THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF MATHEMATICS

The q-Catalan numbers of MacMahon and a related identity of Andrews

SARA COBERN
SPRING 2015

A thesis
submitted in partial fulfillment
of the requirements
for baccalaureate degrees
in Mathematics and Statistics
with honors in Mathematics

Reviewed and approved* by the following:

Ae Ja Yee
Associate Professor of Mathematics
Thesis Supervisor

Victoria Sadovskaya
Associate Professor of Mathematics
Honors Adviser

*Signatures are on file in the Schreyer Honors College.

Abstract

This thesis offers a look into the q-Catalan numbers of MacMahon and their properties. The beginning of the paper explains the Catalan numbers and the lattice path interpretation of them. We then move to the generalized version of the Catalan numbers, the q-Catalan numbers. Much of the work provided in this thesis is related to George E. Andrews' paper titled "q-Catalan Identities." Some of the problems proposed at the end of his paper are answered in this thesis, and the work leading up to those solutions is provided.

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Acknowledgements

First and foremost, I would like to thank Professor Yee for her help and guidance through my thesis work. Having been the one to introduce me to the world of combinatorics, I am grateful that I was able to share this experience with her. Her patience and kindness were extremely important in helping me complete this major task in my life.

I would also like to acknowledge my honors advisor Professor Sadovskaya and Dr. Cheryl Hile, my advisor before my acceptance into Schreyer Honors College. Both of them are important people who have helped me in my success at Penn State and I thank them each for that.

Lastly, I would like to thank those people that have been with me through my thesis work and my college career. To my wonderful parents, sibilings, and extended family which is far too extensive to list here, thank you for encouraging me to strive to do my best and to never give up. I do not thank you enough for all the support you have provided me with over the years, but I hope this can be a small start to my huge thank you. To my friends from home and the ones that I have made over the last four years at Penn State, thank you for your support and making that support known whether I was in the same town as you or not. Without you, I would not be who I am today.

Thank you to all those montioned above and any that may have been missed. I am lucky to have a support group as extensive as it is.

Chapter 1

Introduction

Discovered by Eugene Charles Catalan in Bruges, Belgium, the Catalan numbers appear in many combinatorial problems. They are defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{1.1}$$

where n is a non-negative interger and $C_0 = 1$. They have been studied by many mathematians over the years and can be interpreted in almost two hundred different ways. This thesis will specifically look at lattice path interpretation of the Catalan numbers, but note that many of the other combinatorial interpretations can be found using a bijection from the lattice path explanation.

When given a sequence of numbers in combinatorics, q-analogs of the sequence are often studied. A q-analog of a sequence of numbers is typically a polynomial in the variable q that reduces naturally to the original sequence when $q \to 1$ and it furthermore satisfies versions of some of the algebraic properties, such as recursions, of the sequence carrying some interesting information about the objects counted by the sequence. For instance, the q-analog of a non-negative integer n is defined as

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1},$$

which reduces to n when $q \to 1$.

For this thesis, we are interested in the discussion of the q-Catalan numbers of MacMahon [7]. In [2], George Andrews obtained interesting identities of the q-Catalan numbers. Specifically, this thesis looks at the q-Catalan identity which is a version of Koshy's identity found in Andrews' paper. In this thesis, we give the details of the proof for Andrews' identity and provide solutions to Andrews' first problem and the beginning solutions to the second problem.

This thesis is organized as follows. Chapter 2 focuses on some preliminary work required to begin discussion on the q-Catalan numbers and Andrews' q-Koshy identity. We begin by looking at the binomial coefficients as they are needed to understand the definition of the Catalan numbers. The discussion then moves to the Catalan numbers before moving to their q-analog discussion. Just as the binomial coefficients are used to define the Catalan numbers, the q-binomial coefficients are used to define the q-Catalan numbers. For that reason, Chapter 2 introduces the q-binomial coefficients before ending with the q-Catalan numbers. Through all of these topics in Chapter 2, lattice path interpretations are offered for each of the sequences to help the understaning of the concept.

Chapter 3 begins with Koshy's identity and Andrews' q-analog of Koshy's identity. Before proving Andrews' identity, hypergeometric functions are briefly introduced as they are used to prove the identity. Andrews' identity is the source of the five questions he proposes at the end of his paper, so we move to discuss those questions and to work through them. The first problem is solved in its entirety. For the second problem, we give a solution for a special case and suggest an approach for the general case.

This thesis concludes with Chapter 4 dedicated to the conclusion of the work and future work that could be completed. This section focuses on how to possibly approach the last of the questions posed by Andrews.

Chapter 2

Preliminaries

This chapter begins by looking at the binomial coefficients and their definition since they are a necessary part of the Catalan numbers. The discussion then moves to the Catalan numbers and their lattice path interpretation. Before introducing to the q-Catalan numbers, the q-binomial coefficients are discussed since they are a part of the definition of the q-Catalan numbers. All of these sections also offer a lattice path interpretation of the topic discussed to help link the content together.

2.1 Binomial coefficients and lattice paths

The *binomial coefficients* $\binom{n}{k}$ are defined as follows: For any non-negative integers n and k,

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!}, & \text{for } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 \leq k \leq n$, $\binom{n}{k}$ counts the ways of choosing k elements out of n elements.

We consider lattice paths in the plane consisting of unit horizontal and vertical steps in the positive direction, i.e., each step in the path is either $(0,1)$ (vertical) or $(1,0)$ (horizontal). For instance, the figure below shows such a lattice path from $(0,0)$ to $(5,4)$:

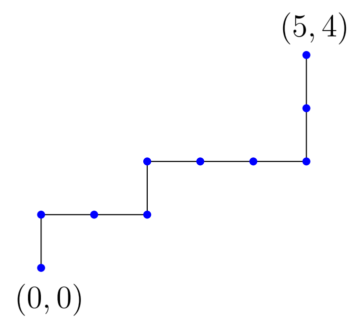

Figure 2.1: Lattice path from $(0,0)$ to $(5,4)$

In the sequel, we shall call them *paths* for short.

For a path from $(0,0)$ to $(n-k,k)$, a total of n steps are made with k of them being vertical ones. Once the placement of k vertical steps is made, the remaining steps are all horizontal, and placing k vertical steps among the total n steps is equivalent to choosing k elements from the set $\{1,2,\ldots,n\}$. Thus we see that the number of paths from $(0,0)$ to $(n-k,k)$ is $\binom{n}{k}$.

Figure 2.2 shows the four different paths from $(0,0)$ to $(3,1)$.

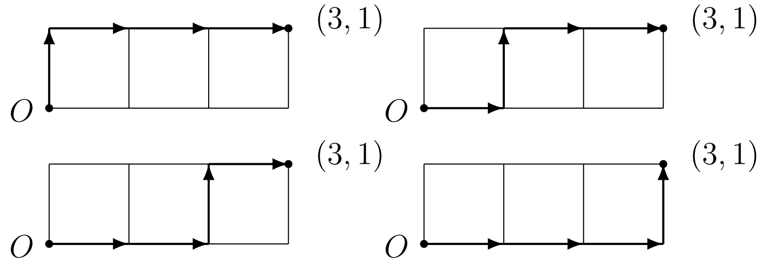

Figure 2.2: All possible paths from $(0,0)$ to $(3,1)$

2.2 Catalan numbers and lattice paths

The *Catalan numbers* C_n are defined in equation (1.1) in the Introduction. Here we relate these C_n with a certain type of paths.

For a positive integer n, we consider paths from $(0,0)$ to (n,n) without moving below the line $y = x$. Throughout this thesis, we will call them Catalan paths. It is known that the total number of Catalan paths equals C_n, a proof of which will be given in Theorem 2.2.1. For instance, from the origin to the point $(3,3)$, we can see below in Figure 2.3 that there are five Catalan paths.

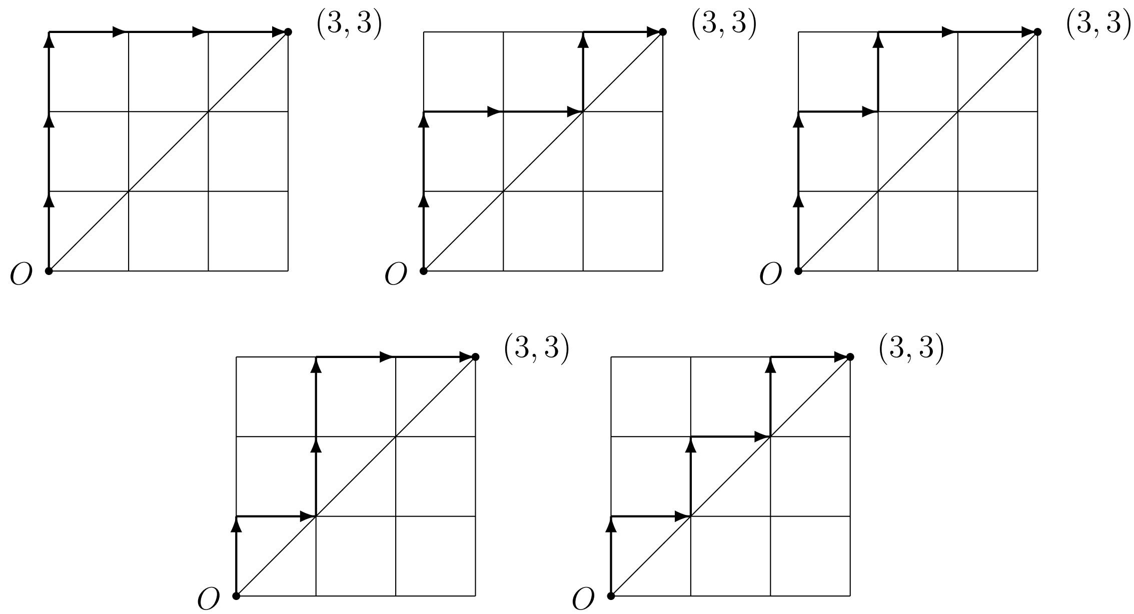

Figure 2.3: Catalan paths for $n = 3$

Here we will prove that the Catalan number C_n counts the number of Catalan paths from $(0,0)$ to (n,n) by following a proof presented in Brualdi [4].

Theorem 2.2.1 *For any positive integer n, the Catalan number C_n counts the number of paths from $(0,0)$ to (n,n) without falling below the line $y = x$.*

Proof. We first consider the set S of all paths from $(0,0)$ to (n,n). There are $\binom{2n}{n}$ paths in S. To prove this theorem, we will subtract from S the paths that fall below the line $y = x$. Namely, we shall show that

$$C_n = \binom{2n}{n} - X_n, \tag{2.1}$$

where X_n is the number of paths that fall below the line $y = x$. The main part of the proof is finding a formula for X_n.

We take a path ℓ that goes below $y = x$ with the first intersection point with $y = x$ at (k,k). Since the path goes below the line, the next step must be horizontal, so the next lattice point, i.e., the point with integral coordinates, is $(k+1,k)$. We now divide the paths into two parts: the first part, ℓ_1, is the section of the lattice path from $(0,0)$ to $(k+1,k)$, and the second part ℓ_2 is the rest of the path from $(k+1,k)$ to (n,n). We now reflect ℓ_1 across the line $y = x-1$ to get the path ℓ_1'. Combining ℓ_1' with ℓ_2 yields a new path from $(1,-1)$ to (n,n) since the origin maps to $(1,-1)$ under the reflection. See Figure 2.4.

This process is reversible. A path from $(1,-1)$ to (n,n) must cross the line $y = x-1$ since $(1,-1)$ is below the line and (n,n) is above the line. Taking the first intersection point of the path and the line, and reflecting the section from $(1,-1)$ to the intersection point, and then combining the reflection image with the unchanged section of the original path gives us a path from $(0,0)$ to (n,n) that falls below the line $y = x$.

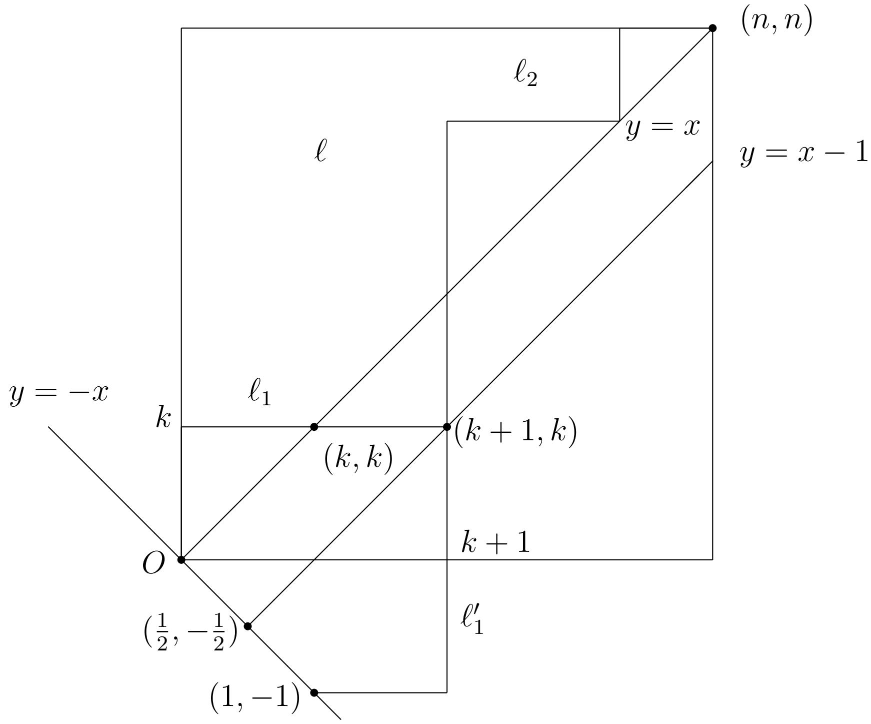

Figure 2.4: Lattice path crossing the line $y = x$

Thus X_n equals the number of all possible paths from $(-1,1)$ to (n,n), which is $\binom{2n}{n+1}$ since there are a total number of $2n$ steps to be made and $n+1$ of those are vertical steps.

Thus we now have the following:

$$\begin{aligned}
\binom{2n}{n} - X_n &= \binom{2n}{n} - \binom{2n}{n+1} \\
&= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\
&= \frac{(2n)!}{n!n!}\frac{(n+1)}{(n+1)} - \frac{(2n)!}{(n+1)!(n-1)!}\frac{n}{n} \\
&= \frac{(2n)!}{(n+1)!n!}(n+1-n) \\
&= \frac{1}{(n+1)}\binom{2n}{n} = C_n.
\end{aligned}$$

Therefore, lattice paths from $(0,0)$ to (n,n) that do not cross $y = x$, but may touch, are counted by the Catalan numbers. □

2.3 q-binomial coefficients and lattice paths

The q-*binomial coefficients* are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$(a;q)_n = (1-a)(1-aq^1)(1-aq^2)...(1-aq^{n-1}),$$

so

$$(q;q)_n = (1-q)(1-q^2)(1-q^3)...(1-q^n).$$

Note that

$$\frac{(q;q)_n}{(1-q)^n} = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) \to n!$$

as $q \to 1$. Thus we can see that $\begin{bmatrix} n \\ k \end{bmatrix}$ reduces to $\binom{n}{k}$ as $q \to 1$. While it is not obvious, $\begin{bmatrix} n \\ k \end{bmatrix}$ also forms a polynomial, which follows from Theorem 2.3.2 below.

These q-binomial coefficients have a combinatorial interpretation as follows. Suppose that $0 \le k \le n$. For a path π from $(0,0)$ to $(n-k,k)$, we define $w(\pi)$ by the number of squares of area one between the path π, the y-axis, and the line $y = k$. We then consider the following sum:

$$\sum_{\pi} q^{w(\pi)}, \tag{2.2}$$

where the sum is over all paths from $(0,0)$ to $(n-k,k)$. Note that the sum above is 1 if $k = 0$ or $k = n$.

To understand this more clearly, we look at the paths from $(0,0)$ to $(3,1)$. Figure 2.5 shows all the paths with the squares of area one above the path highlighted with a blue border. Then the sum in (2.2) becomes

$$\sum_{\pi} q^{w(\pi)} = 1 + q + q^2 + q^3,$$

which equals

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Notice that if we set $q = 1$ then we get the number 4, which represents the number of lattice paths from $(0,0)$ to $(3,1)$.

Before we prove that the sum in (2.2) is indeed the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$, we need the following lemma.

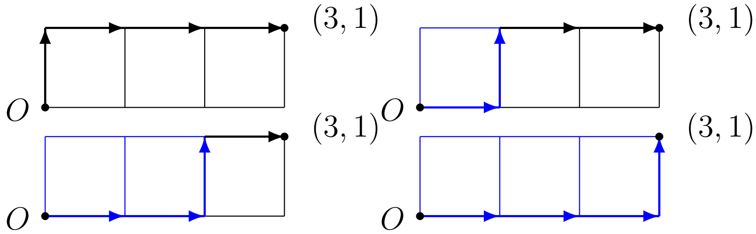

Figure 2.5: Lattice paths to $(3, 1)$ with weight q shown

Lemma 2.3.1 *For $0 \leq k \leq n$, we have*

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} &= q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ &= \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \end{aligned}$$

Proof. To prove Lemma 2.3.1 we show that the two identities are equal.

$$\begin{aligned} q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} &= q^k \frac{(q;q)_{n-1}}{(q;q)_k (q;q)_{n-1-k}} + \frac{(q;q)_{n-1}}{(q;q)_{k-1}(q;q;)_{n-k}} \\ &= \frac{(q;q)_{n-1}}{(q;q)_k (q;q)_{n-k}} (q^k(1-q^{n-k}) + (1-q^k)) \\ &= \frac{(q;q)_{n-1}}{(q;q)_k (q;q)_{n-k}} (1-q^n) \\ &= \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \\ &= \begin{bmatrix} n \\ k \end{bmatrix}. \end{aligned}$$

Note that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ n-k \end{smallmatrix}\right]$. In the first identity, we replace k by $n-k$. Then the second identity immediately follows. □

Theorem 2.3.2 *For any nonnegative integers n, k with $0 \leq k \leq n$,*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\pi} q^{w(\pi)},$$

where the sum is over all paths π from $(0, 0)$ to $(n-k, k)$.

Proof. Let $W_{n,k}(q)$ be the sum on the right side in the right side in Theoerm 2.3.2. We shall show that $W_{n,k}(q)$ satisfies the same recusion as $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ shown in Lemma 2.3.1. In other words,

$$W_{n,k}(q) = q^k W_{n-1,k}(q) + W_{n-1,k-1}(q).$$

We use induction on n, k. For $k = 0$ or $k = n$, it is trivial. We assume that the statement is true for $n-1$ and k with $0 \leq k \leq n-1$. Now consider the case n and k with $0 < k < n$. First, note that there are two possibilities for the first step of lattice paths from the origin to $(n-k,k)$: horizontal or vertical. Let

$$V_n(q) = \sum_{\pi} q^{w(\pi)},$$
$$H_n(q) = \sum_{\pi} q^{w(\pi)},$$

where the first sum is over all paths with the first step being vertical, and the second all paths with the first step being horizontal. Notice that together $H_n(q)$ and $V_n(q)$ comprise all lattice paths from the origin to $(n-k,k)$. Therefore, we are able to write

$$W_{n,k}(q) = V_n(q) + H_n(q).$$

Having already taken one vertical step, we need to count the number of paths from $(0,1)$ to $(n-k,k)$. Moving the path vertically down one step will also show that the path is equivalent to going from the origin to $(n-k,k-1)$. Thus, $V_n(q) = W_{n-1,k-1}(q)$. By assumption, $W_{n-1,k-1}(q) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.

The paths counted by $H_n(q)$ take a horizontal step as its first step, so we must count the number of paths from $(1,0)$ to $(n-k,k)$. Sliding the path to the left one unit to the origin will create a path from $(0,0)$ to $(n-k-1,k)$. These paths are counted by $W_{n-1,k}$. Because of the horizontal step being the first step, we multiply $W_{n-1,k}$ by q^k to account for the sqares of area one directly above the first step to the height of the path, k. Therefore, $H_n(q) = q^k W_{n-1,k}$. Then by assumption, $q^k W_{n-1,k}(q) = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$.

All of this give us the following equation

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} = W_{n,k}(q) = \sum_{\pi} q^{w(\pi)},$$

which is clearly counted by the q-binomial coefficients as shown in Lemma 2.3.1. □

2.4 q-Catalan numbers and lattice paths

There are several q-analogs of the Catalan numbers. Here, we present the one given by MacMahon [7]: For $n \geq 0$,

$$C_n(q) = \frac{1-q}{1-q^{n+1}} \begin{bmatrix} 2n \\ n \end{bmatrix}. \tag{2.3}$$

This will nicely reduce to the Catalan numbers as $q \to 1$. Just like the Catalan numbers, the q-analog of the Catalan numbers similarly counts the Catalan paths with a certain weight.

We first need to introduce a weight function $w_c(\pi)$ defined on the set of paths. For a path π, let $(\pi_x(i), \pi_y(i))$ be the ith point on the path from its initial point, and we define

$$w_c(\pi) = \sum (\pi_x(i) + \pi_y(i)),$$

where the sum is over all i satisfying

$$\pi_x(i-1) + 1 = \pi_x(i) = \pi_x(i+1) \text{ and } \pi_y(i-1) = \pi_y(i) = \pi_y(i+1) - 1. \tag{2.4}$$

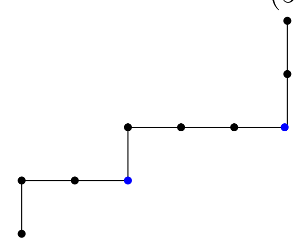

Figure 2.6: Lattice path with weight $w_c = 10$

For instance, let π be the lattice path above. The blue points $(2,1)$ and $(5,2)$ satisfy the conditions in (2.4), so $w_c(\pi) = (2+1) + (5+2) = 10$.

Here we note that the conditions in (2.4) represent points that are obtained by a $(1,0)$ step followed by a $(0,1)$ step.

We are now in position to state the combinatorial interpretation of the q-Catalan numbers given in (2.3). For a positive integer n, we take the following sum:

$$\sum_{\pi} q^{w_c(\pi)}, \tag{2.5}$$

where the sum is over all paths from $(0,0)$ to (n,n) that never fall below the line $y = x$.

We again look at an example to understand this better. Consider the lattice paths from $(0,0)$ to $(3,3)$ counted by the Catalan number C_3. The first path shown in Figure 2.7 has no point satisfying (2.4), but the other paths have such points. Thus, the sum in (2.5) becomes

$$q^0 + q^{2+2} + q^{1+2} + q^{1+1} + q^{(1+1)+(2+2)} = 1 + q^2 + q^3 + q^4 + q^6,$$

which equals $C_3(q)$:

$$\begin{aligned} C_3(q) &= \frac{1-q}{1-q^4}\frac{(1-q^4)(1-q^5)(1-q^6)}{(1-q)(1-q^2)(1-q^3)} \\ &= \frac{(1-q^5)(1-q^6)}{(1-q^2)(1-q^3)} \\ &= \frac{(1-q^5)(1+q^3)}{(1-q^2)} \\ &= \frac{1+q^3-q^5-q^8}{1-q^2} \\ &= \frac{(1-q^8)+q^3(1-q^2)}{1-q^2} \\ &= (1+q^2)(1+q^4)+q^3 \\ &= 1+q^2+q^3+q^4+q^6. \end{aligned}$$

The following theorem is essential in proving that the sum in (2.5) is indeed the q-Catalan number $C_n(q)$. However, the proof of the theorem requires further results that will not be used in the rest of this thesis. Thus, we just state it without proof.

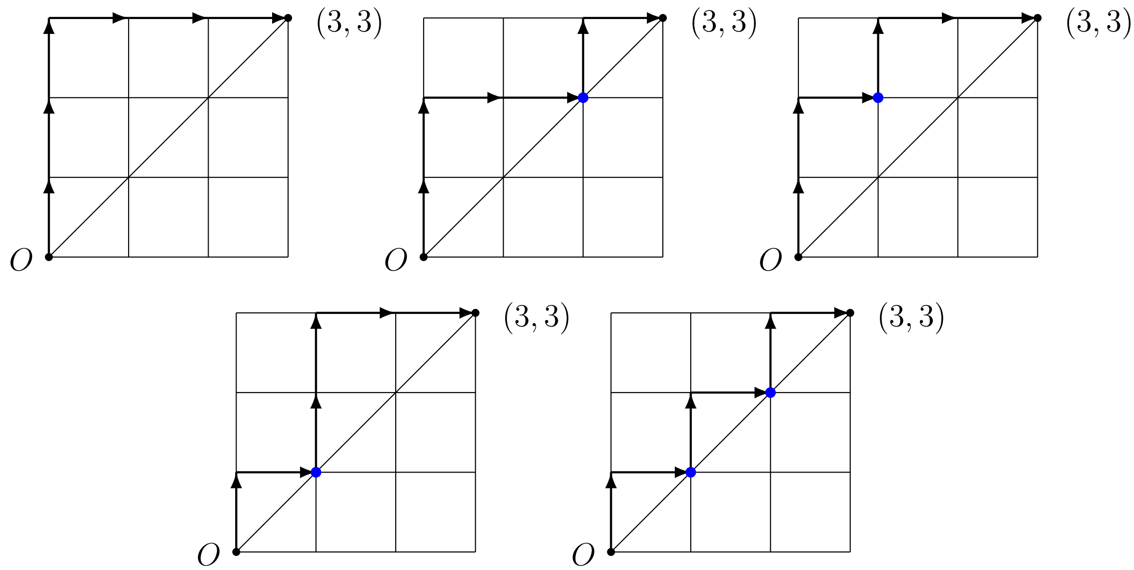

Figure 2.7: q-Catalan lattice paths for $n = 3$

Theorem 2.4.1 (MacMahon [7]) *For any nonnegative integers n, k with $0 \leq k \leq n$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\pi} q^{w_c(\pi)},$$

where the sum is over all paths from $(0, 0)$ to $(n - k, k)$.

Theorem 2.4.2 *For a positive integer n,*

$$C_n(q) = \sum_{\pi} q^{w_c(\pi)},$$

where the sum is over all paths from $(0, 0)$ to (n, n) that never fall below the line $y = x$.

Proof. We present a proof that is given in [3] with a diagram related to it shown in Figure 2.8. As defined in the proof of Theorem 2.2.1, we let X_n be the number of paths from $(0, 0)$ to (n, n) that go below the line $y = x$. Then any path π counted by X_n has at least one point below the line $y = x$. Among those lattice points, let P be the point (x, y) with smallest x for which $x - y$ is maximal, i.e., whose distance from the line $y = x$ in a southeast direction is maximal. Let P' be the lattice point on π before P. There must be a $(1, 0)$ step connecting P' to P. Change this into a $(0, 1)$ step and shift the remainder of the path after P up one unit and left one unit. We now have a path π' from $(0, 0)$ to $(n - 1, n + 1)$, and moreover $w_c(\pi') = w_c(\pi) - 1$.

It is easy to see that this is reversible. Given a path π' from $(0, 0)$ to $(n - 1, n + 1)$, let P' be the point with maximal x-coordinate among those lattice points (x, y) in π' for which $y - x$ is maximal, i.e., whose distance from the line $y = x$ in a northwest direction is maximal. Thus,

$$\sum q^{w_c(\pi)} = \sum q^{w_c(\pi')+1} = q \begin{bmatrix} 2n \\ n+1 \end{bmatrix},$$

where the last equality follows from Theorem 2.4.1. Hence

$$\begin{aligned}\sum_{\pi} q^{w_c(\pi)} &= \sum q^{w_c(\pi)} - \sum q^{w_c(\pi)} \\ &= \begin{bmatrix} 2n \\ n \end{bmatrix} - q \begin{bmatrix} 2n \\ n+1 \end{bmatrix} \\ &= \frac{(q;q)_{2n}}{(q;q)_n (q;q)_n} - q \frac{(q;q)_{2n}}{(q;q)_{n+1}(q;q)_{n-1}} \\ &= \frac{(q;q)_{2n}}{(q;q)_n (q;q)_n} \frac{(1-q^{n+1})}{(1-q^{n+1})} - q \frac{(q;q)_{2n}}{(q;q)_{n+1}(q;q)_{n-1}} \frac{(1-q^n)}{(1-q^n)} \\ &= \frac{(q;q)_{2n}[(1-q^{n+1}) - q(1-q^n)]}{(q;q)_n (q;q)_{n+1}} \\ &= \frac{1-q}{1-q^{n+1}} \begin{bmatrix} 2n \\ n \end{bmatrix},\end{aligned}$$

where the sum on the left hand side is over all paths counted by C_n, the first sum on the right hand side is over all paths from $(0,0)$ to (n,n), and the second sum on the right had side is over all paths conted by X_n. □

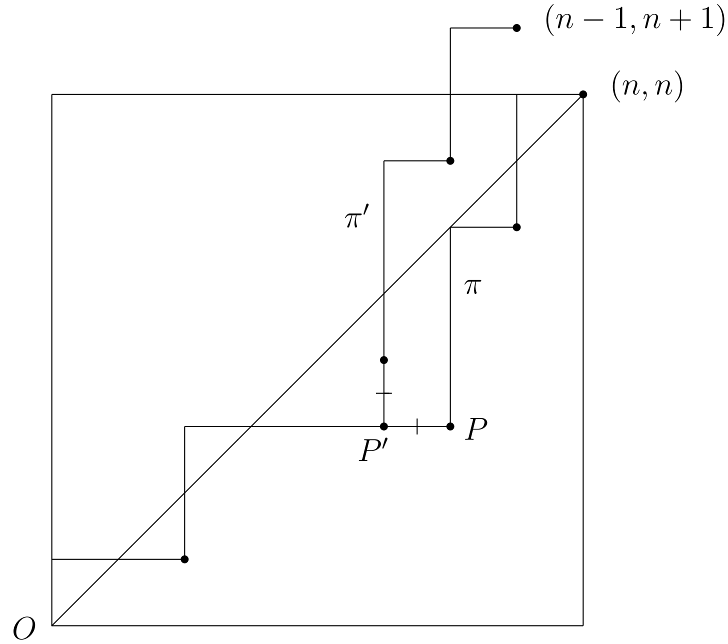

Figure 2.8: Lattice path crossing the line $y = x$ for the q-analog

Chapter 3

Andrews' q-Catalan identity

The Catalan numbers C_n satisfy various identities. In [2], George Andrews investigated some identities of C_n and derived their q-analogs. Our interest focuses on the Koshy identity and its q-analog of Andrews. At the end of his paper, Andrews proposed five problems on his q-analog. The main purpose of this chapter is to answer his first question and to present an attempt for the second question. In Section 3.1, we first provide a detailed proof of the q-Koshy identity. The questions of Andrews are stated in Section 3.2, and the answer to the first question is given in Section 3.3.

3.1 Proof of Andrews' q-Koshy identity

Here we state the recursive formula given by Koshy without proof.

Theorem 3.1.1 (Koshy [6]) *For $n \geq 1$, we have*

$$C_n = \sum_{r=1}^{n} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r}.$$

The object of the paper [2] is to emphasize the methods of applying hypergeometric series for finding q-analogs. As an example, Andrews established the following q-analog of the above formula of Koshy by applying q-series manipulations.

Theorem 3.1.2 *For $n \geq 1$, we have*

$$C_n(q) = \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-2} \begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q C_{n-r}(q) \frac{(-q^{n-r+1};q)_r}{(-q;q)_r}, \tag{3.1}$$

where $C_n(q)$ is the q-Catalan number defined in Chapter 2.

Although this is one of the main results in [2], Andrews gave only a sketch of his proof. Thus we here provide a detailed proof. Before we proceed, we first gather some preliminaries that required for the proof.

For any $a, b, c, z, q \in \mathbb{C}$ with $|z|, |q| < 1$, define

$${}_2\phi_1(a, b; c; q, z) = \sum_{j=0}^{\infty} \frac{(a;q)_j (b;q)_j}{(q;q)_j (c;q)_j} z^j.$$

Theorem 3.1.3 (q-Chu-Vandermonde Summation [5]) *For any positive integer n,*

$${}_2\phi_1(a, q^{-n}; c; q, cq^n/a) = \frac{(c/a;q)_n}{(c;q)_n}.$$

As a special case of the q-Chu-Vandermonde summation formula stated above, we obtain the following result.

Corollary 3.1.4 *For any positive integer n,*

$${}_2\phi_1(q^{-n-1}, q^{-n}; q^{1-2n}; q^2, q^2) = 0.$$

Proof. We first consider the case when $n = 2k$. Setting $a = q^{-2k-1}, c = q^{1-4k}$ and replacing q by q^2 give

$$ {}_2\phi_1(q^{-2k-1}, q^{-2k}; q^{1-4k}; q^2, q^2) = \frac{(q^{2-2k}; q^2)_{2k}}{(q^{1-4k}; q^2)_{2k}} = 0 $$

since

$$ (q^{2-2k}; q^2)_{2k} = (1 - q^{2-2k})(1 - q^{4-2k}) \cdots (1 - q^0)(1 - q^2) \cdots (1 - q^{2k}) = 0. $$

Similarly, for $n = 2k - 1$,

$$ \begin{aligned} {}_2\phi_1(q^{-2k}, q^{-2k+1}; q^{3-4k}; q^2, q^2) &= {}_2\phi_1(q^{-2k+1}, q^{-2k}; q^{3-4k}; q^2, q^2) \\ &= \frac{(q^{2-2k}; q^2)_{2k}}{(q^{3-4k}; q^2)_{2k}} = 0. \end{aligned} $$

□

Proof of Theorem 3.1.2. First,

$$ \begin{aligned} &{}_2\phi_1\left(q^{-n-1}, q^{-n}; q^{1-2n}; q^2, q^2\right) = \sum_{r=0}^{\infty} \frac{(q^{-n-1}; q^2)_r (q^{-n}; q^2)_r q^{2r}}{(q^2; q^2)_r (q^{1-2n}; q^2)_r} \\ &= \sum_{r=0}^{\infty} \frac{(1 - q^{-n-1})(1 - q^{-n+1})...(1 - q^{-n+2r-3})(1 - q^{-n})(1 - q^{-n+2})...(1 - q^{-n+2r-2})q^{2r}}{(q^2; q^2)_r (1 - q^{1-2n})(1 - q^{1-2n+2})...(1 - q^{1-2n+2r-2})} \\ &= \sum_{r=0}^{\infty} \frac{(1 - q^{n+1})(1 - q^{n-1})...(1 - q^{n-2r+3})(1 - q^{n})(1 - q^{n-2})...(1 - q^{n-2r+2})q^{2r}}{(q^2; q^2)_r (1 - q^{2n-1})(1 - q^{2n-3})...(1 - q^{2n-2r+1})} (-1)^r \\ &= \sum_{r=0}^{\infty} (-1)^r q^{r^2-r} \frac{(q^{n-2r+2}; q)_{2r}}{(q^2; q^2)_r (q^{2n-2r+1}; q^2)_r} \\ &= \sum_{r=0}^{\infty} (-1)^r q^{r^2-r} \frac{(q^{n-2r+2}; q)_r (q^{n-r+2}; q)_r}{(q; q)_r (-q; q)_r (q^{2n-2r+1}; q^2)_r} \\ &= \sum_{r=0}^{\infty} (-1)^r q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} \frac{(q^{n-r+2}; q)_r}{(-q; q)_r (q^{2n-2r+1}; q^2)_r} \\ &= 1 + \sum_{r=1}^{\infty} (-1)^r q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} \frac{(q^{n-r+2}; q)_r}{(-q; q)_r (q^{2n-2r+1}; q^2)_r}. \end{aligned} $$

Thus, it follows from Corollary 3.1.4 that

$$ 1 = \sum_{r=1}^{\infty} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} \frac{(q^{n-r+2}; q)_r}{(-q; q)_r (q^{2n-2r+1}; q^2)_r}. $$

Then

$$ C_n(q) = \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} \frac{(q^{n-r+2}; q)_r}{(-q; q)_r (q^{2n-2r+1}; q^2)_r} C_n(q) $$

$$= \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} \frac{(q^{n-r+1};q)_{r+1}}{(-q;q)_r (q^{2n-2r+1};q^2)_r (1-q^{n-r+1})} \frac{1-q}{1-q^{n+1}} \begin{bmatrix} 2n \\ n \end{bmatrix}$$
$$= \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} \frac{1-q}{1-q^{n-r+1}} \frac{(q^{n-r+1};q)_{r+1}}{(q^{2n-2r+1};q^2)_r} \frac{(q;q)_{2n}}{(q;q)_n (q;q)_n (1-q^{n+1})} \frac{1}{(-q;q)_r}$$
$$= \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} \frac{1-q}{1-q^{n-r+1}} \frac{(q^{n-r+1};q)_{r+1}}{(q^{2n-2r+1};q^2)_r} \frac{(q;q)_{2n-2r} (q^{2n-2r+1};q)_{2r}}{(q;q)^2_{n-r} (q^{n-r+1};q)^2_r (1-q^{n+1})} \frac{1}{(-q;q)_r}$$
$$= \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} \frac{1-q}{1-q^{n-r+1}} \begin{bmatrix} 2(n-r) \\ n-r \end{bmatrix} \frac{(q^{n-r+1};q)_{r+1} (q^{2n-2r+1};q)_{2r}}{(q^{2n-2r+1};q^2)_r (q^{n-r+1};q)^2_r (1-q^{n+1})} \frac{1}{(-q;q)_r}$$
$$= \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} C_{n-r}(q) \frac{(q^{2n-2r+1};q)_{2r}}{(q^{2n-2r+1};q^2)_r (q^{n-r+1};q)_r} \frac{1}{(-q;q)_r} \frac{(-q^{n-r+1};q)_r}{(-q^{n-r+1};q)_r}$$
$$= \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} C_{n-r}(q) \frac{(q^{2n-2r+2};q^2)_r}{(q^{n-r+1};q)_r (-q^{n-r+1};q)_r} \frac{(-q^{n-r+1};q)_r}{(-q;q)_r}$$
$$= \sum_{r=1}^{n} (-1)^{r-1} q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix} C_{n-r} \frac{(-q^{n-r+1};q)_r}{(-q;q)_r},$$

as desired. □

3.2 Questions posed by Andrews

By Theorem 3.1.2, we can rewrite (3.1) as

$$C_n(q) = \sum_{r=1}^{n} (-1)^{r-1} T_r(n,q) \tag{3.2}$$

where

$$T_r(n,q) = q^{r^2-r} \begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q C_{(n-r)}(q) \frac{(-q^{n-r+1};q)_r}{(-q;q)_r}. \tag{3.3}$$

We list Andrews' five questions below.

1. Show that $T_r(n,q)$ is a polynomial.
2. If $2r \leq n$, show that all the coefficients in $T_r(n,q)$ are nonnegative.
3. Show that $T_{r+1}(2r+1,-q)$ has non-negative coefficients.
4. Provide a partition-theoretic interpretation of $T_r(n,q)$ for $2r \leq n$ and for $T_{r+1}(2r+1,-q)$.
5. In light of the fact that $C_n(q)$ generates the Catalan partitions with largest part $< n$ and number of parts $\leq n$, show by using Problem 4 to interpret the right-hand side of (5.2), that a sieve process eliminates all non-Catalan partitions.

3.3 Showing that $T_r(n,q)$ is a polynomial

$T_r(n,q)$ alone is an interesting equation to consider, so we turn our attention to it and its properties. The first problem proposed by George E. Andrews at the end of his paper is the to "show that $T_r(n,q)$ is a polynomial."

Using manipulation, the definitions of the q-Catalan numbers, the q-binomial coefficients, and $(q;q)_n$, the problem can be easily solved. While the expanded form of $(q;q)_n$ will not be used in this proof, it is important to understand its definition.

We begin by expanding $T_r(n,q)$ using (3.3) and the definition of the q-binomial coefficients and q-Catalan numbers.

$$
\begin{aligned}
T_r(n,q) &= q^{r^2-r}\begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q C_{(n-r)}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r} \\
&= \frac{(q;q)_{n-r+1}(1-q)(q;q)_{2(n-r)}(-q^{n-r+1};q)_r}{(q;q)_r(q;q)_{n-2r+1}(1-q^{n-r+1})(q;q)_{n-r}(q;q)_{n-r}(-q;q)_r} \\
&= \frac{(q^{n-2r+2};q)_r(1-q)(q^{n-r+1};q)_{n-r}(-q^{n-r+1};q)_r}{(q;q)_r(1-q^{n-r+1})(q;q)_{n-r}(-q;q)_r} \\
&= \frac{(q^{n-2r+2};q)_r(1-q)(q^{n-r+1};q)_{n-r}(-q^{n-r+1};q)_r}{(q^2;q^2)_r(1-q^{n-r+1})(q;q)_{n-r}} \\
&= \frac{(q^{n-2r+2};q)_r(1-q)(q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r}}{(q^2;q^2)_r(1-q^{n-r+1})(q;q)_{n-r}} \\
&= \frac{(q^{n-2r+2};q)_{r-1}(1-q)(q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r}}{(q^2;q^2)_r(q;q)_{n-r}} \\
&= \frac{(1-q)(q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r}}{(q^2;q^2)_r(q;q)_{n-2r+1}} \\
&= \frac{[(1-q^{2n-2r+1})-q(1-q^{2n-2r})](q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r}}{(q^2;q^2)_r(q;q)_{n-2r+1}} \\
&= \frac{(1-q^{2n-2r+1})(q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r}}{(q^2;q^2)_r(q;q)_{n-2r+1}} - q\frac{(1-q^{2n-2r})(q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r}}{(q^2;q^2)_r(q;q)_{n-2r+1}} \\
&= \frac{(q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r+1}}{(q^2;q^2)_r(q;q)_{n-2r+1}} - q\frac{(1-q^{2n-2r})(q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r}}{(q^2;q^2)_r(q;q)_{n-2r+1}} \\
&= \frac{(q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r+1}}{(q^2;q^2)_r(q;q)_{n-2r+1}} - q\frac{(1-q^{2n})(q^{2(n-r)};q^2)_r(q^{n+1};q)_{n-2r}}{(q^2;q^2)_r(q;q)_{n-2r+1}} \\
&= \frac{(q^{2(n-r+1)};q^2)_r(q^{n+1};q)_{n-2r+1}}{(q^2;q^2)_r(q;q)_{n-2r+1}} - q\frac{(1+q^{n})(q^{2(n-r)};q^2)_r(q^{n};q)_{n-2r+1}}{(q^2;q^2)_r(q;q)_{n-2r+1}} \\
&= \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}\begin{bmatrix} 2n-2r+1 \\ n-r \end{bmatrix}_q - q(1+q^n)\begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2}\begin{bmatrix} 2(n-r) \\ n-1 \end{bmatrix}_q.
\end{aligned}
$$

The q-binomial coefficients each form a polynomial as shown in Theorem 2.3.2. We also know that when two polynomials are multiplied together, another polynomial is formed. Because the above equation is comprised of only q-binomial terms and two polynomials q and $(1+q^n)$, it is easily seen that $T_r(n,q)$ is a polynomial.

3.4 Coefficients in $T_r(n,q)$

The second problem proposed by Andrews is in regards to the coeffients of $T_r(n,q)$. When $2r \le n$, the coefficients are all non-negative and we are asked to show that. In Section 3.3, we found a different version of $T_r(n,q)$:

$$T_r(n,q) = \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2n-2r+1 \\ n-r \end{bmatrix}_q - q(1+q^n) \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2(n-r) \\ n-1 \end{bmatrix}_q ,$$

which we will use to begin our analysis. On the right hand side, the products of two q-binomial coefficients can be interpreted as a generating function of pairs of lattice paths. If we show that for a fixed weight there are more pairs counted by the first product than by the second product, this would show the non-negativity question. We couldn't show it in the general case yet. Instead, here we show that the total number of pairs of paths counted by the first product is greater than or equal to the total number of paths counted by the second product. Namely, we set $q = 1$ and show that

$$\binom{n}{r}\binom{2n-2r+1}{n} - 2\binom{n-1}{r}\binom{2n-2r}{n-1}$$

is non-negative.

$$\begin{aligned}
&\binom{n}{r}\binom{2n-2r+1}{n} - 2\binom{n-1}{r}\binom{2n-2r}{n-1} \\
&= \frac{n!}{r!(n-r)!}\frac{(2n-2r+1)!}{n!(n-2r+1)!} - 2\frac{(n-1)!}{r!(n-r-1)!}\frac{(2n-2r)!}{(n-1)!(n-2r+1)!} \\
&= \frac{(2n-2r+1)!}{r!(n-r)!(n-2r+1)!} - 2\frac{(2n-2r)!}{r!(n-r-1)!(n-2r+1)!} \\
&= \frac{(2n-2r+1)(2n-2r)!}{r!(n-r)(n-r-1)!(n-2r+1)!} - 2\frac{(2n-2r)!}{r!(n-r-1)!(n-2r+1)!} \\
&= \frac{(2n-2r)!}{r!(n-r-1)!(n-2r+1)!}\left[\frac{2n-2r+1}{n-r} - 2\right] \\
&= \frac{(2n-2r)!}{r!(n-r-1)!(n-2r+1)!}\left[\frac{2(n-r)}{n-r} + \frac{1}{n-r} - 2\right] \\
&= \frac{(2n-2r)!}{r!(n-r-1)!(n-2r+1)!(n-r)} \\
&= \frac{(2n-2r)!}{r!(n-r)!(n-2r+1)!} \ge 0.
\end{aligned}$$

With $2r \le n$, all factorials will be positive allowing for the whole equation to be positive also.

Chapter 4

Conclusion and Future Work

This thesis introduced many topics with the q-Catalan numbers and Andrews' identity at the focal point. Chapter 2 presented the definitions and meanings of the binomial coefficients, the Catalan numbers, and the q-analog for each. With all of these, we also described their lattice path interpretation to help with the understanding of the concepts. Disscution regarding the Catalan numbers and their q-analog went into the most detail with proofs of theroerms and concepts being provided. Chapter 3 begins with an introduction to Andrews' q-Koshy identity and a proof of its derivation from the q-Chu-Vandermonde Summation. From there, the questions posed by Andrews' at the end of his paper are listed and then explored. A complete solution to the first problem is provided in Section 3.3. A solution of the second problem for a special case of $q = 1$ is given in Section 3.4, and some insight is offered as how to approach the general case.

Aside from completing Problem 2, the last three problems are all open to exploration for solutions. The third problem asks to show that $T_{r+1}(2r + 1, -q)$ has non-negative coefficients. This can be shown when $q = 1$ in a similar way as the second problem was shown. It is not clear how to prove the general q case though. It is possible however, that this may be shown using the same process used in the second problem.

The fourth problem asks for a partition-theoretic interpretation of $T_r(n, q)$ and $T_{r+1}(2r+1, -q)$. Using the idea of lattice paths from the second problem is a perfect place to start with the interpretation, especially if they are a successful explanation of the non-negative coefficients for Problems 2 and 3.

The fifth and final problem posed by Andrews uses Problem 4 to show that a sieve process eliminates all non-Catalan partitions. This is where the lattice path interpretation may be useful again, especially if we are able to show that the non-Catalan partitions are the lattice paths that cross the main diagonal for the paths. This work and the work provided in the previous paragraphs is just the beginning of the future work with Andrews' q-Koshy identity, but a good start for now.

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ACADEMIC VITA

Sara Catherine Cobern
sara.cobern.12@gmail.com

Education

The Pennsylvania State University ***August 2011- Spring 2015***

Bachelor of Science, Mathematics, Systems Analysis Option
Bachelor of Science, Statistics, Applied
Minor in Economics

- Schreyer Honors Scholar
- PMASS Program, Spring 2013
- President's Award, Freshmen Year
- Mary Lister McCammon Scholarship, Mathematics
- Dean's List, all semesters

Semester at Sea ***Fall 2013***

Study Abroad

- Traveled Atlantic continents for four months via ship while taking classes on the ship
- Credit awarded through the University of Virginia

Activities

Joint Program in Survey Methodology Intern ***Summer 2014***

- Contracted to work with the National Center for Education Statistics (NCES) through American Institutes for Research (AIR)
- Created a working aid looking at state and regional schools and agencies
- Completed an Excel document for LEA membership working group

Kappa Phi Club President ***May 2014 – May 2015***
Secretary and Treasurer ***May 2012 – May 2013***

- Planned and facilitated weekly meetings and rituals
- Wrote and distributed reports after each meeting in a timely fashion
- Prepared the budget and kept track of all monetary expenses and incomes using Excel

Abba Java Coffeehouse Volunteer ***August 2014 – May 2015***

- Volunteered time weekly at student coffeehouse
- Collected data on attendance for different times of day
- Operated the counter and improved appearance of area consistently

Penn State Math Proctor and Grader ***August 2012 – May 2015***

- Graded papers weekly in a timely manner
- Assisted professors in proctoring exams