \epsilon \leq z \\ 0 & \text{otherwise} \end{cases}$$

To obtain the joint density $f(x,y,z)$ from $f(x,y,z,\epsilon)$, we solve the indefinite integral with respect to the bounds of $f(x,y,z,\epsilon)$. For $z \leq y \leq x \leq 1$,

$$\begin{aligned} f(x,y,z) &= \int_0^1 f(x,y,z,\epsilon)\mathrm{d}\epsilon \\ &= \int_0^z \mathrm{d}\epsilon \\ &= z \end{aligned}$$

We offer the other possible cases without proof. Combining the cases not shown with the one above, when $z \leq y \leq x$,

$$f_{XYZ}(x,y,z) = \begin{cases} z & \text{if } z \leq y \leq x \leq 1 \\ 2-x & \text{if } 1 \leq z \leq y \leq x \\ 1+z-x & \text{if } z \leq 1 \leq y \leq x \leq z+1 \\ 1+z-x & \text{if } z \leq y \leq 1 \leq x \leq z+1 \\ 0 & \text{otherwise} \end{cases}$$

Like in the prior example, the other possible cases can be determined symmetrically. The joint density $f(x, y)$ requires another integration,

$$
\begin{aligned}
f_{XY}(x, y) &= \int_{-\infty}^{\infty} f(x, y, z)\,\mathrm{d}z \\
&= \int_{0}^{y} z\,\mathrm{d}z \\
&= \frac{y^2}{2}
\end{aligned}
$$

Again, since $X_i = S_i + \epsilon$ the probability density function of $X_i$ over the interval $[0, 2]$ is (from a standard application of the convolution formula)

$$
f_X(x) = \begin{cases} x & \text{if } x \leq 1 \\ 2 - x & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}
$$

Now, for $z \leq y \leq x \leq 1$ (the other cases are symmetric about 1) the conditional distribution is

$$
f_{Y|X}(y|x) = \frac{y^2}{2x}
$$

and for all $y \in [0, 2]$, the cumulative distribution function $F(y|y; \theta)$ is

$$
\begin{aligned}
F(x|x) &= \int_{0}^{x} f(y|x)\mathrm{d}y \\
&= \int_{0}^{x} \frac{y^2}{2x} \\
&= \frac{y^3}{6x}\Big|_0^x \\
&= \frac{x^2}{6}
\end{aligned}
$$

Thus,

$$\frac{f_{Y_1}(x|x)}{F_{Y_1}(x|x)} = \frac{3}{x} \tag{30}$$

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## Education

B.S. Economics, Penn State University, May, 2013, *with Honors*

## Professional Experience

**Distinguished Undergraduate Researcher** — 2012 - Present
Applied Research Laboratory, Penn State University — University Park, PA

## Affiliations

Penn State Investment Association, 2011-Current

## Awards, Grants and Scholarships

1. Schreyer Honors Scholar
2. Dean's List

## Publications

*Journal Articles*

1. R. Tatko and C. Griffin. Game Theoretic Formation of a Centrality Based Network. ASE Human Journal, 1(1):40-51, 2012.
   · 3% of manuscripts submitted were selected for publication in the ASE Human Journal.

*Conference Articles (Peer Reviewed)*

1. R. Tatko and C. Griffin. Game Theoretic Formation of a Centrality Based Network. ASE International Conference on Social Informatics, Washington DC, December 14 - 16, 2012.

**Abstract**

We model the formation of networks as a game where players aspire to maximize their own centrality by increasing the number of other players to which they are path-wise connected, while simultaneously incurring a cost for each added adjacent edge. We simulate the interactions between players using an algorithm that factors in rational strategic behavior based on a common objective function. The resulting networks exhibit pairwise stability, from which we derive necessary stable conditions for specific graph topologies. We then expand the model to simulate non-trivial games with large numbers of players. We show that using conditions necessary for the stability of star topologies we can induce the formation of hub players that positively impact the total welfare of the network.

## Presentations

1. MIT Interdisciplinary Workshop on Information and Decision in Social Networks, Cambridge, MA, November 8-9, 2012.

## Papers (In Progress)

1. Cursed Equilibrium Refinements in Sequential Auction Games (Senior Thesis advised by Vijay Krishna)

**Abstract**

In this paper, we apply the concept of a *cursed equilibrium* to a sequential auction environment. In a cursed equilibrium, players fail to take into consideration the relationship between their opponents' strategies and types with probability $\chi$. In auctions, bidders who exhibit this behavior are said to suffer from the winner's curse: the price they pay upon winning the auction exceeds the true value of the object. As an extension of Eyster and Rabin's original paper, we derive symmetric equilibria strategies for a two-round sequential auction as well as the corresponding price series for any fixed level of cursedness. In addition, we posit a plausible connection between the winner's curse and the downward price anomaly in sequential common-value auctions.