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STOCHASTIC MODELING AND PRICING OF MORTALITY-LINKED SECURITIES

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ABSTRACT

This paper investigates the application of stochastic mortality model and the Wang transform to mortality-linked securities by incorporating an additional process. The Lin and Cox model, which takes extreme events into consideration, is used to model future mortality rates. The Wang transform leads to the final price of the pure mortality bond. With the help of the statistical software R, we use the maximum likelihood estimation to estimate the values of parameters in the Lin and Cox model and use the Wang transform to evaluate the price of the Swiss Re mortality bond. Also, by using a similar method we find the price of a pure mortality bond, which is designed to resemble the Swiss Re bond. Our result can be used as a reference for companies that issue such bonds.

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Chapter 1

Introduction

Insurance companies typically face mortality risks. Those that sell insurance have losses when death rates are unexpectedly high due to sudden events. Those that sell annuities suffer losses when mortality rates decrease unexpectedly. According to Cox, Lin and Pedersen (2010), US mortality data indicate there were two significant events in the U.S. history, 1918 flu pandemic and mortality decrease around 1970.

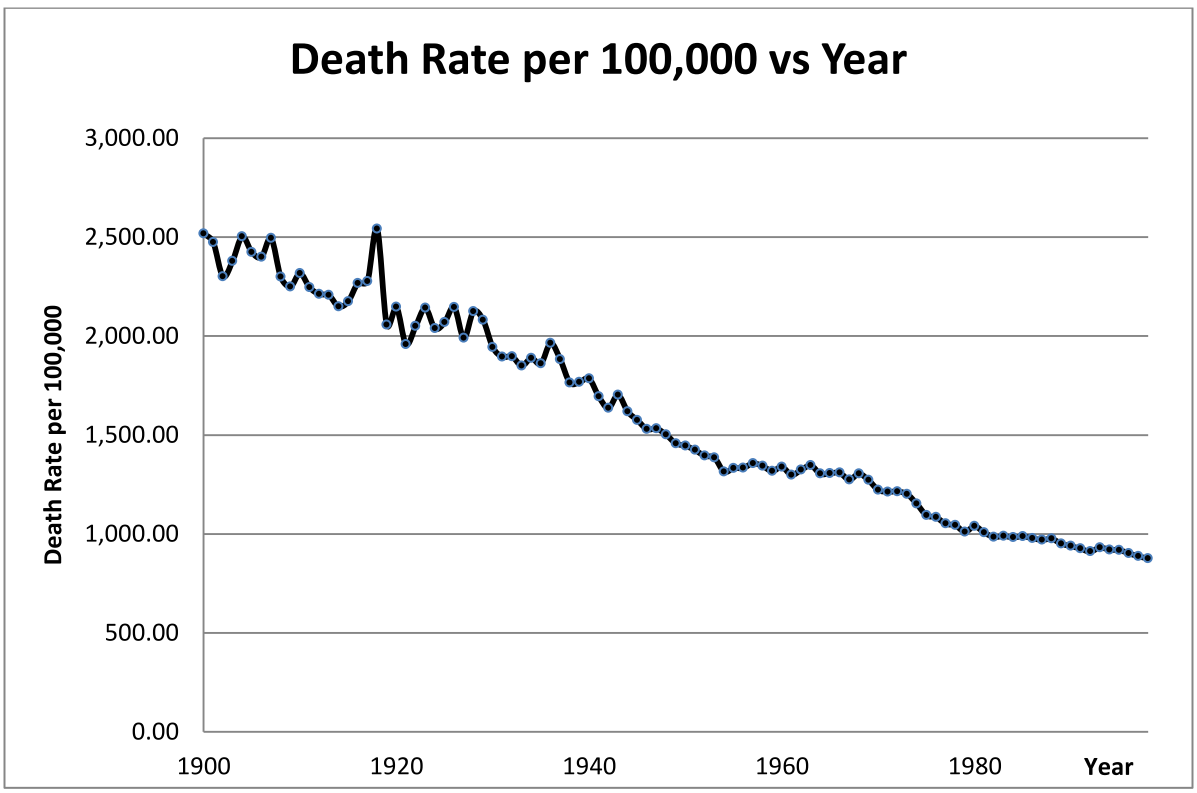

Figure 1: 1900-1998 US age-adjusted death rate per 100,000.[1]

From figure 1, we can clearly find that the death rates increased dramatically around the year 1918 due to the 1918 flu pandemic. It was reported that nearly 20% to 40% of the world

[1] Source: http://www.cdc.gov/.

population was infected. The number of people being killed by the 1918 flu is even greater than the number that died in World War I. In the United States, the devastating epidemic caused nearly 675,000 deaths. Even in today's society, flu viruses are still threatening people's lives. During the year 2009-2010, an estimated 10,000 people died in the global H1N1 flu pandemic.[2] Events like devastating flu pandemics and earthquakes causing millions of deaths influences insurance companies' revenues. Even worse, there is a chance that a company is unable to pay claims when too many claims occur at the same time and becomes bankrupted.

On the other hand, if death rates decline unexpectedly, annuity carriers have to pay more annuities and face the danger to close their companies. A substantial reduction in mortality rates in the 1970s can be observed. The report delivered by Andreev and Vaupel (2005) indicates that the rapid mortality decline occurring in the 1970s took place among both genders at a rate 1.5-2% per annum or higher. Scholars have some guesses and assumptions to explain the dramatic decline. According to David N. Wylde (2012), a medical research group concluded that the introduction of coronary care and heart-related drug regimens introduced in the 1970s was likely to contribute to the significant decrease in cardiovascular mortality. It is reasonable to infer that, with today's rapid technological and medical improvements, there are some possibilities that some cancers could be cured. Therefore, longevity risks can create severe solvency issues to pension plans and insurance companies because they need to pay annuities for a longer period.

Mortality risks, either the pure mortality risks or the longevity mortality risks, are one of the risks that insurance companies face typically. In a financial dimension, risks can be categorized as systematic risks and unsystematic risks. Systematic risks can also be referred as market risks. It refers to events that will influence the whole market. The stock market crash in 2008 is a good example of a systematic risk. Most companies suffered a huge loss from the crash. Some leading financial companies even announced their bankruptcy, such as American

[2] Source: http://www.flu.gov/pandemic/history/.

International Group (AIG). Similarly, mortality risks can also be categorized into two types, systematic mortality risk and unsystematic mortality risk. According to Dahl, Mechior and Moller (2008), systematic mortality risks refer to the risks associated with changes in the underlying mortality intensity. A big earthquake and tsunami with thousands of deaths are examples of systematic mortality risks. On the contrary, unsystematic mortality risks, explained by Dahl, Mechior and Moller (2008), refer to the risks associated with the randomness of deaths in an insurance portfolio with known mortality rates. A natural death of a person or a traffic accident can be examples of unsystematic mortality risks. Generally, actuaries pay more attention to systematic mortality risks because systematic mortality risks cannot be diversified. Such risks cannot be eliminated even within a large group of people. Therefore, actuaries try to use different ways to hedge systematic mortality risks.

Mortality securitization is a tool to hedge mortality risks, both pure mortality risks and longevity risks. Securities, also known as financial instruments, are broadly categorized as debt securities (such as bonds), equity securities (such as common stocks) and derivative contracts (such as forward, futures, options and swaps). Similarly, mortality-linked securities include pure mortality bonds, longevity bonds, mortality swaps, mortality futures and mortality options. Here, we will discuss two particular bonds, the Swiss Re mortality bond and the European Investment Bank (EIB) bond. The three-year Swiss Re bond issued in December 2003 was the first pure death-risk linked deal. It is a hedge against catastrophic losses of insured lives that might result from natural or man-made disasters in the US or Europe. In 2004, EIB issued the first pure longevity-risk linked deal, a 25-year 540 million pound bond. The EIB bond is a security to transfer longevity risks to bondholders.

This paper intends to apply the stochastic mortality model introduced by Lin and Cox (2008) and the Wang transform (2002) to the pure mortality bond. It is a tradition that actuaries use deterministic mortality intensity, a function of age only, and a constant interest rate to determine premiums and reserves. However, life insurance companies are exposed to systematic mortality risks in a long term. A stochastic process helps to capture two features: time dependency and uncertainty of the future development (Dahl, 2004). Therefore, a stochastic process is able to show the path of unsystematic and systematic mortality risks during different time intervals.

The paper proceeds as follows. Chapter 2 introduces the designs of the Swiss Re bond and EIB bond, Chapter 3 introduces stochastic processes, some commonly-used mortality models and the Wang transform, Chapter 4 uses the maximum likelihood estimation and the Wang transform to price the Swiss Re bond under the Lin and Cox model, Chapter 5 applies the same approach to price a pure mortality bond designed to resemble the Swiss Re bond, and Chapter 6 concludes.

Chapter 2

Mortality-Linked Securities

The most common way for insurance companies to hedge mortality risks is through reinsurance. The reinsurer who enters into a reinsurance agreement pays a share of claims, which reduces risks insurance companies may suffer. The advantage of the reinsurance is that it can cover both the unsystematic and systematic mortality risks. However, many life insurance companies are hesitant to purchase reinsurance contracts because of the high risk of reinsurance business. Through purchasing reinsurance contracts, insurance companies still face undiversified risks because risks are not eliminated, which could lead to unexpectedly high losses.

In order to hedge the undiversified mortality risks, people begin to consider securitizations. Blake and Burrow (2001) are the first to consider securitization as a tool to hedge mortality risks. They argue that the decrease in mortality had been underestimated. In order to hedge those aggregate mortality risks, they introduce a new type of bond named survivor bond, which is a longevity bond we mentioned in the introduction chapter. Securitizations enjoy advantages over reinsurance contracts because securitizations cover systematic mortality risks by trading securities in the financial market depending on the general mortality change (Dahl, 2004). Further, securitizations are able to bring more capital and provide innovative contracting features. It also diversifies the market participants because mortality securities are uncorrelated with financial market (Lin & Cox, 2008).

2.1 Mortality Securitization Markets

First, let us focus on a big picture, the insurance securitization market. The insurance securitization market surges with the convergence between the insurance industry and capital markets. The insurance securitization market has grown rapidly. According to Lane and Beckwith (2005 & 2007), $1.9 billion of insurance-linked securities (ILS) were issued in 2004. The amount increased to $5.6 billion in 2007.

There are several types of ILS available in the market. Two important segments are catastrophe bonds (cat bonds) and life insurance securitizations. Cat bonds hedge catastrophic risks, such as hurricanes and earthquakes. They are issued by property and casualty insurers or reinsurers. Life insurance securitizations include mortality and longevity risk securitizations as discussed in the introduction chapter. In terms of the market share of the two segments, cat bonds play a major role in the market. According to National Association of Insurance Commissioners (2014), at the end of 2011, life insurers held $379 million in ILS, of which 80% were catastrophic risk deals and 20% were mortality risk deals. From the view of the number of securitization issued, we can see an increasing expansion of the market. From April 2004 to March 2005, there were only 15 catastrophe issues and 6 mortality based securities. From April 2006 to March 2007 the issues increased to 61 and 13, respectively. Two important events we will focus are the issuance of the Swiss Re mortality bond and the European Investment Bank (EIB) bond. In December 2003, Swiss Re issued the first pure death-linked deal. In November 2004, the EIB introduced its own longevity bond. The mechanism of these two deals is discussed in the next two sections.

2.2 Swiss Re Mortality Bond

In December 2003, Swiss Re, the world's second largest reinsurance company, issued a three-year pure mortality bond that matured on January 1, 2007. It helps Swiss Re hedge against catastrophic events. The bond is issued via a special purpose entity (SPE; or special purpose vehicle/SPV) called Vita Capital, which is a legal entity used by companies to isolate the firm from financial risk. Blake, Cairns and Dowd (2006) use the following figure to clarify the structure of the Swiss Re bond:

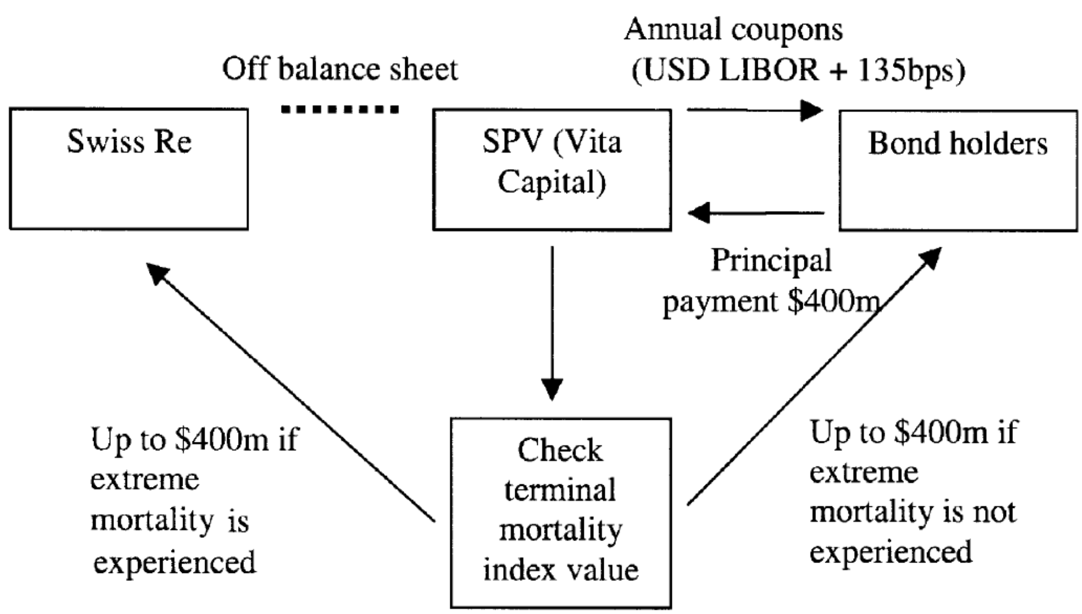

Figure 2: The structure of the Swiss Re mortality bond (Blake, Cairns and Dowd, 2006).

From the above figure, we can find that the total face amount of the Swiss Re mortality bond is $400 million. For investors, they receive quarterly coupons and principal at maturity. According to Blake, Cairns and Dowd (2006), the quarterly coupons are set at three-month US Dollar LIBOR + 135 basis points. LIBOR stands for London Interbank Offered Rate, which is the average interest rate estimated by leading banks in London. We provide an example to clarify the coupon rates. Suppose that at the coupon reset date the three-month LIBOR is 1.3%. Then the quarterly coupon rates become LIBOR plus the basis points, which is $1.3\% + 1.35\% = 2.65\%$.

However, the principal depends on the actual mortality rates q_t ($t = 2004, 2005$ or 2006) and is not guaranteed to be paid back. Specifically, q_t refers to the mortality rates in year t in five countries: the US, UK, France, Italy and Switzerland. It is a weighted average mortality rate, 70% of US, 15% of UK, 7.5% of France, 5% of Italy and 2.5% of Switzerland. Investors will receive a full principal if q_t is less than 130% of the actual 2002 level, denoted as q_0. If q_t exceeds $1.3q_0$, investors will receive reduced principal or even receive nothing at all. The principal loss percentage in year t, $loss_t$, is showed in the following equations:

$$loss_t = \begin{cases} 0 & if\ q_t \leq 1.3q_0 \\ 1 - \dfrac{1.5q_0 - q_t}{0.2q_0} & if\ 1.3q_0 < q_t < 1.5q_0 \\ 1 & if\ q_t \geq 1.5q_0 \end{cases} \tag{2.2-1}$$

Then the payment at maturity is equal to

$$400{,}000{,}000 \times \begin{cases} 100\% - \displaystyle\sum_{t=2004}^{2006} loss_t & if \displaystyle\sum_{t=2004}^{2006} loss_t < 100\% \\ 0 & if \displaystyle\sum_{t=2004}^{2006} loss_t \geq 100\% \end{cases} \tag{2.2-2}$$

The probability of having two catastrophe events in three years is extremely low and we can neglect it. So the payment at maturity becomes

$$400{,}000{,}000 \times \begin{cases} 1 & if\ q \leq 1.3q_0 \\ \dfrac{1.5q_0 - q}{0.2q_0} & if\ 1.3q_0 < q < 1.5q_0 \\ 0 & if\ q \geq 1.5q_0 \end{cases} \tag{2.2-3}$$

where $q = \max(q_{2004}, q_{2005}, q_{2006})$.

2.3 EIB Longevity Bond

In November 2004, the European Investment Bank (EIB) issued a longevity bond targeted at pension plans and other annuity carriers. The bond was not as popular as the Swiss Re bond and it was withdrawn in late 2005. However, it is still beneficial to investigate the design of the longevity bond for future reference.

The value of the EIB bond is £540 million and it is 25-year long. Investors receive annual coupon payments depending on the survival rates of a specific group of people, who are 65-year old English and British males in 2002. The initial coupon paid by EIB was set at £50 million. The future coupon payments equal to £50 million multiplied by the corresponding survival rate of the reference people. Suppose that $t = 0$ on December 31, 2004, $t = 1$ on December 31, 2005, etc. Then when $t = 1$, the survival rate is p_{65}, which is the probability of the reference people who can survive at least one year. When $t = 2$, the survival rate is ${}_2p_{65}$, which is the probability of the reference population who can survive at least two years. In general, the survival rate is ${}_tp_{65}, t = 1,2, \ldots, 25$. The survival rate is calculated from the crude death rate published by the Office for National Statistics. Let q_x be the published death rate of the reference population aged x who are unable to survive one year. Then ${}_tp_{65} = (1 - q_{65}) \times (1 - q_{66}) \times \ldots \times (1 - q_{65+t-1})$. Generally, we can calculate the coupon payment at time t as

$$P(t) = 50{,}000{,}000 \times \prod_{1}^{t} (1 - q_{65+t-1}), t = 1,2, \ldots, 25 \qquad (2.3\text{-}1)$$

Blake, Cairns and Dowd (2006) construct a simple diagram to show the cash flows from the EIB bond, which is shown as follows:

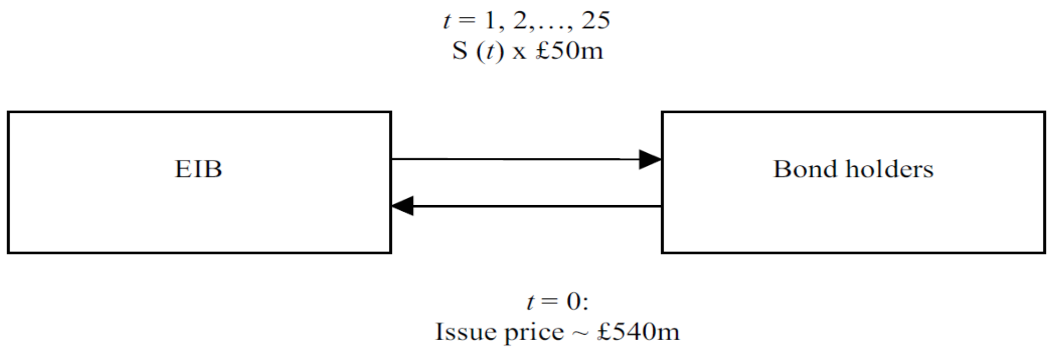

Figure 3: The cash flows from the EIB bond (Blake, Cairns and Dowd, 2006).

Chapter 3

Stochastic Modeling and the Wang Transform

In the chapter 2, we have a brief introduction about the mechanism of mortality-linked securities, especially the pure mortality bond and longevity bond. In this chapter, we are going to investigate the methods to price those securities.

3.1 Deterministic Modeling

It is a tradition that actuaries use the deterministic mortality intensity, which is a function of age only, and a constant interest rate to determine premiums. Basically, we use the Equivalence Principle to determine premiums. The principle requires that the expected loss of an insurance policy to the insurer at the issuance should be equal to zero. That is,

$$E[loss] = 0$$

The insurers receive premiums and pay benefits. The differences between them are the loss. According to the Equivalence Principle,

$$E[present\ value\ of\ benefits] = E[present\ value\ of\ premiums]$$

Here is a simple example to demonstrate how to use the Equivalence Principle to determine premiums. Let us work on fully discrete net annual premiums where both benefits and premiums are paid discretely. Benefits are paid at the end of the year when benefits are claimed and premiums are paid at the beginning of the policy year. In this case, we do not consider other expenses in order to make the calculation simpler. According to the Equivalence Principle, we have

$$P_x * \ddot{a}_x = A_x$$

where $\ddot{a}_x$ is the whole life annuity due with annual payments at beginning of years and A_x is the actuarial present value of discrete whole life insurance. Hence, we can get the premium

$$P_x = \frac{A_x}{\ddot{a}_x}$$

where

$$A_x = \sum_{K_x=0}^{\omega-x-1} v^{k+1} \cdot {}_{k|}q_x$$

$$\ddot{a}_x = \sum_{K_x=0}^{\omega-x-1} v^k$$

From the equations provided above, we can find that in the traditional way the premium is determined by the interest rate and mortality rates from the mortality table.

There are some deterministic mortality models, such as the Gompertz's law and the Makeham's law. Under the Makeham's law, the mortality μ_x can be expressed as $\mu_x = A + Bc^x$, where $A, B,$ and c are constants. The Gompertz's law is a special case of the Makeham's law. Under the Gompertz's law, the constant A in the Makeham's law equals to zero, and hence, $\mu_x = Bc^x$.

With deterministic models, extreme events are not taken into consideration. Huge losses of insurance companies could occur if there are systematic mortality risks in the future. So we need to find a more reliable way to estimate the future mortality.

3.2 Stochastic Modeling

The stochastic modeling is the opposite of the deterministic modeling. Unlike the deterministic modeling where the mortality rates are a specific set of values, stochastic modeling considers randomness and predictability when presenting or predicting data. Because of the specific property, stochastic processes are becoming important in the insurance industry. With stochastic processes, predictions become more reliable because most insurance risks evolve as time passes and with stochastic processes we can clearly see the path of those risks during different time intervals.

A stochastic process is defined as a collection of random variables indexed by time. $\{X_n\}_{n=0}^{\infty}$ represents a discrete time stochastic process, and $\{X_t\}_{t\geq 0}$ is a continuous time stochastic process. In this paper, a discrete time stochastic process is used because we are interested in annual death rates. A simple example of stochastic processes is flipping a fair coin and counting the number of heads in the first n flips. In other words, X_n is the number of heads in the first n flips, n could be 0, 1, 2, etc. Another example is the Markov model. A Markov model is a stochastic model which follows the Markov property. The Markov property indicates that the conditional property of any future event depends on the current state of the system only. Numerically,

$$\Pr\{X_{t+1} = j | X_t = i_t, X_{t-1} = i_{t-1}, \dots, X_0 = i_0\} = \Pr\{X_{t+1} = j | X_t = i_t\}$$

where $t = 0, 1, \dots$.

Stochastic processes are utilized by some scholars and they have developed some stochastic mortality models which aim to improve the accuracy of the prediction for the mortality. The Lin and Cox (2008) model, which is introduced in the following sections, is a special one of the Markov model. Other existing stochastic mortality models are the Lee-Carter model, the Cairns, Blake and Dowd model and so on. Some models are introduced in the following sections.

3.3 The Lee-Carter Model

Lee and Carter (1992) uses time series methods to estimate age-specific mortality in the United States from 1990 to 2065. They propose the logs of the age-specific death rates model with unknown index and parameters. In their model, $m(x,t)$ is the death rate for age x in year t. Then the death rate $m(x,t)$ can be modeled by the following equation:

$$\ln\big(m(x,t)\big) = a_x + b_x k_t + \varepsilon_{x,t}$$

or

$$m(x,t) = e^{a_x + b_x k_t + \varepsilon_{x,t}}$$

where $\{a_x\}$ and $\{b_x\}$ are sets of age-specific constants, k_t is the time-varying index, and $\varepsilon_{x,t}$ is the error term with mean 0 and variance σ_ε^2. More specifically, the set of $\{a_x, x = 0,1,2,\ldots\}$ reflects the general shape of the mortality schedule. $\{b_x\}$ indicates the sensitivity of the death rates to k_t at age x. You can refer Lee and Carter (1992) to find more information on getting parameter values in the Lee and Carter model.

The model is used to estimate long-run age-specific mortality rates in the U.S. from 1990 to 2065. Based on the U.S. death rates from 1933 to 1987, Lee and Carter conclude that the performance of the model is insensitive to the decrease in the length of the base period from 90 to 30 years. Further, they estimate that the life expectancy increases to 86.05 years in 2065 and 46% of the U.S. population will survive to age 90.

One advantage of the Lee-Carter model is that it is parsimonious and the number of parameters is small. However, the model is constrained to reflect extreme events, which defeats the original intention to use the stochastic processes. Therefore, we need a more complicated model to reflect the extreme events.

3.4 Lin and Cox Model

Compared with the Lee-Carter Model, Lin and Cox model (2008) is more appropriate since Lin and Cox use an additional random variable Y_t to estimate the performance of extreme events. In this way, we can successful at hedging systematic mortality risks. They use a Brownian motion and a discrete Markov Chain with log-normal jump size distribution. In their model, the mortality rate, $q_{t+h}$is modeled by the following equation:

$$\tilde{q}_{t+h}|\mathcal{F}_t = q_t \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)h + \sigma(W_{t+h} - W_t)\right]Y_{t+h} \tag{3.4-1}$$

In order to understand what those parameters stand for, we need to learn some basic concepts. Brownian motion is a special case of stochastic process. If $\{B_t, t \geq 0\}$ is a Brownian motion, an important property of the process is that for every $t > 0$, B_t is normally distributed with mean 0 and variance $\sigma^2 t$. When $\sigma = 1$, it is called the standard Brownian motion. If $\{W_t, t \geq 0\}$ is a standard Brownian motion, then $\{Y_t, t \geq 0\}$ is called a Brownian motion process with drift coefficient μ and variance parameter σ^2 if $Y_t = \mu t + \sigma W_t$. This implies that Y_t is normally distributed with mean μt and variance $t\sigma^2$. We say $\{X_t, t \geq 0\}$ is a geometric Brownian motion if $dX_t = \mu X_t dt + \sigma X_t dW_t$. The solution to this stochastic differential equation is $X_t = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$, where X_0 is the initial value of X_t.

Looking back at the model, when considering without extreme events, the mortality index q_t follows a geometric Brownian motion, W_t is a standard Brownian motion with mean 0 and variance t, α is the expected force of the US population mortality index, and σ is the volatility of the mortality index. Conditioning on no extreme events, we can have

$$q_t = q_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t} \tag{3.4-2}$$

Generally, if there is no extreme event in $(t - h, t)$,

$$q_t = q_{t-h}e^{\left(\alpha-\frac{\sigma^2}{2}\right)h+\sigma(W_t-W_{t-h})} \quad (3.4\text{-}3)$$

Now when considering extreme events, we need a new random variable Y_t. Let us assume that the probability of having an extreme event is p. Y_t is defined by the following equations:

$$Y_t = \begin{cases} e^{m+sU_t} & with\ probability\ of\ p \\ 1 & with\ probability\ of\ 1-p \end{cases} \quad (3.4\text{-}4)$$

where U_t is a standard normal random variable. We can find that when there is an extreme event, Y_t is log-normally distributed with parameters m and s.

The U.S. population mortality index $\tilde{q}_t$ becomes q_tY_t, which is equation 3.4-1. In the next chapters, we will show how to derive those parameters in the equation and also how to apply those results to the pricing of the mortality-linked securities.

Before showing how to get the values of those parameters and the distribution of mortality, we need to find an appropriate way to price the bonds. In Lin and Cox (2008), they use the two-factor Wang transform to build the pricing model.

3.5 The Wang Transform

Proposed by Wang (2000), the Wang transform (2000) can price financial and insurance products in a universal framework. The one-factor Wang transform of a distribution function F is defined as follows:

$$F^*(x) = \Phi\left[\Phi^{-1}\left(F(x)\right) + \lambda\right] \quad (3.5\text{-}1)$$

where Φ follows the standard normal cumulative distribution and λ is the market price of risk. The market price of risk measures the level of systematic risk.

For an asset X with cumulative distribution function (cdf) $F(x)$, by the application of the one-factor transformation, we can get a risk-adjusted cdf, $F^*(x)$. Then the price of the asset X equals to the discounted expected value of X under the transformed distribution,

$$v^t E^*(X) = v^t \int x dF^*(x) \tag{3.5-2}$$

where v^t is the discount factor for risk-free bonds at time 0.

Since it is unrealistic to assume that the true underlying distribution $F(x)$ is known, Wang (2002) makes adjustments of the one-factor transformation to accommodate model uncertainty. Suppose that we have observations from a population with a normal distribution with the mean μ and variance σ^2. The mean and variance cannot be observed from the population. We can use the sample mean and sample variance to estimate their values. Therefore, a Student's t-distribution, which is generally used when the underlying distribution is unknown, takes place of a standard normal distribution. It is rare to know the underlying distribution of observations in real life applications. So after the modification, the transformation is as follows:

$$F^*(x) = \mathrm{Q}\left[\Phi^{-1}\big(F(x)\big) + \lambda\right] \tag{3.5-3}$$

where Q follows a Student's t-distribution with k degrees of freedom. This is also called the two-factor the Wang transform.

In order to determine the degrees of freedom, Wang (2004) uses the historical data of corporate bonds, the default frequencies and probabilities, and tries different combinations of λ and k to find the best fit. At six degrees of freedom, Wang gets the best fit. We follow Wang (2004) and Lin and Cox (2008) to use six degrees of freedom.

In the following chapter, we are going to investigate how to use the Lin and Cox model and the Wang transform to evaluate the pure mortality bond price. In this paper, we focus on the pricing of Swiss Re bond, which is a three-year bond issued in December 2003. We also try to find a way to evaluate the price of a three-year pure mortality bond which is issued in January 2014.

Chapter 4

Pricing of Swiss Re Bond

In the last sections, we introduced Lin and Cox mortality model, the basis of pricing. We also have a brief understanding of the pricing model, the Wang transform. In the following sections, we try to use historical data to solve all the parameters in Lin and Cox model and apply the Wang transform to evaluate the price of the Swiss Re bond.

4.1 Maximum Likelihood Estimation

In this section, we use the historical U.S. mortality data from 1900-1998 to estimate the parameters in Lin and Cox model. Recall the stochastic mortality model, given by equations 3.4-1 and 3.4-4,

$$\tilde{q}_{t+h}|\mathcal{F}_t = q_t \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)h + \sigma(W_{t+h} - W_t)\right]Y_{t+h}$$

where

$$Y_t = \begin{cases} e^{m+sU_t} & with\ probability\ of\ p \\ 1 & with\ probability\ of\ 1-p \end{cases}$$

Thus, we have five parameters to be estimated, $\sigma, \alpha, p, m,$ and s. The maximum likelihood estimation (MLE) is a common way to estimate parameters in a statistical model. With MLE, we can get estimators that have the best chance to produce the data.

Suppose $X_i, i = 1,2, \ldots n$ are independent and identically distributed (iid) random variables, which all follow a distribution that depends on one or more unknown parameters,

$\theta_1, \theta_2, \dots, \theta_j$, with probability mass function (pmf) or probability density function (pdf), denoted by $f(x_i; \theta_1, \theta_2, \dots, \theta_j)$. Then the joint pmf or pdf of X_i is

$$L(\theta_1, \theta_2, \dots, \theta_j) = f(x_1; \theta_1, \theta_2, \dots, \theta_j) f(x_2; \theta_1, \theta_2, \dots, \theta_j) \dots f(x_n; \theta_1, \theta_2, \dots, \theta_j)$$

which is called the likelihood function.

Next, we want to find out the values of $\theta_1, \theta_2, \dots, \theta_j$ at which the log-likelihood function is maximized. For this purpose, we set the first derivative of the log-likelihood function to zero, and solve for $\theta_i, i = 1,2, \dots j$. The obtained estimators are called maximum likelihood estimators.

We need a different treatment for our model due to its complexity. First, take the logarithm of equation 3.4-1, and we have

$$log\tilde{q}_{t+h} = logq_t + \left(\alpha - \frac{1}{2}\sigma^2\right)h + \sigma(W_{t+h} - W_t) + logY_{t+h} \qquad (4.1\text{-}1)$$

Let $Z_t = \nabla log\tilde{q}_{t+h}$. Then we have

$$Z_t = log\tilde{q}_{t+h} - log\tilde{q}_t \qquad (4.1\text{-}2)$$

Since $log\tilde{q}_t = logq_t Y_t = logq_t + logY_t$,

$$\begin{aligned} Z_t &= log\tilde{q}_{t+h} - logq_t - logY_t \\ &= \left(\alpha - \frac{1}{2}\sigma^2\right)h + \sigma(W_{t+h} - W_t) + logY_{t+h} - logY_t \end{aligned} \qquad (4.1\text{-}3)$$

The conditional distribution of Z_t is $Z_t|\mathcal{F}_t$, where $\mathcal{F}_t$ denotes the information up until time t. Therefore, there are a total of four cases as listed:

- no extreme event in $[t-h, t]$, and no extreme event in $[t, t+h]$, which has probability $(1-p)^2$
- no extreme event in $[t-h, t]$, and extreme event occurs in $[t, t+h]$, which has probability $(1-p)p$
- extreme event occurs in $[t-h, t]$, and no extreme event in $[t, t+h]$, which has probability $p(1-p)$
- extreme event occurs in both $[t-h, t]$ and $[t, t+h]$, which has probability p^2

Now we need to determine the distribution of $Z_t|\mathcal{F}_t$ based on these four cases. As an example we show the calculations for Case I. In Case I there is no extreme event in the interval

$[t-h, t]$ and $[t, t+h]$. In other words, both Y_t and Y_{t+h} equal 1 with probability $(1-p)^2$. Then we have $Z_t = \left(\alpha - \frac{1}{2}\sigma^2\right)h + \sigma(W_{t+h} - W_t)$. We obtain Z_t follows a normal distribution with

$$E(Z_t|Case\ \mathrm{I}) = E\left[\left(\alpha - \frac{1}{2}\sigma^2\right)h + \sigma(W_{t+h} - W_t)\right]$$

$$= \left(\alpha - \frac{1}{2}\sigma^2\right)h$$

$$Var(Z_t|Case\ \mathrm{I}) = Var\left[\left(\alpha - \frac{1}{2}\sigma^2\right)h + \sigma(W_{t+h} - W_t)\right]$$

$$= \sigma^2 h$$

Considering the rest of the three situations, we conclude that

Table 1: Mean and variance of Z_t

Extreme Events	$E(Z_t\|\mathcal{F}_t)$	$Var(Z_t\|\mathcal{F}_t)$	*Probability*
N, N	$\left(\alpha - \frac{1}{2}\sigma^2\right)h$	$\sigma^2 h$	$(1-p)^2$
Y, N	$\left(\alpha - \frac{1}{2}\sigma^2\right)h - m$	$\sigma^2 h + s^2$	$p(1-p)$
N, Y	$\left(\alpha - \frac{1}{2}\sigma^2\right)h + m$	$\sigma^2 h + s^2$	$(1-p)p$
Y, Y	$\left(\alpha - \frac{1}{2}\sigma^2\right)h$	$\sigma^2 h + 2s^2$	p^2

We are dealing with the annual mortality rate, and hence $h = 1$. Since we know that $Z_t|\mathcal{F}_t$ follows a normal distribution, and we have the mean and variance of $Z_t|\mathcal{F}_t$, Lin and Cox (2008) shows that we can obtain the pdf of Z_t in terms of the conditional density function of $Z_t|\mathcal{F}_t$ as follows:

$$f_Z(z_t) = \left(\frac{1}{S_{nn}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z_t - M_{nn}}{S_{nn}}\right)^2}\right)(1-p)^2$$
$$+\left(\frac{1}{S_{yn}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z_t - M_{yn}}{S_{yn}}\right)^2}\right)p(1-p)$$
$$+\left(\frac{1}{S_{ny}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z_t - M_{ny}}{S_{ny}}\right)^2}\right)(1-p)p \qquad (4.1\text{-}4)$$
$$+\left(\frac{1}{S_{yy}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z_t - M_{yy}}{S_{yy}}\right)^2}\right)p^2$$

If we have K observations of historical mortality data, we have $K-1$ observations for Z_t. The likelihood function is

$$L = \prod_{t=1}^{K-1} f_Z(z_t)$$

Taking the logarithm of the likelihood function, we have

$$logL = log\prod_{t=1}^{K-1} f_Z(z_t)$$

$$= \sum_{t=1}^{K-1} \log\left\{\left(\frac{1}{S_{nn}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z_t-M_{nn}}{S_{nn}}\right)^2}\right)(1-p)^2\right.$$

$$+\left(\frac{1}{S_{yn}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z_t-M_{yn}}{S_{yn}}\right)^2}\right)p(1-p)$$

$$+\left(\frac{1}{S_{ny}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z_t-M_{ny}}{S_{ny}}\right)^2}\right)(1-p)p$$

$$\left.+\left(\frac{1}{S_{yy}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z_t-}{S_{yy}}\right)^2}\right)p^2\right\}$$

$$= \sum_{t=1}^{K-1} \log\left\{\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z_t-\left(\alpha-\frac{1}{2}\sigma^2\right)}{\sigma}\right)^2}\right)(1-p)^2\right. \qquad (4.1\text{-}5)$$

$$+\left(\frac{1}{\sqrt{(\sigma^2+s^2)2\pi}}e^{-\frac{1}{2}\left(\frac{z_t-\left(\alpha-\frac{1}{2}\sigma^2-m\right)}{\sqrt{\sigma^2+s^2}}\right)^2}\right)p(1-p)$$

$$+\left(\frac{1}{\sqrt{(\sigma^2+s^2)2\pi}}e^{-\frac{1}{2}\left(\frac{z_t-\left(\alpha-\frac{1}{2}\sigma^2+m\right)}{\sqrt{\sigma^2+s^2}}\right)^2}\right)(1-p)p$$

$$\left.+\left(\frac{1}{\sqrt{(\sigma^2+2s^2)2\pi}}e^{-\frac{1}{2}\left(\frac{z_t-\left(\alpha-\frac{1}{2}\sigma^2\right)}{\sqrt{\sigma^2+2s^2}}\right)^2}\right)p^2\right\}$$

We use the statistical software R to maximize the log-likelihood function and solve for those estimators.

Lin and Cox (2008) use the data from the Vital Statistical of the United States (VSUS). The VSUS reports the U.S. age-adjusted death rate for selected causes by race and sex. The reason that we choose the age-adjusted death rate in place of the crude death rate is as follows: the crude death rates show the absolute risk of death in a population; age adjustment changes the amount that each age group contributes to the overall rate in each community. The age adjustment is accomplished when different weighted rates are applied to different age groups. Since the risks of deaths are closely related to the age, age-adjusted death rates reflect the true risk of deaths among different groups of population.

We collect the published U.S. age-adjusted death rates from 1900-1998. Also, we use the equation $Z_t = log\tilde{q}_{t+h} - log\tilde{q}_t$ to transform the data. Those data are available in the Appendix A.

So far, we have our data prepared. The next step is to find the best function in R to estimate the parameters we need. The built-in functions non-linear minimization (nlm) and linear constrained optimization (constrOptim) are used to derive the estimated values of parameters. These two functions both have strengths and weaknesses.

The basic format of the function nlm is $nlm(f, p, \ldots)$ where we need to specify the function to be minimized, f, and the starting parameter values for the minimization, p. If necessary, we can also specify the range of step length by stepmax and steptol option. The function nlm not only offers us the estimated values of those parameters, but also provides the gradient at the estimated minimum of f. The gradient is significant when we evaluate the accuracy of the estimation. A gradient which is close to zero indicates that the estimation is probably the solution. We have obtained the log likelihood function from the previous calculation. The starting parameter is critical in the nlm function. First, we need to consider whether the starting values are appropriate. In our function, σ cannot be zero since it is in the denominator. Also, you can change the values of the starting parameter by observing the values of gradient. Since we are dealing with the death rates, we believe those parameters tend to be small. By specifying the length step, we are able to get estimations more accurately. After trying several times in order to determine the appropriate starting values, stepmax, and steptol values, we have the following result:

```
$minimum
[1] -189.8882

$estimate
[1]  0.031004855 -0.009598911  0.011487760  0.040393647  0.149182796

$gradient
[1] -7.344170e-05  1.430180e-04  3.089440e-05 -1.460876e-05  4.121148e-
06

$code
[1] 1

$iterations
[1] 37
```

We see that the gradients are very close to zeroes, which indicate that the estimated values are reasonable. In a summary, the estimated values are as follows:

Table 2: Estimated values of parameters in the stochastic mortality model

Parameters	α	σ	p	m	s
Estimated Values	-0.009598911	0.031004855	0.011487760	0.040393647	0.149182796

We should notice that the value of α is negative. This is because the mortality is improved year by year generally. p is relatively small because the probability of having an extreme event is small.

We mentioned that the function constrOptim in R can give us the same result. One advantage of this function is that we can define constraints for those parameters. In our model, p represents the probability of having extreme events, which should be between zero and 1. With the nlm function, we fail to limit the value of p. With the constrOptim function, you can use a matrix to define the constraints. As a result, the constrOptim function gives us less warning messages than the nlm function does. However, both functions are sensitive with respect to the starting value. We still need to try several times to find a proper combination of starting parameter values.

4.2 Application of the Wang Transform

Recall equation 3.5-2, which is the following pricing model,

$$v^t E^*(X) = v^t \int x dF^*(x)$$

where X is the random variable of the payment and v^t is the price of zero coupon bond. As the equation 2.2-3 shows, the payment function of the Swiss Re bond is

$$400{,}000{,}000 \times \begin{cases} 1 & if\ q \leq 1.3q_0 \\ \dfrac{1.5q_0 - q}{0.2q_0} & if\ 1.3q_0 < q < 1.5q_0 \\ 0 & if\ q \geq 1.5q_0 \end{cases}$$

where q_0 is known as the 2002 weighted average mortality rate in five developed countries and $q = \max(q_{2004}, q_{2005}, q_{2006})$. Apparently, the payment function is a function of mortality $q_t, t = 2004, 2005, 2006$. Hence, in order to find the expected value of the payment, we need to find the distribution of the mortality.

With the MLE estimations, we have a complete formula to derive q_t. The mortality rate without extreme events, and $\tilde{q}_{t+h}$, the mortality rate with extreme events are

$$q_t = q_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

$$\tilde{q}_{t+h} | F_t = q_t \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)h + \sigma(W_{t+h} - W_t)\right] Y_{t+h}$$

which are shown in the section 3.4. In our case, h is equal to 1 because we use the annual mortality rate and $t = 1,2,3$ ...

In the software R, we are able to simulate values of q_t and $\tilde{q}_{t+h}$. Since W_t follows a normal distribution with mean 0 and variance t, we can simulate the values of W_t with the command $rnorm(n, mean, sd)$.We can specify the number of simulations we need, the mean and standard deviation of the normal distribution. R will give us a vector of n terms. Applying the vector to the formula for q_t, we get n simulated values of $q_t, t = 1,2,3,4$, which corresponds to the mortality in year 2003, 2004, 2005, and 2006. Similarly, we can get simulations for $\tilde{q}_{t+1}, t = 1,2,3$. We combine $\tilde{q}_2, \tilde{q}_3, \tilde{q}_4$ as a matrix and use the function $apply(mat, 1, max)$ to show the result of q, which is the maximum of $\tilde{q}_2, \tilde{q}_3, \tilde{q}_4$. So far, we have 10,000 values of q. We need to find the distribution of q. Empirical distribution is an important way when we do not know the underlying distribution of the random variable. The CDF of empirical distribution is

$$F_n(t) = \frac{number\ of\ elements\ in\ the\ sample\ \leq\ t}{n}$$

With 10,000 simulations run, the CDF plot of q is shown below, which is the red line in the figure.

The next step is to apply the result to the Wang transform, equation 3.5-3, to fully evaluate the risk,

$$F^*(x) = \mathrm{Q}\left[\Phi^{-1}\left(F(x)\right) + \lambda\right]$$

where Φ follows the standard normal distribution and Q follows the t distribution with six degrees of freedom. In Lin and Cox (2008), the market price of risk λ_{SR} is evaluated as -1.3603. Since we do not have the data of Swiss Re bond price, we will use the $\lambda_{SR} = -1.3603$ for the following calculation. With R, we can work it out by defining a function, which is named $Fstar$. The result of $Fstar$ can be obtained by the command, $pt(qnorm(Fq(x)) + 1.3603{,}6)$, where $Fq(x)$ is the original empirical CDF, $qnorm$ is the corresponding quantile for a particular probability under the normal distribution, pt is the cumulative probability under a student t distribution, and the number 6 in the pt function represents the degree of freedom. Hence, we have the transformed CDF function of the mortality, which is plotted in the figure 4 as the green line.

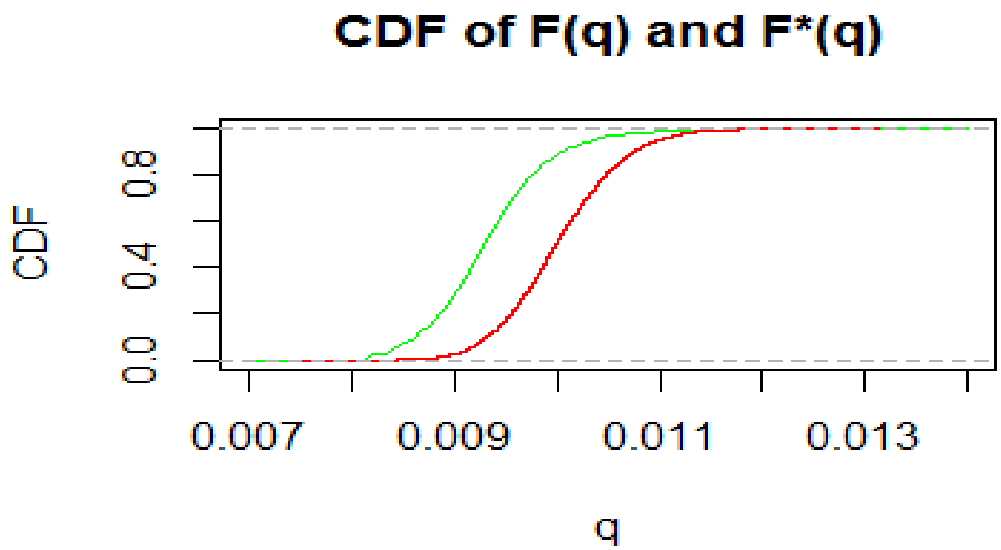

Figure 4: CDF plot of $F(q)$ and $F^{*}(q)$ for the Swiss Re bond

The transformed CDF shifts to the left, which indicates that either investors have underestimated the morality rates or they are risk prone to the mortality rates.

In order to find the price of the bond, we need to find the expected value of the payment with mortality rates following $Fstar$. Since we have the distribution of the mortality, we are close to getting the result. An intuitive way is to define a payment function in R. In the function, we compare each transformed q with $1.3q_0$ and 1.5 q_0 using if statement. If q is less than $1.3q_0$, the payment is 1. If q is greater than $1.5q_0$, the payment is 0. Otherwise, the payment is $\frac{1.5q_0-q}{0.2q_0}$. With our simulated result, the expected value of the payment is 0.9627. Since the payment is received at the end of the third year after the issue of the bond, the final step of the pricing is to discount the expected value to get the present value of the bond. We use $i = 1.12\%$, which is the interest rate of the 3-year U.S. Treasury Zeros bond.[3]

In conclusion, the price of a Swiss Re mortality bond is $0.9311 for $1 face value. In the next chapter, we will use the same method to evaluate the price for a similarly designed three-year pure mortality bond which was issued in January, 2014.

[3] Source: https://fixedincome.fidelity.com.

Chapter 5

Pricing of Pure Mortality Bonds

Assume a three-year pure mortality bond issued by Firm A on January 1, 2014 is currently in the market. We call it Firm A bond. Investors will get their payment on January 1, 2017. The design of the bond is similar to the Swiss Re bond. The payment of the bond with a face value of $1 is

$$\begin{cases} 1 & if\ q \le 1.3q_0 \\ \dfrac{1.5q_0 - q}{0.2q_0} & if\ 1.3q_0 < q < 1.5q_0 \\ 0 & if\ q \ge 1.5q_0 \end{cases}$$

where q_0 is 2011 U.S. mortality index, and q is the maximum death rate with extreme events during three years 2014, 2015, and 2016. We are going to investigate at which price potential investors will be attracted to purchase the Firm A bond. In order to find the proper price of the particular bond, we need to employ Lin and Cox model and the Wang transform. The first step we need to take is to estimate the parameters in Lin and Cox model.

Currently, the U.S. age-adjusted death rates are available from year 1900-2011. We add the data from year 1999-2011 to the original Z_t. The values are:

Table 3: 1999-2011 U.S. age-adjusted death rates and the corresponding Z_t

Year	death rate per 100,000	death rate (%)	Z_t	Year	death rate per 100,000	death rate (%)	Z_t
1999	875.6	0.008756	-0.00757	2006	776.5	0.007765	-0.02122
2000	869	0.00869	-0.01683	2007	760.2	0.007602	-0.00198
2001	854.5	0.008545	-0.01082	2008	758.7	0.007587	-0.02361
2002	845.3	0.008453	-0.01502	2009	741	0.00741	0.006993
2003	832.7	0.008327	-0.03906	2010	746.2	0.007462	-0.00753
2004	800.8	0.008008	-0.0025	2011	740.6	0.007406	-
2005	798.8	0.007988	-0.02831				

With the updated value of Z_t, the maximum likelihood estimation gives us a different combination of parameter values:

Table 4: Maximum likelihood estimation of parameters with updated Z_t

Parameters	α	σ	p	m	s
Estimated Values	-0.009883599	0.028990203	0.013892815	0.056953810	0.123195069

Compared with the values in table 3, the absolute value of α increases slightly, which indicates the tendency of death rates is to decrease annually. The value of p, which is the probability of extreme events occurrence, increases by approximately 20%. The increase in frequencies of extreme events, such as devastating earthquakes and tsunamis, is the major reason. In 2005, over 1,800 people lost their lives in Hurricane Katrina.

Since the base mortality rate is q_{2011} and the data we need are q_{2014}, q_{2015}, and q_{2016}, the simulation process needs to be modified. Using the equation 3.4-2,

$$q_t = q_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

where $q_0 = q_{2011}$, we can simulate four vectors $q_{2012}, q_{2013}, q_{2014}$ and q_{2015}. Then applying those values to the equation 3.4-1,

$$\tilde{q}_{t+h}|F_t = q_t \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)h + \sigma(W_{t+h} - W_t)\right]Y_{t+h}$$

we are able to get the mortality with extreme events in year 2013, 2014, 2015 and 2016. The data we need is the mortality in year 2014, 2015, and 2016. The CDF of the original mortality and the transformed mortality are:

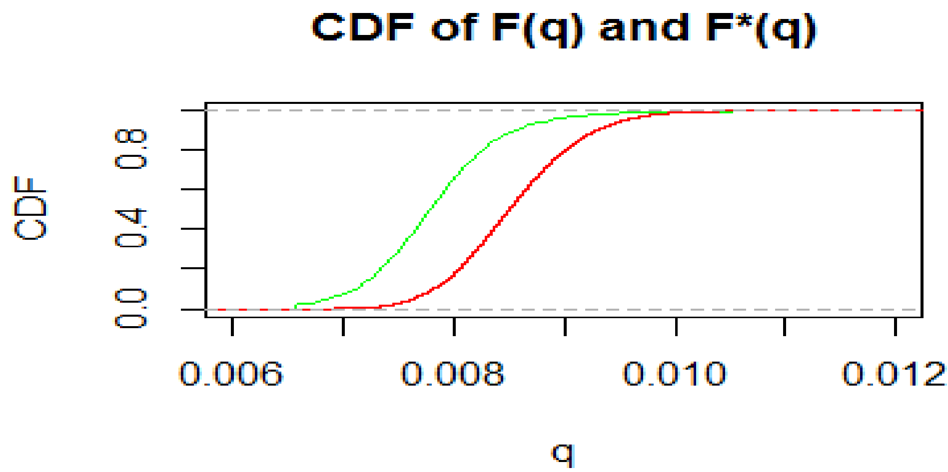

Figure 5: CDF plot of $F(q)$ and $F^{*}(q)$ for Firm A bond

The following steps are very similar to the process when we simulate the price of the Swiss Re bond. Since the design of this bond is very close to that of Swiss Re bond, we still adopt the market price of the risk obtained in Lin and Cox (2008), which is $\lambda = -1.3603$. For the Firm A bond with a face value of $1, the bond price is $0.9383.

Chapter 6

Conclusion

With the application of the Lin and Cox model and the Wang transform, we successfully get the price of the Swiss Re mortality bond and the Firm A mortality bond. The price of the Firm A mortality bond is \$0.9383 with a face value of \$1, which can be an important reference for companies who are going to issue a three-year pure mortality bond. If the company sells the bond over than \$0.9383 with \$1 face value, the bond may not be attractive to the public.

However, since the historical data we used to conduct the simulation is based on year 1900-2011, the simulation result could be less accurate for the present time. It is necessary to conduct the simulation again if the data are updated. Also, when we price the Firm A bond, we still use the Swiss Re bond’s market price of risk, which could result in inaccuracy. Further, the result is constrained to be applied to bonds that have a similar design to the Swiss Re bond. Yet we have demonstrated that the Lin and Cox model and the Wang Transform are powerful tools in pricing mortality-linked securities.

Appendix A

1900-1998 US death rates and values of Z_t

Year	death rate per 100,000	death rate (%)	Z_t	Year	death rate per 100,001	death rate (%)	Z_t
1900	2518	0.02518	-0.01799	1950	1446	0.01446	-0.01568
1901	2473.1	0.024731	-0.072	1951	1423.5	0.014235	-0.02051
1902	2301.3	0.023013	0.033206	1952	1394.6	0.013946	-0.00647
1903	2379	0.02379	0.05061	1953	1385.6	0.013856	-0.05245
1904	2502.5	0.025025	-0.03199	1954	1314.8	0.013148	0.013222
1905	2423.7	0.024237	-0.01024	1955	1332.3	0.013323	0.00105
1906	2399	0.02399	0.038996	1956	1333.7	0.013337	0.017098
1907	2494.4	0.024944	-0.08162	1957	1356.7	0.013567	-0.00985
1908	2298.9	0.022989	-0.02186	1958	1343.4	0.013434	-0.01962
1909	2249.2	0.022492	0.029785	1959	1317.3	0.013173	0.016488
1910	2317.2	0.023172	-0.03148	1960	1339.2	0.013392	-0.03063
1911	2245.4	0.022454	-0.01512	1961	1298.8	0.012988	0.018915
1912	2211.7	0.022117	-0.00235	1962	1323.6	0.013236	0.017005
1913	2206.5	0.022065	-0.02627	1963	1346.3	0.013463	-0.03208
1914	2149.3	0.021493	0.011794	1964	1303.8	0.013038	0.002069
1915	2174.8	0.021748	0.041344	1965	1306.5	0.013065	0.001912
1916	2266.6	0.022666	0.004095	1966	1309	0.01309	-0.0271
1917	2275.9	0.022759	0.110418	1967	1274	0.01274	0.023658
1918	2541.6	0.025416	-0.21145	1968	1304.5	0.013045	-0.02539
1919	2057.2	0.020572	0.042772	1969	1271.8	0.012718	-0.03945
1920	2147.1	0.021471	-0.09209	1970	1222.6	0.012226	-0.0078
1921	1958.2	0.019582	0.04557	1971	1213.1	0.012131	0.0014
1922	2049.5	0.020495	0.043864	1972	1214.8	0.012148	-0.01126
1923	2141.4	0.021414	-0.04949	1973	1201.2	0.012012	-0.042
1924	2038	0.02038	0.014951	1974	1151.8	0.011518	-0.05112
1925	2068.7	0.020687	0.036778	1975	1094.4	0.010944	-0.00946
1926	2146.2	0.021462	-0.07582	1976	1084.1	0.010841	-0.03044
1927	1989.5	0.019895	0.0657	1977	1051.6	0.010516	-0.00754
1928	2124.6	0.021246	-0.02064	1978	1043.7	0.010437	-0.03193
1929	2081.2	0.020812	-0.0683	1979	1010.9	0.010109	0.027514
1930	1943.8	0.019438	-0.02537	1980	1039.1	0.010391	-0.03128
1931	1895.1	0.018951	0.001055	1981	1007.1	0.010071	-0.02219

1932	1897.1	0.018971	-0.02509	1982	985	0.00985	0.005063
1933	1850.1	0.018501	0.020384	1983	990	0.0099	-0.0076
1934	1888.2	0.018882	-0.01499	1984	982.5	0.009825	0.005684
1935	1860.1	0.018601	0.0542	1985	988.1	0.009881	-0.00966
1936	1963.7	0.019637	-0.04218	1986	978.6	0.009786	-0.00883
1937	1882.6	0.018826	-0.0649	1987	970	0.0097	0.005859
1938	1764.3	0.017643	0.001473	1988	975.7	0.009757	-0.02617
1939	1766.9	0.017669	0.010192	1989	950.5	0.009505	-0.01249
1940	1785	0.01785	-0.05197	1990	938.7	0.009387	-0.01416
1941	1694.6	0.016946	-0.03531	1991	925.5	0.009255	-0.0159
1942	1635.8	0.016358	0.039907	1992	910.9	0.009109	0.022363
1943	1702.4	0.017024	-0.05054	1993	931.5	0.009315	-0.01221
1944	1618.5	0.016185	-0.02699	1994	920.2	0.009202	-0.00185
1945	1575.4	0.015754	-0.02944	1995	918.5	0.009185	-0.01768
1946	1529.7	0.015297	0.001502	1996	902.4	0.009024	-0.01687
1947	1532	0.01532	-0.01998	1997	887.3	0.008873	-0.01305
1948	1501.7	0.015017	-0.03001	1998	875.8	0.008758	-
1949	1457.3	0.014573	-0.00778				

Appendix B

R Command: Swiss Re Bond Pricing Simulation

Here is the R program used to evaluate the price of the Swiss Re bond:

```
zvec=scan('death.txt')
fn <- function(a) {
 sigma <- a[1]
 alpha <- a[2]
 prob <- a[3]
 sd <- a[4]
 mu <- a[5]
 -sum(log(
  1 / (sigma*sqrt(2*pi)) * exp(-0.5 * ((zvec-alpha+0.5*sigma^2)/sigma)^2) * (1-prob)^2 +
   1 / (sqrt(sigma^2+sd^2) * sqrt(2*pi)) *
   exp(-0.5 * ((zvec-alpha+0.5*sigma^2+mu) / sqrt(sigma^2+sd^2))^2) * (1-prob)*prob +
   1/ (sqrt(sigma^2+sd^2) * sqrt(2*pi)) *
   exp(-0.5 * ((zvec-alpha+0.5*sigma^2-mu) / sqrt(sigma^2+sd^2))^2) * prob*(1-prob) +
   1/ (sqrt(sigma^2+2*sd^2) * sqrt(2*pi)) *
   exp(-0.5 * ((zvec-alpha+0.5*sigma^2) / sqrt(sigma^2+2*sd^2))^2) * prob^2
 )
 )
}
mle <- nlm(fn, a <- c(0.1,0.1,0.1,0.1,0.1), stepmax=0.1, steptol=0.00000001, iterlim = 500)
mle
sigma <- mle$estimate[1]
```

```
alpha <- mle$estimate[2]
prob <- mle$estimate[3]
sd <- mle$estimate[4]
mu <- mle$estimate[5]
#Use the constrOptim function
Amat <- rbind(c(-1,0,0,0,0),c(0,-1,0,0,0),c(0,0,-1,0,0),c(0,0,0,-1,0),c(0,0,0,0,-1))
bmat <- cbind(rep(-1,5))
constrOptim(a <- c(0.1,0.1,0.1,0.1,0.1), fn, NULL, ui = Amat, ci = bmat, method = "Nelder-
Mead")
#The mortality in 2002 is the base mortality
q0 = 845.3/100000
#Application of morality model without extreme events
q2003 <- q0*exp(alpha-0.5*sigma^2+sigma*rnorm(10000))
q2004 <- q2003*exp(alpha-0.5*sigma^2+sigma*rnorm(10000))
q2005 <- q2004*exp(alpha-0.5*sigma^2+sigma*rnorm(10000))
q2006 <- q2005*exp(alpha-0.5*sigma^2+sigma*rnorm(10000))
#Application of morality model with extreme events
q1 <- q0*exp(alpha-0.5*sigma^2+sigma*rnorm(10000)+mu+sd*rnorm(10000))
q2 <- q2003*exp(alpha-0.5*sigma^2+sigma*rnorm(10000)+mu+sd*rnorm(10000))
q3 <- q2004*exp(alpha-0.5*sigma^2+sigma*rnorm(10000)+mu+sd*rnorm(10000))
q4 <- q2005*exp(alpha-0.5*sigma^2+sigma*rnorm(10000)+mu+sd*rnorm(10000))
mat <- cbind(q2,q3,q4)
q <- apply(mat,1,max)
Fq <- ecdf(q)
#Define the transformed CDF F*(q)
```

```
Fstar <- function(x){
  result <- pt(qnorm(Fq(x))+1.3603,6,lower.tail = TRUE, log.p = FALSE)
  return(result)
}
plot(Fstar, xlim=c(0.007, 0.014), col="green", ylab="CDF")
lines(ecdf(q),col="red")
#Define payment function based on F*(q)
pmt <- function(x){
  q0 = 845.3/100000
  qstar <- quantile(q, Fstar(x))
  if (qstar <= 1.3*q0){
    y = 1
  }
  else if (qstar >1.3*q0 & qstar <1.5*q0){
    y = (1.5*q0-qstar)/(0.2*q0)
  }
  else{
    y = 0
  }
  return (y)
}
#Apply the payment function to the vector q
payment <- sapply(q, pmt)
E <- mean(payment)
i <- 1.10
```

```
price <-  (1/i)^3*E

price
```

Appendix C

R Command: Firm A Bond Pricing Simulation

Here is the R program used to estimate the price of the Firm A bond:

```
zvec=scan('newdeath.txt')
fn <- function(a) {
  sigma <- a[1]
  alpha <- a[2]
  prob <- a[3]
  sd <- a[4]
  mu <- a[5]
  -sum(log(
    1 / (sigma*sqrt(2*pi)) * exp(-0.5 * ((zvec-alpha+0.5*sigma^2)/sigma)^2) * (1-prob)^2 +
      1 / (sqrt(sigma^2+sd^2) * sqrt(2*pi)) *
      exp(-0.5 * ((zvec-alpha+0.5*sigma^2+mu) / sqrt(sigma^2+sd^2))^2) * (1-prob)*prob +
      1/ (sqrt(sigma^2+sd^2) * sqrt(2*pi)) *
      exp(-0.5 * ((zvec-alpha+0.5*sigma^2-mu) / sqrt(sigma^2+sd^2))^2) * prob*(1-prob) +
      1/ (sqrt(sigma^2+2*sd^2) * sqrt(2*pi)) *
      exp(-0.5 * ((zvec-alpha+0.5*sigma^2) / sqrt(sigma^2+2*sd^2))^2) * prob^2
  )
  )
}
#Use the nlm function
mle <- nlm(fn, a <- c(0.1,0.1,0.1,0.1,0.1), stepmax=0.1, steptol=0.00000001, iterlim = 500)
```

```
mle
sigma <- mle$estimate[1]
alpha <- mle$estimate[2]
prob <- mle$estimate[3]
sd <- mle$estimate[4]
mu <- mle$estimate[5]
#The mortality in 2011 is the base mortality
q0 = 740.6/100000
#Application of morality model without extreme events
q2012 <- q0*exp(alpha-0.5*sigma^2+sigma*rnorm(10000))
mean(q2012)
q2013 <- q2012*exp(alpha-0.5*sigma^2+sigma*rnorm(10000))
mean(q2013)
q2014 <- q2013*exp(alpha-0.5*sigma^2+sigma*rnorm(10000))
mean(q2014)
q2015 <- q2014*exp(alpha-0.5*sigma^2+sigma*rnorm(10000))
mean(q2015)
#Application of morality model with extreme events
q1 <- q2012*exp(alpha-0.5*sigma^2+sigma*rnorm(10000)+mu+sd*rnorm(10000))
mean(q1)
q2 <- q2013*exp(alpha-0.5*sigma^2+sigma*rnorm(10000)+mu+sd*rnorm(10000))
mean(q2)
q3 <- q2014*exp(alpha-0.5*sigma^2+sigma*rnorm(10000)+mu+sd*rnorm(10000))
mean(q3)
q4 <- q2015*exp(alpha-0.5*sigma^2+sigma*rnorm(10000)+mu+sd*rnorm(10000))
```

```
mean(q4)
mat <- cbind(q2,q3,q4)
q <- apply(mat,1,max)
Fq <- ecdf(q)
#Define the transformed CDF F*(q)
Fstar <- function(x){
  result <- pt(qnorm(Fq(x))+1.3603,6,lower.tail = TRUE, log.p = FALSE)
  return(result)
}
plot(Fstar,xlim=c(0.006, 0.012),col="green", xlab="q", ylab="CDF", main="CDF of F(q) and
F*(q)")
lines(ecdf(q),col="red")
#Define payment function based on F*(q)
pmt <- function(x){
  q0 = 740.6/100000
  qstar <- quantile(q, Fstar(x))
  if (qstar <= 1.3*q0){
    y = 1
  }
  else if (qstar >1.3*q0 & qstar <1.5*q0){
    y = (1.5*q0-qstar)/(0.2*q0)
  }
  else{
    y = 0
  }
```

```
  return (y)
}
#Apply the payment function to the vector q
payment <- sapply(q, pmt)
E <- mean(payment)
E
i <- 1.12/100
price <-  (1/(1+i))^3*E
price
```

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ACTUARIAL EXAMS

Passed: Exam P/1, Exam FM/2 — *January-February 2013*
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PROFESSIONAL EXPERIENCE

China International Capital Corporation, Shanghai, China — *July-August 2013*
Internship – Investment Banking Department
- Investigated the iron and steel industry and analyzed financial performance of top five companies
- Assisted in developing strategies for issuing exchangeable bonds and asset-backed securities to discover new prospective clients

Ping An Property and Casualty Insurance Company, Hangzhou, China — *May-July 2013*
Internship – Strategic Planning Department
- Reported weekly performance of insurance products over the market using Excel
- Verified the accuracy of premiums collected from local account to clarify major competitors' market share
- Developed data templates using management information system to accelerate data search process

Hexing Electrical Co., Ltd, Hangzhou, China — *July 2011*
Internship – Foreign Trade Department
- Translated English financial statements into Chinese and helped with administrative duties
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RESEARCH EXPERIENCE

Project in Simulation of Markov Chains in Life Insurance, University Park, PA — *Fall 2013*
- Derived a simulation of mortality process under the specific assumption using R
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Johnson & Johnson Case Competition, University Park, PA — *February 2013*
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