THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF MATHEMATICS

A MODEL OF ECONOMIC GROWTH DRIVEN BY COMPETITIVE INNOVATION

SIJIA SHEN
Spring 2013

A thesis
submitted in partial fulfillment
of the requirements
for a baccalaureate degree in Mathematics
with honors in Mathematics

Reviewed and approved* by the following:

Alberto Bressan
Eberly Family Professor
Thesis Supervisor

Luen-Chau Li
Professor
Honors Advisor

*Signatures are on file in the Schreyer Honors College.

ABSTRACT

The Piazza model is a simplified economic growth model describing the competition between a leader company and a follower company within the same sector. It is described by a system of generalized evolution equations. We split the admissible domain into four regions and in each domain, the variables evolve in time according to a system of ODEs. We examine the dynamics of the system in each domain and try to find a periodic trajectory of the system. We find that such an orbit cannot exist in domain one or four by examining the equilibrium points; therefore it can only switch between regime two and three. To do so, we research on the vector fields of these two ODE systems, looking for a location where the vectors of the two dynamical systems point in opposite directions. But finally we prove that in nowhere the vector fields behave so with the assigned parameters, which means, there exists no periodic trajectory of this model under the given parameters. We also give conditions that the parameters have to meet in order to allow a periodic orbit to exist.

CONTENTS

LIST OF FIGURES

1 Introduction

Within each industry, leaders and followers play a game of innovation, and their expectation of each other's future strategies determine the current strategies on research and development (R&D).

We say the industry is either at a contestable or non-contestable state. In the contestable state the leader and the follower have the same level of technology that can be used for innovation. As a result, the follower will stay involved in R&D and so will the leader in order to distance himself enough from the follower. We refer to this state as the state where the distance between the leader and the follower is one; in the non-contestable state the technology mastered by the follower is so unproductive that the follower chooses not to innovate and therefore he no longer threatens the leader. As a result, the leader also stops investing on R&D. We refer to this state as the state where a follower is two steps behind the leader.

Somehow even in the uncontestable state there will be a natural spillover of technology from the leader to the follower. To quantify this factor, we assume the distance between leader and follower is reduced from two to one step at a constant rate τ, where $0 < \tau < 1$.

2 A dynamical system generated by an innovation game

We consider a model of economic growth recently introduced by R. Piazza in [1].

It is assumed that the economy involves a continuum of productive sectors. In each sector there is a competition between two companies: a leader, and a follower. In this simplified model, the technological gap between leader and follower can be either 1 or 2. If the gap is 2, then it is regarded as unsurmountable, and both companies stop doing research and development. If the gap is 1, then both companies may engage in R&D, because follower hopes to take the lead, while the leader tries to widen the technological gap to 2 units.

The leading company has more profits, because of the technologically superior product that can be sold at a higher price. The is a natural spillover of technology from leader to follower, that occurs slowly in time (patents expire, information is spread, etc...).

The model is formulated in terms of the following variables, which we regard as functions of time:

$\alpha \in [0,1]$ = portion of productive sectors where the technological gap between leader and follower is 1,

$1-\alpha \in [0,1]$ = portion of productive sectors where the technological gap between leader and follower is 2,

V_1^l = value of having a technological lead of 1 unit,

V_2^l = value of having a technological lead of 2 units,

$L \in [0,1]$ = percentage of workers employed in manufacturing,

$1-L \in [0,1]$ = percentage of workers employed in research and development,

λ_*^l = research effort by the leaders,

λ_*^f = research effort by the followers.

The model also contains the following constants:

ρ = discount rate,

τ = rate at which technology spills over from leaders to followers,

$m > 1$ = instantaneous profit ratio between a leader and a follower.

A reasonable choice of these constants is

$$\tau \approx 0.2\,, \qquad m \approx 1.4\,, \qquad \rho \approx 0.1\,. \tag{2.1}$$

As shown in [1], the evolution of the economy is described by the following system of generalized evolution equations.

$$\dot V_1^l(t) = \left(\frac{\dot L}{L}+\rho\right)V_1^l-(m-1)L+\lambda_*^f V_1^l-\lambda_*^l(V_2^l-V_1^l-1)\,, \tag{2.2}$$

$$\dot V_2^l(t) = \left(\frac{\dot L}{L}+\rho\right)V_2^l-(m-1)L+\tau(V_2^l-V_1^l)\,, \tag{2.3}$$

$$L = 1-(\lambda_*^f+\lambda_*^l)\alpha\,, \tag{2.4}$$

$$\dot\alpha = \tau(1-\alpha)-\lambda_*^l\alpha\,. \tag{2.5}$$

From an additional evolution equation for the value function of the follower, it is known that λ_*^f satisfies

$$V_1^l<1 \quad\Longrightarrow\quad \lambda_*^f=0\,. \tag{2.6}$$

$$V_1^l=1 \quad\Longrightarrow\quad \lambda_*^f = (m-1)L-\left(\frac{\dot L}{L}+\rho\right)\,. \tag{2.7}$$

In addition, we have

$$V_2^l-V_1^l < 1 \qquad\Longrightarrow\qquad \lambda_*^l = 0\,, \tag{2.8}$$

$$V_2^l-V_1^l \le 1\,, \qquad\qquad V_1^l \le 1\,. \tag{2.9}$$

We can split the admissible domain into four regions (see Fig.1).

CASE 1: $V_1<1$, $V_2-V_1<1$. In this case, $\lambda_*^f=\lambda_*^l=0$ and the system reduces to

$$\begin{cases}\dot V_1 = \rho V_1-(m-1)\,,\\ \dot V_2 = \rho V_2-(m-1)\,,\end{cases}\qquad \begin{cases} L = 1\,,\\ \dot\alpha = \tau(1-\alpha)\,.\end{cases} \tag{2.10}$$

CASE 2: $V_1=1$, $V_2-V_1<1$. Then $\lambda_*^l=0$ and

$$\begin{cases}\lambda_*^f = \dfrac{1-L}{\alpha}\,,\\[2ex] \dot V_2 = \left((m-1)L-\dfrac{1-L}{\alpha}+\tau\right)V_2-(m-1)L-\tau\,,\end{cases}\qquad \begin{cases}\dfrac{\dot L}{L} = (m-1)L-\dfrac{1-L}{\alpha}-\rho\,,\\[2ex] \dot\alpha = \tau(1-\alpha)\,.\end{cases} \tag{2.11}$$

CASE 3: $V_1=1$, $V_2=2$. Then

$$\begin{cases}\lambda_*^f = \dfrac{(m-1)L+\tau}{2}\,,\\[2ex] \lambda_*^l = \dfrac{1-L}{\alpha}-\dfrac{(m-1)L+\tau}{2}\,,\end{cases}\qquad \begin{cases}\dfrac{\dot L}{L} = \dfrac{(m-1)L-\tau}{2}-\rho\,,\\[2ex] \dot\alpha = \tau\Big(1-\dfrac{\alpha}{2}\Big)-(1-L)+(m-1)\dfrac{\alpha}{2}L\,.\end{cases} \tag{2.12}$$

CASE 4: $V_1 < 1$, $V_2 - V_1 = 1$. Then (2.2)-(2.5) yield $\lambda_*^f = 0$ and

$$\begin{cases} \lambda_*^l & = \dfrac{1-L}{\alpha}\,, \\[2ex] \dot V_1 & = \dot V_2 \;=\; -\tau V_1 - (m-1)L\,, \end{cases} \qquad\qquad \begin{cases} \dfrac{\dot L}{L} & = -\tau - \rho\,, \\[2ex] \dot\alpha & = \tau(1-\alpha) + L - 1\,. \end{cases} \tag{2.13}$$

Restricted to each of the four domains $\Omega_1, \ldots, \Omega_4$ (Fig. 1), corresponding to CASES 1–4, the variables V_1, V_2, α, L evolve in time according to a system of ODEs. We seek solutions globally defined for all $t \geq 0$.

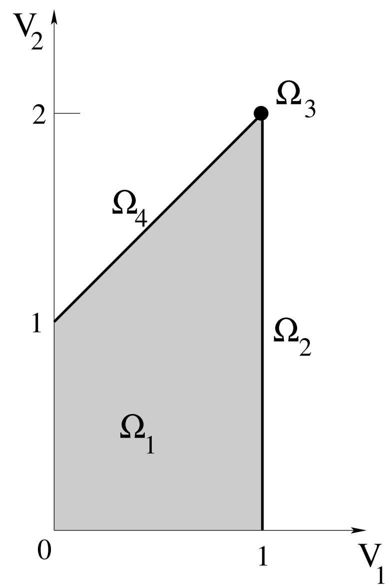

Figure 1: The stratified domain for the variables $V_1 = V_1^l$, $V_2 = V_2^l$.

3 Dynamics of the system

In this section we analyze the dynamics in the four regions $\Omega_1, \ldots, \Omega_4$, and determine the equilibrium points.

CASE 1 has a unique equilibrium point

$$V_1 \;=\; V_2 \;=\; \frac{m-1}{\rho}\,. \tag{3.1}$$

The dynamics is totally unstable. All trajectories move away from this point, along straight lines. With the choice (2.1) of the constants, this point lies outside the domain Ω_1, and the dynamics is shown in Fig. 2.

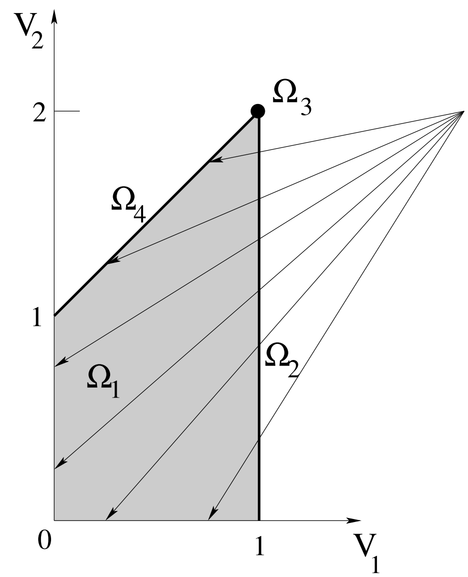

Figure 2: Trajectories of the system (2.10).

CASE 2 has the only equilibrium point

$$\overline{P} \;=\; (\bar{L}, \bar{\alpha}) \;=\; \left(\frac{1+\rho}{m}\,,\;1\right). \tag{3.2}$$

This is meaningful as long as $\rho \leq m-1$.

A sketch of the trajectories of (2.11) is given in Fig. 3. Notice that $\dot{L} = 0$ when

$$(m-1)L - \frac{1-L}{\alpha} - \rho \;=\; 0\,.$$

Along the hyperbola

$$\gamma \;=\; \left\{(L,\alpha)\,;\quad \alpha \;=\; \frac{1-L}{(m-1)L-\rho}\right\} \;=\; \left\{(L,\alpha)\,;\quad (m-1)\alpha L - \rho\alpha + L \;=\; 1\right\}$$

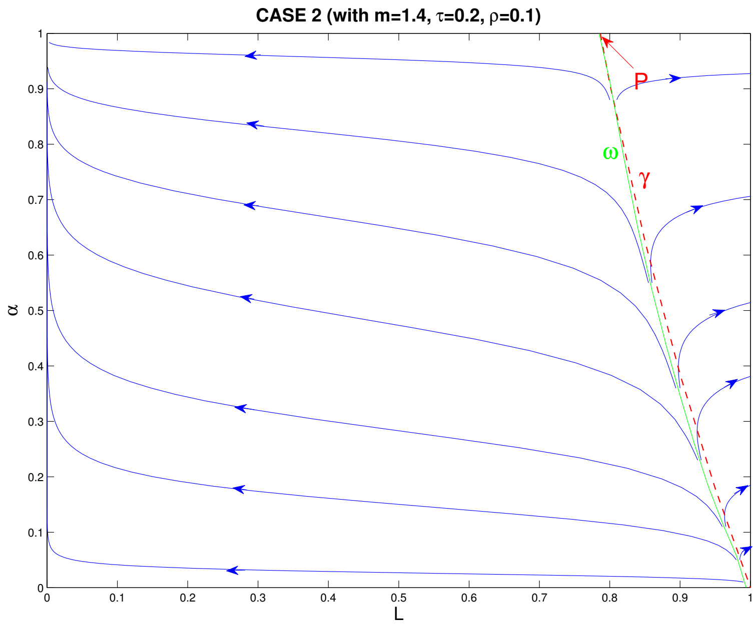

Figure 3: Trajectories of the system (2.11). Along the dashed curve γ the velocity field is vertical. The trajectory ω is the stable manifold for the equilibrium point $\overline{P}$.

one has $\dot{L} = 0$, hence the flow points vertically upward. To the left of γ there is a stable orbit ω approaching the equilibrium point $\overline{P}$.

CASE 3 (Fig. 4). When $V_1 = 1$, $V_2 = 2$, the variables L, α evolve according to the ODEs

$$\left\{ \begin{array}{rcl} \dot{L} & = & \dfrac{(m-1)L^2}{2} - \left(\rho + \dfrac{\tau}{2}\right) L \,, \\[3mm] \dot{\alpha} & = & \Big[(m-1)L - \tau\Big] \dfrac{\alpha}{2} + \tau - 1 + L \,. \end{array} \right. \tag{3.3}$$

In this case, one has $\dot{L} = 0$ along the vertical line

$$\omega \;=\; \left\{ (L, \alpha) \,; \quad L = \frac{2\rho + \tau}{m-1} \right\}.$$

Moreover, $\dot{\alpha} = 0$ on the hyperbola

$$\gamma' \;=\; \left\{ (L, \alpha) \,; \quad \alpha \;=\; 2 \cdot \frac{1 - \tau - L}{(m-1)L - \tau} \right\} \;=\; \left\{ (L, \alpha) \,; \quad \frac{m-1}{2} \alpha L - \frac{\alpha\tau}{2} + L \;=\; 1 - \tau \right\}.$$

We thus have one equilibrium point E, whose coordinates satisfy

$$\left\{ \begin{array}{rcl} (m-1)L - \tau & = & 2\rho \,, \\[3mm] [(m-1)L - \tau] \dfrac{\alpha}{2} & = & 1 - \tau - L \,. \end{array} \right. \tag{3.4}$$

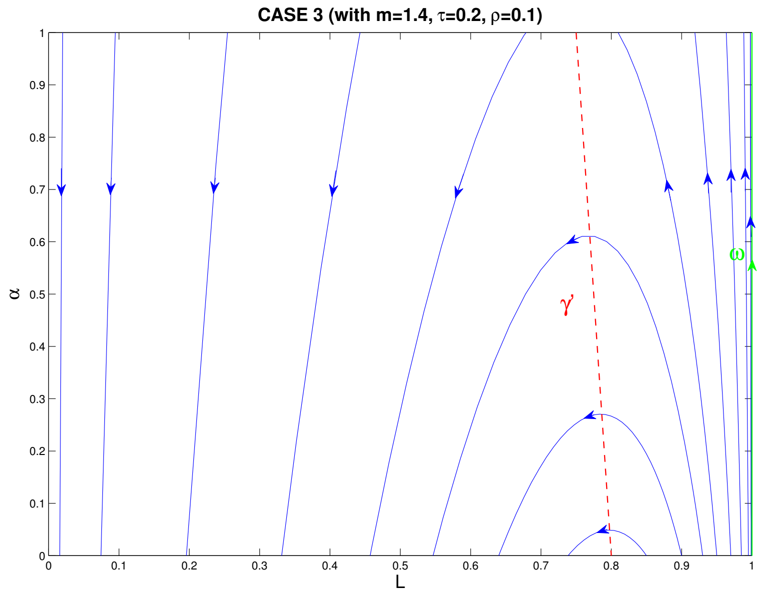

Figure 4: Trajectories of the system (2.12). Along the dashed curve γ' the velocity field is horizontal. The vertical trajectory ω is the stable manifold for the equilibrium point E.

$$\left\{ \begin{array}{rcl} L & = & \dfrac{2\rho+\tau}{m-1}\,, \\ \\ \alpha & = & \dfrac{1-\tau-L}{\rho} \;=\; \dfrac{(m-1)-\tau m-2\rho}{(m-1)\rho}\,. \end{array} \right. \tag{3.5}$$

Since $L,\alpha\in[0,1]$, this equilibrium point E lies inside our domain only if

$$m\tau \;\leq\; (m-1)-2\rho \;\leq\; m\tau+(m-1)\rho\,. \tag{3.6}$$

CASE 4 has no equilibrium points. By (2.13) both components V_1, V_2 are strictly decreasing. If $L(0)>0$, then

$$L(t) \;=\; e^{-(\tau+\rho)t}L(0) \;>\; 0 \qquad \text{for all} \quad t>0\,. \tag{3.7}$$

Hence the component V_1 becomes negative in finite time.

4 Absence of periodic orbits

Consider the two vector fields on $Q = [0,1] \times [0,1]$ defined as the right hand sides of (2.11)-(2.12), namely

$$\mathbf{v} \;=\; \begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix} \;=\; \begin{pmatrix} \left[(m-1)L - \dfrac{1-L}{\alpha} - \rho\right] L \\ \\ \tau(1-\alpha) \end{pmatrix}, \tag{4.1}$$

$$\mathbf{w} \;=\; \begin{pmatrix} w_1 \\ \\ w_2 \end{pmatrix} \;=\; \begin{pmatrix} \left[\dfrac{(m-1)L-\tau}{2} - \rho\right] L \\ \\ \tau\Big(1 - \dfrac{\alpha}{2}\Big) - (1-L) + (m-1)\dfrac{\alpha}{2}\, L \end{pmatrix}. \tag{4.2}$$

We now show that, if there is no point (L, α) at which these two vector fields are parallel and point to opposite directions, then no periodic orbit can exist.

Lemma 1. *Assume that there exists a periodic orbit $t \mapsto X(t) = (L(t), \alpha(t)) \in Q$, with*

$$\dot{X}(t) \;\in\; \{\mathbf{v}, \mathbf{w}\}\,, \qquad\qquad X(0) = X(T). \tag{4.3}$$

Then there exists a point $(\overline{L}, \bar{\alpha}) \in Q$ at which the vector fields $\mathbf{v}, \mathbf{w}$ point to opposite directions:

$$\langle \mathbf{v}\,,\, \mathbf{w}\rangle \;\leq\; 0\,, \qquad\qquad \mathbf{v}\wedge\mathbf{w} \;=\; det\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \;=\; 0\,. \tag{4.4}$$

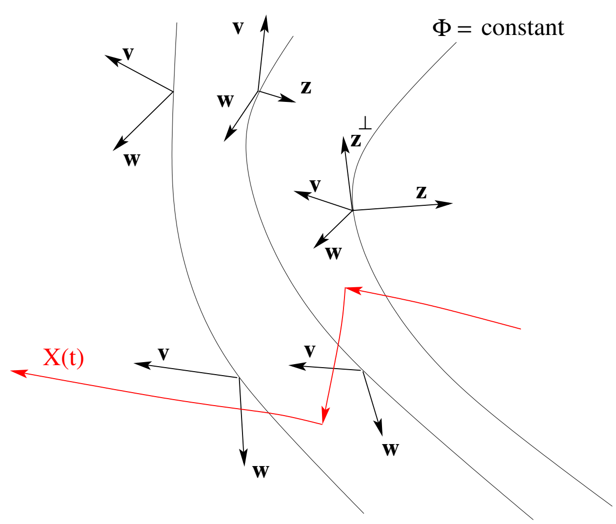

Figure 5: If the vector fields $\mathbf{v}, \mathbf{w}$ never point in opposite directions, one can construct a Lyapunov function Φ which is strictly decreasing along trajectories of $\mathbf{v}$ or $\mathbf{w}$.

Proof. 1. Assume that Q contains no points where (4.4) holds. Then for every $X = (L, \alpha) \in Q$ we can define the vector field

$$\mathbf{z} \;=\; \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \;\; -\frac{\mathbf{v}}{|\mathbf{v}|} - \frac{\mathbf{w}}{|\mathbf{w}|} \;\;=\;\; -\frac{1}{\sqrt{v_1^2+v_2^2}}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \frac{1}{\sqrt{w_1^2+w_2^2}}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Notice that

$$\langle \mathbf{v}\,,\,\mathbf{z}\rangle \ <\ 0, \qquad\qquad \langle \mathbf{w}\,,\,\mathbf{z}\rangle \ <\ 0. \tag{4.5}$$

We now construct a Lyapunov function Φ such that $\nabla\Phi = \lambda\mathbf{z}$ for some strictly positive function λ. The condition $\mathrm{curl}\,(\lambda\mathbf{z}) = 0$ yields the scalar linear first order PDE

$$\lambda_{x_2} z_1 + \lambda z_{1,x_2} - \lambda_{x_1} z_2 - \lambda z_{2,x_1} \ =\ 0\,.$$

Setting $\mathbf{z}^\perp \doteq (-z_2, z_1)$, this can be written as

$$\nabla\lambda \cdot \mathbf{z}^\perp \ =\ \lambda(z_{1,x_2} - z_{2,x_1}). \tag{4.6}$$

If λ is any smooth, strictly positive solution of (4.6), then $\mathrm{curl}\,(\lambda\mathbf{z}) = 0$ and hence there exists a function Φ defined on a neighborhood of the convex set Q such that $\nabla\Phi = \lambda\mathbf{z}$.

2. By (4.5) and the fact that $\lambda > 0$, for any solution of

$$\dot X(t) \ \in\ \{\mathbf{v}, \mathbf{w}\}$$

we have

$$\frac{d}{dt}\Phi(X(t)) \ =\ \nabla\Phi \cdot \dot X \ =\ \lambda\mathbf{z}\cdot\dot x \ <\ 0\,. \tag{4.7}$$

If now $X(T) = X(0)$, this would imply $\Phi(X(T)) = \Phi(X(0))$, in contradiction with (4.7). □

To prove that there are no periodic orbits, it suffices to show that (4.4) never happens.

Theorem 1. *Assume that the constants τ, m, ρ in (2.1) satisfy the inequality*

$$\tau + \frac{\tau + 2\rho}{m-1} \ \geq\ 1\,, \tag{4.8}$$

Then no periodic solution of $\dot X \in \{\mathbf{v}, \mathbf{w}\}$ can exist.

Proof. By the previous lemmas, if a periodic orbit exists, then there exists at least one point $(\overline{L}, \bar\alpha)$ where v, w point to opposite directions. We claim that this cannot happen, if the assumption (4.8) holds. Two cases will be considered.

CASE 1: $w_1 < 0 < v_1$. In this case we have

$$\frac{(m-1)L - \tau}{2} - \rho \ <\ 0\ <\ (m-1)L - \frac{1-L}{\alpha} - \rho\,.$$

This implies

$$\frac{(m-1)L}{2} - \frac{1-L}{\alpha} \ >\ -\frac{\tau}{2}\,,$$

$$\frac{(m-1)\alpha L}{2} - (1-L)\ > -\frac{\alpha\tau}{2} \tag{4.9}$$

If (4.9) holds, then

$$w_2 \;=\; \tau\Big(1-\frac{\alpha}{2}\Big)-(1-L)+(m-1)\frac{\alpha}{2}L \;>\; \tau\Big(1-\frac{\alpha}{2}\Big)-\frac{\alpha\tau}{2} \;=\; \tau(1-\alpha) \;=\; v_2 \;>\; 0\,.$$

Therefore, v_2 and w_2 are both positive, showing that the vectors v and w cannot point to opposite directions.

CASE 2: $v_1 \leq 0 \leq w_1$. In this case we have

$$(m-1)L-\frac{1-L}{\alpha}-\rho \;\leq\; 0 \;\leq\; \frac{(m-1)L-\tau}{2}-\rho\,. \tag{4.10}$$

The second inequality in (4.10) yields

$$(m-1)L-\tau \;\geq\; 2\rho\,, \qquad\qquad L \;\geq\; \frac{2\rho+\tau}{m-1}\,.$$

Hence

$$w_2 \;=\; \tau+\frac{\alpha}{2}[(m-1)L-\tau]-(1-L) \;\geq\; \tau+\alpha\rho-(1-L) \;\geq\; \tau+\alpha\rho-1+\frac{2\rho+\tau}{m-1}\,.$$

Since $v_2 = \tau(1-\alpha) > 0$, if (4.8) holds, then $w_2 \geq 0$ and hence v, w cannot point to opposite directions. An application of Lemma 1 rules out the possibility of periodic orbits.

Remark 1. In particular, if the parameters take the values (2.1), then

$$\tau+\frac{\tau+2\rho}{m-1} \;=\; 0.2+\frac{0.2+0.2}{1.4-1} \;=\; 1.2 \;>\; 1,$$

and the inequality (4.8) is satisfied. Therefore, no periodic orbit can exist.

Remark 2. The above analysis also shows that, if (4.5) holds, then any solution will eventually reach the boundary of Q. In particular, it cannot keep switching between Ω_2 and Ω_3.

References

[1] R. Piazza, Leadership contestability, monopolistic rents and growth. Preprint, 2012.

ACADEMIC VITA

Sijia Shen
(814)441-9280
szs5386@psu.edu

Education

Penn State University, May 2013
Bachelor of Science in Mathematics, Systems Analysis Option
Honors in Mathematics
Thesis Titile: A Model of Economic Growth Driven by Competitive Innovation
Thesis Supervisor: Alberto Bressan

Experience

Mathematics Advanced Study Semesters (MASS) Program, Penn State University, Fall 2012

PSU-PKU Summer Program in Dynamical Systems, Peking University, Beijing, Summer 2012

Consulting Intern, Mercer Consulting (China) Ltd., Shanghai, China, June 2012

Pre Mathematics Advanced Study Semester (PMASS) Program, Penn State University, Fall 2011

Analyst (Intern), Haitong Securities Co., Ltd, Shanghai, China, Summer 2011

Awards

Evan Pugh Scholar Award 2012, Penn State University
Member of Phi Beta Kappa Honors Society
The President's Freshman Award 2010, Penn State University
Dean's List