THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF MATHEMATICS

APPLICATIONS OF GAME THEORY TO RISK MANAGEMENT

FANCHEN YANG
SPRING 2014

A thesis
submitted in partial fulfillment
of the requirements
for a baccalaureate degree
in Mathematics
with honors in Actuarial Science

Reviewed and approved* by the following:

Eli Christopher C Byrne
Assistant Professor of Mathematics
Thesis Supervisor

Ron Gebhardtsbauer
Clinical Associate Professor of Actuarial Science
Honors Adviser

* Signatures are on file in the Schreyer Honors College.

Abstract

Insurance pricing is an important part of discussing actuarial science and this topic is a very important part of doing research in this particular field. This paper will show how using game theory methods and models we can create strategies that result in a different perspective in not only pricing, but in various other aspects of actuarial science and the insurance field. First we take a look at different game theory models and concepts to establish the suitability of applying this particular field of mathematics to actuarial science. We then see if our work will result in a difference from actuarial methods, whether that be in premiums or maybe in ratio of accept or reject in life insurance. There are many different fields in actuarial science and the applications of game theory can address problems that apply to all of them.

Table of Contents

List of Figures

Chapter 1 – Introduction

Why Use Game Theory?

Game theory is an application of mathematics to the decision making processes of the cooperation and conflict of rational bodies. For example, these bodies could be the insurer and the insured. Using game theory models as a way to predict behavior of the bodies involved gives a more detailed look at the rationale behind decisions being made by either party. Looking at it from a new perspective results in a more accurate picture being drawn of certain situations and creates possibilities of not just having arbitrary rules but mathematically explained solutions to problems that may arise at any point in insurance pricing. The main point is that game theory is designed with conflict of interest in mind and is flexible enough to explore the rewards of both sides of a relationship assuming rational decision making.

The concepts introduced here could be applied to various fields of actuarial science and indeed, many of them are in use in the business world as we know it. As an example, swaps, options, and forwards all have their theoretical base in zero-sum games. To give a simple example, swaps (more specifically interest rate swaps) are agreements to exchange future interest payments based on a specified principal amount. Generally speaking, the two parties in the exchange trade fixed-rate and variable-interest rate such that one party can hedge risks associated with the variable interest rate and the other can receive potentially higher rewards while holding a conservative asset. In essence, this is a zero-sum monetary exchange that is realized in the future since neither firm knows exactly how the rates will change. In a game that uses present value terms, this is played as a cooperative game that realizes win-win measured by different utility values on the risk of the swap.

Game Theory Introduction

Let us assume that there are two players, represented by P1 and P2. A common representation of a zero-sum game is P1 and P2 placing a penny on a table with the payoff dependent on whether or not the pennies match. We can then create a payoff matrix for this game and represent it as follows:

P1\P2	Heads	Tails
Heads	a_{hh},b_{hh}	a_{ht},b_{ht}
Tails	a_{th},b_{th}	a_{tt},b_{tt}

Figure 1

According to Fig. 1, we have our payoffs represented by a_{ij} and b_{ij} where $i, j \epsilon \{h, t\}$ depending on whether or not the two faces of the coins are the same. If they are the same, then P1 gets a penny and P2 gets the penny if the faces differ. For example $a_{hh} = 1$ while $b_{hh} = -1$ and $a_{th} = -1$ while $b_{th} = 1$. As is evident, the combined payoff for P1 and P2 in all the cells in the matrix is zero, thus the name zero-sum game. As a note, remember that this is a model and models map decision trees, not what the players will actually do.

Now we consider decision rules, or various ways to come to an optimal decision. In Figure 1, this particular game has a very simple decision tree because of how the payoffs are symmetrical. One way we can make a decision is the minimax criterion, which is the tendency for both players to attempt to obtain the largest guaranteed pay-off. This means that they are minimizing the possible loss in the worst case (*max*imum loss) scenario. Note that this is equivalent to maximizing the minimum gain (or maximin) for the game that we have created here. We therefore get:

$$\max_{i\epsilon\{h,t\}} \min_{j\epsilon\{h,t\}} a_{ij} = \max_{i\epsilon\{h,t\}} \min_{j\epsilon\{h,t\}} b_{ij}$$

The above equation depending on whose perspective is being observed and whether or not you are maximizing the gain or minimizing the loss. Either way, the optimal strategy would be a mix of both strategies with probability one-half and this results in the value of the game being 0.

The minimax criterion is based on the assumption that the player is attempting to maximize their gain. In the context of a zero-sum game, when a player maximizes their gain it results in the other player taking a loss. In other words, the goals of our two players are strictly at odds. However, I will now look at Bayes criterion, which is where one party assumes the other party has committed to a fixed a priori strategy. That is, one player will assume that the other will have a certain ratio of decisions and will be able to use those assumptions to simplify the decision making process. The assumption has to be a realistic proportion of decisions made, otherwise if the assumption is that the other player will always make a certain move then the problem becomes very unrealistic and trivial. This particular sort of analysis is useful if one player is very familiar with how the other player thinks or if the fixed player is in a market situation that renders them too big to react quickly.

Chapter 2 – Pricing Models

Cournot and Bertrand

There are two kinds of models that will be discussed in this chapter: Bertrand and Cournot oligopoly. An oligopoly is a type of market that is dominated by a small number of sellers and this makes our models a bit easier to work with. A Bertrand oligopoly is where insurers set the premium and Cournot is where insurers choose optimal values of insurance coverage. There are several differences between the two models while they have similar assumptions. Bertrand assumes firms compete on price while the consumers choose the quantities at the prices set. Cournot has the firms compete on the output they produce and they choose these values independently and simultaneously.

The Bertrand model is a non-cooperative game where all prices are set simultaneously and consumers buy everything from the company that offers their product at the lower price and randomly if all the companies charge the same price. This model considers a duopoly (market completely dominated by two companies) where firm A’s price depends on where it believes firm B will set its price. One firm can take over the market by pricing just below the other, but this actually is not optimal if the other firm is pricing below marginal cost since both would receive negative profits. This particular model also comes with a paradox called the Bertrand Paradox which states that it only takes two firms to obtain perfect competition. Obviously in a vacuum without any external factors this could occur, but is not particularly realistic. Thus this model pushes the market price towards marginal cost.

The Cournot model is a non-cooperative game with a fixed number of firms in which firms simultaneously choose production levels (output) and the total production across all firms affects the market price. There is a variant of this model that has two stages, the first where companies choose capacities and in the second they compete using the Bertrand model. In addition, as the number of firms tends towards infinity, the market price is pushed down toward marginal utility as each firm is competing for market share and we end up with the same result as the Bertrand model.

Hotelling

Another type of model that we will look at is the Hotelling model. This model expands on our Bertrand model by taking into account and allowing some differences in the service of the company. This service does not necessarily mean what it is they're selling in this case, but things such as customer service, product design, location, etc. In the case of location, we can imagine two stores on a street that sell the exact same commodity for the exact same price. The customer will generally choose the closer store since the product itself is the same. The other factors like customer service or product design can also be pictured as being placed on a line and the consumer can visualize where they would fall on that line and see how "close" they are to a particular firm.

While the Hotelling model does hold the product between firms the same at least from the perspective of the consumer, it is possible to apply product differentiation between firms while also following parts of this model. Bear in mind that one of the ideas addressed was the various factors of "closeness". Companies can effectively sell the same thing as their peers, but throw their own twist by changing one of those factors. If your product has some dependency on those factors then marketing could differentiate the product from those of competitors while keeping the core of the product identical. I came across the company JetBlue that markets itself as a cheaper airline while also offering similar services and flight schedules. In this example the factor of "closeness" is price. In addition, the consumer also values what they give up for this price, in this case comfort and this can be taken into account as an additional cost for a cheaper monetary price.

The result of this model, in the case of a duopoly, is that both players are pushed towards the middle of the location line. In the example of two shops on a street, if distance is the only difference between them then customers will go to whichever is closer. If the two take up the two ends of the street then they share more or less 50% of the market if customers are uniformly distributed along the street. There is an invisible line down the middle of the street that indicates where customers are more or less indifferent to either store. In order to capitalize on the potential customer pool that exists in the middle one or both of the stores will attempt to shift their location to be closer to that pool and take more than 50% of the market share. The closer to the middle, the more market share either store will receive assuming that the other store does not move as well. Thus both stores will end up in the middle if the game proceeds naturally. When that happens, distance is no longer a factor of “closeness” by most standards of measurement. Ideally, we would want both stores to be equidistant from the end and the middle of the street, or a quarter of the way down from the end. This way a customer on the street would have to walk the least distance to either store and market share for both is still 50%. However, distrust between the two players would prevent this as there’s always the possibility that one will move their location in an attempt to gain more market share.

Chapter 3 – Life Insurance Applications

Life Insurance Underwriting

Life insurance applications of game theory and the ideas introduced in this section will mostly come from the paper by Jean Lemaire titled *A Game Theoretic Look at Life Insurance Underwriting*. This particular paper mostly focuses on life insurance underwriting, which determines whether or not a person is accepted or not for insurance. The mathematics contained within the paper is fairly simple, but it serves as a very important introduction to game theory concepts as they are applied to the insurance field. We start with establishing a payoff matrix for this example from the perspective of the insurer:

P1\P2	Healthy Proposer	Ill Proposer
Accept	A	C
Reject	B	D

Figure 2

In the above figure, P1 is the insurer (player 1) and P2 is the proposer (player 2). Their payoffs are represented by A, B, C, and D. For now, we will look at the payoffs with respect to player 1, the insurer. Obviously the worst result for an insurer is to accept a proposer that is ill, thus C is the lowest payoff. In turn, D > B and A > B. As a note, there can be arguments made for whether rejecting a bad risk is better than accepting a good one but it is not relevant to this particular discussion.

We now analyze the decision making process of our two players by first looking at the minimax criterion. The map of our payoffs for this zero-sum game can be expressed graphically.

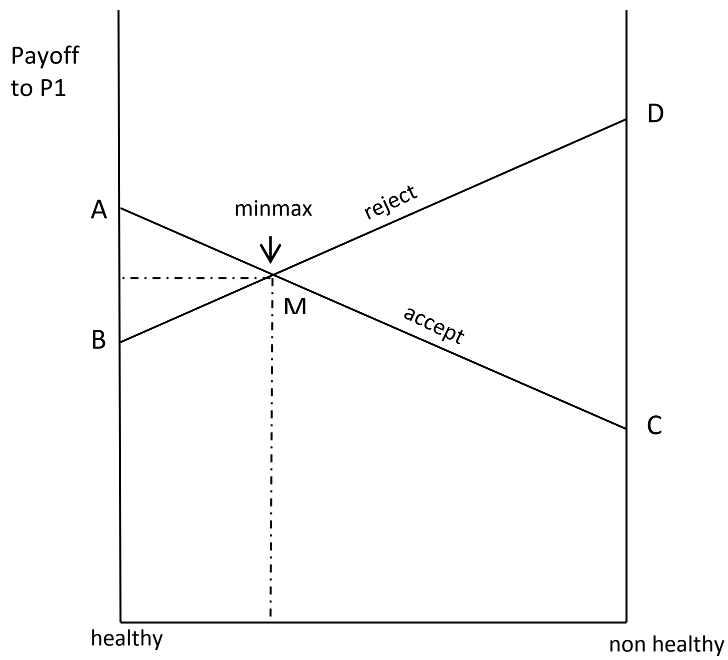

Figure 3

Using the payoffs presented in Figure 2, we can build a graph based on the payoffs of P1. The horizontal axis in Figure 3 represents P2's strategy.

This model is very naïve and does not take into account the ability of the players to gather outside information. For example, P1 (the insurer) could collect medical information on P2 and make a decision based on this factor. If we assume that there are only two characteristics of the obtained medical information: p_s (successfully noticing a bad risk) and p_f (failing to notice a bad risk), then we can introduce a third pure strategy for P1. This strategy is simply following the medical information and the payoffs E and F can be expressed as follows:

$$E = D * p_s + C(1 - p_s) \quad and \quad F = (1 - p_f)A + p_f * B$$

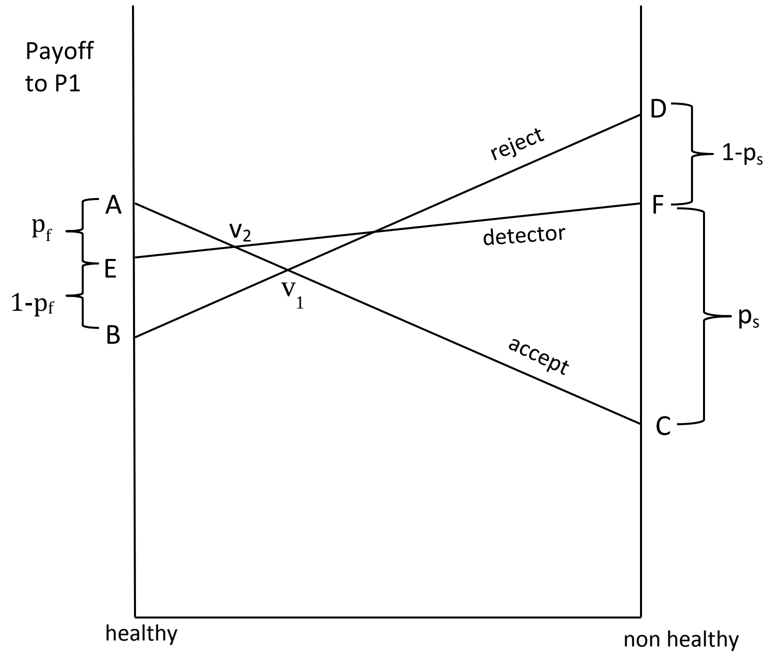

Figure 4

The payoffs for P1 can be optimized by mixing the strategies of "accepting" and "following the medical information". If we end up with medical information such that the line EF ends up below the intersection between AC and BD, then we have medical information so weak that it's completely useless.

Insurance Pricing Competition

So far we have discussed the game where it is strictly insurer vs. consumer. In order to analyze the case of firm vs. firm, we use a Hotelling model, based upon the fact that consumers buy insurance despite moderate price differences. Our two players are firms H and J. Assume that the consumers' preferences are uniformly distributed on an interval [0, 1]. Our products possess gross utility *v*. If there is a location *x* which represents where on the line an arbitrary consumer is then there is a utility cost of *kx* that is incurred from purchasing from firm H and utility cost of *(1-k)x* when purchasing from firm J if both companies are at the extremes of the interval. The symbol *k* is, in the original Hotelling model, a parameter for location. In this particular model, I will use it as a metaphor for consumer's preferences and/or characteristics. This interpretation will guide the firms toward product differentiation and specialization of their products. Assume the prices of the products from firm H and J are $p_H\ and\ p_J$ respectively.

Let's create an arbitrary firm g, whose position is located at location l_g, $g \epsilon \{H, J\}$ then we can denote net utility using the following functions:

$$net\ utility = gross\ utility - price - utility\ cost$$

$$\mu_H(p_H, x) = v - p_H - k|x - l_H|^\gamma\ where\ \gamma > 0$$

$$\mu_J(p_J, x) = v - p_J - k|x - l_J|^\gamma\ where\ \gamma > 0$$

The above equation will allow us to calculate the utility gained from the products of these firms even if their location is not at the extremes of the line. Observe from these utility functions that when $\gamma = 0$, the purchase cost becomes the same regardless of where the firm is located. When $\gamma = 1$, then the utility cost is linear and when $\gamma = 2$ the utility cost is quadratic, which means that utility is reduced more strongly as the distance between their preference and the product increases. For now, we assume $\gamma = 1$ for simplicity's sake.

Consider v to be a reserve price. If u_g is negative, then the consumer does not buy any products. Then if the prices are not too high relative to the price *v* we have the market covered. We then assume that $0 \leq l_H \leq x \leq l_J \leq 1$ so we can find the marginal consumer *x* who is indifferent between the two firms and we get (from equating the two net utility functions):

$$\mu_H(p_H, x) = \mu_J(p_J, x)$$

$$v - p_H - k|x - l_H| = v - p_J - k|x - l_J|$$

$$x = \frac{\left(p_J - p_H + k(l_H + l_J)\right)}{2k}$$

The above equation represents our midpoint, which is the location on the interval that represents where the consumer chooses a firm at random. Now we introduce D, the variable for demand. Demand in this case represents the number of people in the market pool. Our demand functions $Q_H \; and \; Q_J$ represent the number of people that would favor one firm over the other out of a pool of D people. We can then derive the demand functions for the two firms by multiplying D by the midpoint between our two firms:

$$Q_H(p_H, p_J) = Dx = D\frac{\left(p_J - p_H + k(l_H + l_J)\right)}{2k}$$

where $p_H \leq p_J + k(l_H + l_J)$ and $p_H \leq v.$ If $p_H > v$ or $p_H > p_J + k(l_H + l_J)$ then $Q_H = 0$.

$$Q_J(p_H, p_J) = D(1 - x) = D\frac{\left(p_H - p_J + k(2 - l_H - l_J)\right)}{2k}$$

where $p_J \leq p_H + k(l_H + l_J)$ and $p_J \leq v.$ If $p_J > v \; or \; p_J > p_H + k(l_H + l_J)$ then $\mathrm{Q_J} = 0$.

We now introduce our set of strategies S which is the set of prices *p* that will allow our two firms to maximize profit. If the firms' products have unit costs c_H and c_J respectively, then our profit functions π are:

$$profit = (price - cost) * Demand$$

$$\pi_H(p_H, p_J) = (p_H - c_H)Q_H(p_H, p_J) = (p_H - c_H)\left(p_J - p_H + k(l_H - l_J)\right)\frac{1}{2k}$$

and

$$\pi_J(p_H, p_J) = (p_J - c_J)Q_J(p_H, p_J) = (p_J - c_J)\left(p_H - p_J + k(2 - l_H - l_J)\right)\frac{1}{2k}$$

The best response functions for our two firms can be done by maximizing the two profit functions by setting their derivatives to zero and solving. This gives us:

$$\frac{\partial \pi_H(p_H, p_J)}{\partial p_H} = -(p_H - c_H) + \left(p_J - p_H + k(l_H - l_J)\right) = 0$$

$$p_H = \frac{\left(p_J + c_H + k(l_H + l_J)\right)}{2}$$

and

$$\frac{\partial \pi_J(p_H, p_J)}{\partial p_J} = -(p_J - c_J) + \left(p_H - p_J + k(2 - l_H - l_j)\right)\frac{1}{2k} = 0$$

$$p_J = \frac{\left(p_H + c_J + k(2 - l_H - l_J)\right)}{2}$$

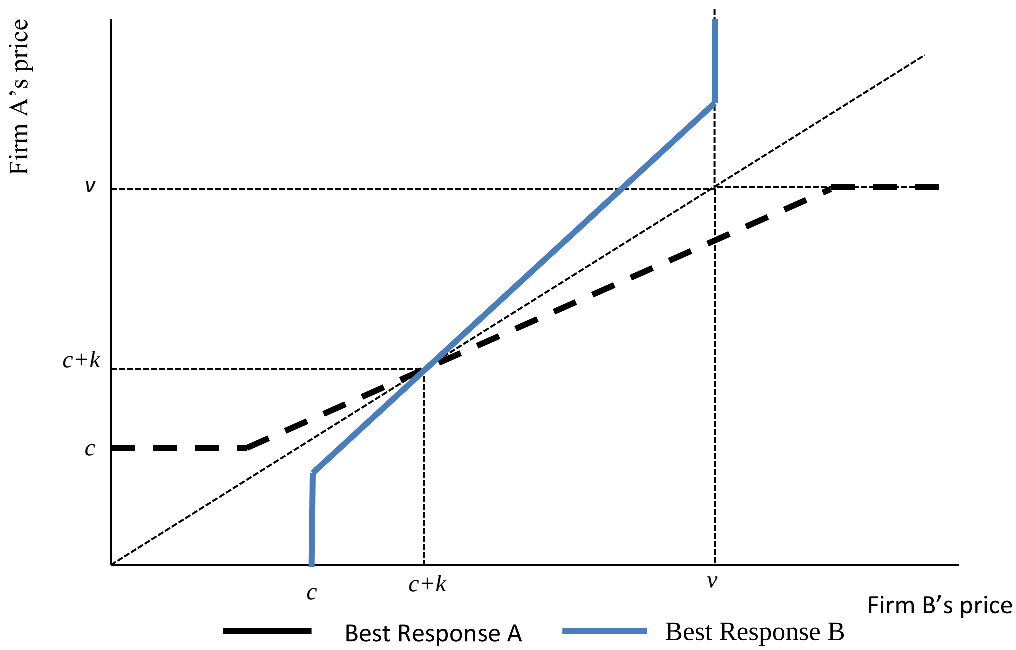

Figure 5

The solution to this game can be obtained using the Nash Equilibrium. This equilibrium profile consists only of the best moves (price to be set for the product) for our two firms. In this model, our Nash Equilibrium represents the price that will yield the most profit given that both firms know the location of the other firm and the distribution of their consumers. What this means is that given these prices and assumptions, neither of our firms can change the price in such a way that will yield them a greater profit than what they will already receive. The Nash

Equilibrium can then be calculated as the intersection of these two equations, which occurs at the following two prices of the firms:

$$p_H^* = \frac{(2c_H + c_J)}{3} + \frac{k(2 + l_H + l_J)}{3}$$

and

$$p_J^* = \frac{(2c_J + c_H)}{3} + \frac{k(4 - l_H - l_J)}{3}$$

Figure 5 graphically illustrates the case for the two firms when the costs are equal. With a larger *k*, the products are more differentiated and the equilibrium prices are higher. Similarly, when $k = 0$ our prices get smaller and it gets close to a Bertrand competition with homogeneous products barring any other factors that need to be taken into account.

Building upon this Hotelling model, we note that insurance is a way to pool risk. In this case, the firms might find themselves in a situation where equally splitting the market share down the middle does not absorb enough risk to be sustainable. It is possible that this might occur in an area with a small enough population or there could be other factors. In such a situation, the firms may end up needing to specialize in their type of insurance and adding in various incentives to split the market in a different way in order to absorb risk. In the case of taking into account more than one factor, note that a two dimensional model of the baseline Hotelling model does not necessarily move both companies into the center of a square. This is because if the two companies stay in the center of the square it lowers the cost of entry into the market by a third firm that can provide that specialized care and be closer than the other two without having as high a cost. The *k* is very important to this model as it dictates how much the product for our two firms should differ in order to maximize profit, not just market share. Note that this means that some degree of specialization occurs in our metaphor for location.

Chapter 4 – Non-life Insurance Applications

Game theory has further applications to non-life insurance and indeed, even the models discussed in the life insurance section have applications in most fields of actuarial science. The life insurance underwriting model in particular could be used in say medical insurance underwriting. Most general applications can be done with changes in the calculation and additions made to other factors that need to be taken into account. Specific models will not be covered in this paper, but I can recommend the paper by Dutang, Albrecher, and Loisel (2013) as a very good resource that contains an in-depth look at game theory approach towards this field.

One of the areas of insurance that should be mentioned is risk loads for property catastrophe insurance (Mango 1998). In the referenced paper by Donald Mango, he talks about how game theory applications to real world issues can also be applied to this particular field. There are mentions of tax allocations, maintenance costs, financing of large projects, construction fees, etc. This particular model is known as "cooperative games with transferable utilities" and is used when the players need to share money between them. Whether they want to maximize or minimize their money in this game is dependent on the type of game that they are addressing. Applying this to risk load gives us a pretty easy fit where our "players" want to minimize allocations of the portfolio risk load. That being said, the paper does acknowledge that there are small details that involve more complex mathematics that would require more research in order to make the game fit more easily with this particular problem.

Another field of study where game theory has applications is reinsurance. We start with the primary insurance market with an arbitrary number of players and then extend this model by creating "levels" of reinsurance by its distance from the primary insurance market (Powers and Shubik 2006). Level "1" is the reinsurance of the primary insurers and level "2" the reinsurance of those reinsurers. Each level is a stage of a game and we end up with an (r+1)–stage game depending on how many levels of reinsurance there are. The equilibrium in this case is one that is “perfect” in the sense that the equilibrium of the primary game will also be the same for each sub game. In addition, there are things other than the price of insurance that can be analyzed in this model. One of the questions asked is the optimal number of reinsurers because like most things, the reinsurance market can become saturated in a way that the marginal utility of additional levels is so low that it is not desirable.

Chapter 5 – Conclusion

In this paper I have given a brief introduction to game theory and analyzed various applications of game theory in the insurance industry. I proposed an adaptation of the Hotelling model using the Hotelling's original location parameter as a metaphor for consumer preferences and/or other characteristics. An outline is presented of how this adaptation can provide a methodology for finding equilibrium for product differentiation i.e. specialized policies for increased profit for insurance companies. I also survey additional applications of game theory for insurance and present a few computational examples on how they might be modeled. While the models presented are simple, the various complexities that reality adds onto any model can create interesting situations. In this case, the way that I have set up the model, additional factors can be taken into account and added into the game theory model in a fairly obvious way. The field of game theory still has a lot of different areas that could be explored and its applications are still wide and varied.

Bibliography

Borch, Karl. "Application of Game Theory to Some Problems in Automobile Insurance." *ASTIN Bullentin* 2.2 (1962): 208-221. Print.

Brockett, Patrick L., and Xiaohua Xia. "Operations research in insurance: A review." *Transactions of society of actuaries* 47 (1995): 7-87. Print.

Chentsov, N.N.. "Statistical decision theory." *Encyclopedia of Mathematics*. N.p., n.d. Web. 7 Apr. 2014. <http://www.encyclopediaofmath.org/index.php/Statistical_decision_theory>.

Dutang, Christophe, Hansjeorg Albrechor, and Stéphane Loisel. "Competition Among Non-life Insurers Under Solvency Constraints: a Game-Theoretic Approach." *European Journal of Operational Research* 231 (2013): n. pag. *EJOR*. Web. 26 Nov. 2013.

Lemaire, Jean. "A Game Theoretic Look at Insurance Underwriting." *ASTIN Bulletin* 11.1 (1980): 1-16. *AAI-IAA*. Web. 19 Nov. 2013.

Mango, Donald. "An Application of Game Theory: Property Catastrophe Risk Load." *Proceedings of the Casual Actuarial Society* 85 (1998): 157-186. Print.

Mao, Hong, and Krzysztof Ostaszewski. "Application of Game Theory to Pricing of Participating Deferred Annuity." *Journal of Insurance Issues* 30 (2007): 102-122. Print.

Polborn, Mattias K.. "A Model of an Oligopoly in an Insurance Market." *The GENEVA PAPERS on Risk and Insurance Theory* 23.1 (1998): 41-48. Print.

Powers, Michael R., and Martin Shubik. "A "square-root rule" for reinsurance." *Revista Contabilidade & FinanÃ§as* 17.spe2 (2006): 101-107. Print.

Warren, Ryan, Ji Yao, Tim Rourke, and Jan Iwanik. "Brian Hey Prize - Game Theory in General Insurance." *Home*. Brian Hey Prize, n.d. Web. 7 Apr. 2014.

<http://www.actuaries.org.uk/research-and-resources/documents/brian-hey-prize-game-theory-general-insurance>.

"Zero-Sum Game." *Investopedia*. N.p., n.d. Web. 7 Apr. 2014. <http://www.investopedia.com/terms/z/zero-sumgame.asp>.

Fanchen Yang

825 Bellaire Ave, Apt. X224, State College, PA 16801
Phone: (570)-854-7662
e-mail: fqy5013@psu.edu

Sex: Male
Birth date: 12/28/1991
Citizenship: USA

Education:

Pennsylvania State University Schreyers Honors College
Mathematics B.S. with an Actuarial Sciences Option *August 2010-May 2014*

Academic Courses and Training:

- Compound Interest/Annuities
- Stochastic Modeling
- Probability Theory
- C++ and Java
- Actuarial/Financial Mathematics

Actuarial Tests:

Exam P/1 Date Passed: November 18, 2012
VEE requirements Met upon Graduation: Statistics, Economics

Work Experience:

- Volunteer at Weis Research Center, Geisinger Medical Center *June 2008-July 2008*
  - Assisted research scientists to perform biomedical research
  - Learned how to design experiments and analyze data
  - Performed molecular and biochemical experiments in a lab environment

- Internship at National Cancer Institute (NCI) *June 2012-August 2012*
  - Programmed in Perl and C++
  - Worked with researchers and performed bioinformatics projects
  - Learned to work as part of a team on scientific research

Honors and Awards:

- Academic Excellence Scholarship Penn State University *August 2010-present*
- Dean's List Penn State University *August 2010-present*
- President Award Penn State University *March 2011*