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Fock Spaces

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Abstract

Fock spaces were introduced by Vladimir Aleksandrovich Fock in 1932, in order to model the quantum states of a variable number of particles when the energy allows particle creation and annihilation. In this paper we present the particle and wave interpretations of Fock spaces. Fock spaces can be interpreted as spaces of waves, or as spaces of an indeterminate number of particles, which makes them useful in quantum field theory. We prove the equivalence of these two formulations, and furthermore show an equivalence between two formulations of the inner product on Fock spaces.

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Chapter 1

Introduction

In theoretical physics, quantum mechanics describes a fixed number of particles, for instance an atom. In contrast to it, quantum field theory describes a variable number of particles with all their interactions. The greatest open problem in theoretical science is an exact model for quantum field theory and gravity. In what follows, we shall describe mathematical aspects of the first interesting quantum field, namely the free (that is, non-interacting) bosonic field, which describes a variable number of particles of a single kind, which are bosons, i.e. they are described by a function which is symmetric when the particles are interchanged. The quantum field theory we are studying is perturbative, that is, we use Taylor series around classical solutions.

The mathematical structure of a bosonic field is called the bosonic Fock space and can be described in three very different but equivalent forms. Our main goal in this paper is to describe and construct the mathematical equivalence between these three points of view.

If a finite set X describes the positions of an object, the corresponding quantum system is described by a unit vector in $H = \mathbb{C}^X$, the set of functions from X to $\mathbb{C}$, which we view as linear combinations of classical positions with complex coefficients having the sum of the square moduli equal to 1. For continuous spaces X, the vector space $\mathbb{C}^X$ is replaced by $H = L^2(X, \mu, \mathbb{C})$ for a measure μ on X. Thus H is the space of square-integrable complex-valued functions, which describes one particle. In this case, $H \otimes H$ will have basis $X \times X$, the set of pairs of positions in X, and thus will describe two particles. The two particles are non-interacting (free), else $H \otimes H$ would be reduced to a subspace. In a feature very specific to quantum mechanics, n particles of the same kind interact in the sense that they cannot be distinguished. Moreover, either the vectors in $H^{\otimes n}$ describing n particles are **symmetric**, such as a vector of the form $v \otimes w + w \otimes v$, in which case the particle is called a **boson**, or the vectors are **antisymmetric**, such as $v \otimes w - w \otimes v$, in which case the particle is called a **fermion**. No other type of behavior of particles has been experimentally observed. Bosonic quantum fields can be described either as formal exponentials, corresponding to waves, or as vectors in the symmetric Fock space, corresponding to particles, or as limits of polynomial functions on the Hilbert space H. We shall construct correspondences between these three structures and prove that these maps are isomorphisms.

Thus the passage from the classical set X of states to this quantum version $H = \mathbb{C}^X$ is mathematically an exponentiation, which is called the **first quantization**. The passage from a single particle to an arbitrary number of particles is mathematically also an exponentiation, from H to a symmetric version of e^H. This procedure is called the **second quantization**.

Chapter 2

The Quantum Field Theory State Spaces

2.1 The Tensor Product, or Particle Space

The first representation of Fock spaces we will consider is used to model the quantum state of an indeterminate number of particles. The major idea of this representation is that we will construct a Fock space as an infinite direct sum of component spaces, each of which will model a possible state of a fixed number of particles. In that way we can have vectors, consisting of components from each of these component spaces, that will model an indeterminate number of particles. This idea motivates the following definitions, which will be used to rigorously define this interpretation of Fock spaces.

Definition [1]: A *Hilbert space* H is a vector space over $\mathbb{C}$ (resp. $\mathbb{R}$), equipped with an inner product $\langle .,. \rangle : H \times H \to \mathbb{C}$ (resp. $\mathbb{R}$) that satisfies the following conditions: $\forall$ $x, y, z \in H$ and $\forall$ a, b $\in \mathbb{C}$ (resp. $\mathbb{R}$) we have:
1) $\overline{\langle x, y \rangle}$ = $\langle y, x \rangle$
2) $\langle ax + bz, y \rangle = a\langle x, y \rangle + b\langle z, y \rangle$
3) $\langle x, x \rangle \geq 0$, with equality if and only if $x = 0$
Furthermore, H must be complete with respect to the metric induced by the following norm: $\|x\|$:= $\sqrt{\langle x, x \rangle}$.

Given two vector spaces *H* and *K* over a field $\mathbb{F}$, we define the *tensor product*, denoted *H* $\otimes$ *K*, to be the vector space whose elements are equivalence classes of vectors in the free vector space $F(H \times K)$ subject to the following equivalence relations:
1) $(v, w) + (u, w) \equiv (u + v, w)$ $\forall u, v \in H$ and $w \in K$
2) $(v, w) + (v, y) \equiv (v, w + y)$ $\forall v \in H$ and $w, y \in K$
3) $c(v, w) \equiv (cv, w) \equiv (v, cw)$ $\forall c \in \mathbb{F}$, $v \in H$ and $w \in K$

The definition above holds for any Hilbert space. If we take H and K to be finite-dimensional vectors spaces, with bases B_H and B_K, then $B_H \times B_K$ is a basis for $H \otimes K$, that is, we can take vectors in our tensor product to be linear combinations of the product of vectors in the bases of H and K, subject to the equivalence relations above.

If H, K are Hilbert spaces, we define the inner product on $H \otimes K$ as follows: for $h, k \in v \otimes v$ such that $h = \sum_{i=1}^{n} a_i v_i, k = \sum_{j=i}^{m} b_j w_j$, where $a_i, b_j \in \mathbb{C}$ and $v_i = v_{i_1} \otimes v_{i_2}, w_j = w_{j_1} \otimes w_{j_2}$, then we let $\langle h, k \rangle = \sum_{i,j} a_i \overline{b_j} \langle v_{i_1}, w_{j_1} \rangle \langle v_{i_2}, w_{j_2} \rangle$ (where $\langle v_{i_1}, w_{j_1} \rangle$ and $\langle v_{i_2}, w_{j_2} \rangle$ denote the inner products in H, K respectively). Essentially, given two pure tensors, the inner product is the product of the inner products of the first and second component vectors, respectively.

This inner product is exactly the inner product we want to use for symmetric tensors. If we take $H \otimes H$, a symmetric tensor is one of the form $v = v_1 \otimes v_2 + v_2 \otimes v_1$. If we have symmetric v, w, then the inner product $\langle v, w \rangle$ is

$$\langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle + \langle v_1, w_2 \rangle + \langle v_2, w_1 \rangle$$

Which is symmetric, that is, it is preserved under permutation of the indices of v and w.

For $v \in V$ and $w \in W$ we denote the equivalence class of $v \times w$ as $v \otimes w$. Thus $V \otimes W$ is spanned by elements of the form $v \otimes w$ for $v \in V$ and $w \in W$. Such elements are called pure tensors.

In order to prove that the tensor product of Hilbert spaces is itself a Hilbert space, we need the following definitions:

Definition: Given $n \times m$ matrices A, B, we define the *Hadamard product* of A, B, denoted $A \circ B$, to be their element-wise product.

For example, given the 2×2 matrices

$$A = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 9 \\ -1 & 6 \end{pmatrix}$$

Then their Hadamard product is

$$A \circ B = \begin{pmatrix} 2 \times 1 & 5 \times 9 \\ 3 \times -1 & 7 \times 6 \end{pmatrix} = \begin{pmatrix} 2 & 45 \\ -3 & 42 \end{pmatrix}.$$

Definition: A Hermitian $n \times n$ matrix A over $\mathbb{C}$ is called *positive-definite* (resp. *positive-semidefinite*) if $\forall v \in \mathbb{C}^n, v \neq 0$, $v^*\mathrm{A}v > 0$ (resp. $v^*\mathrm{A}v \geq 0$), where v^* denotes the conjugate transpose of v.

Along with these definitions, we need the following theorem:

Schur Product Theorem: If A, B are $n \times n$ positive-definite matrices (resp. positive-semidefinite) over $\mathbb{C}^n$, then $A \circ B$ is positive definite (resp. positive-semidefinite).

Proof. [2] Suppose A, B are positive-semidefinite. For a nonzero vector a, $a^*(A \circ B)a = Tr(diag(\overline{a})Adiag(a)\overline{B})$, where $diag(a)$ is the diagonal nxn matrix with the components of a as its entries. Because B is positive-semidefinite, it has a positive-semidefinite square root, $\sqrt{B}$. Then $Tr(diag(\overline{a})Adiag(a)\overline{B}) = Tr(\sqrt{\overline{B}}diag(\overline{a})Adiag(a)\sqrt{\overline{B}}) = Tr(C^*AC)$, where we let $C = diag(a)\sqrt{\overline{B}}$. It follows that C^*AC is positive-semidefinite, so it has nonnegative trace, and thus $a^*(A \circ B)a \geq 0$. Thus $A \circ B$ is positive-semidefinite.

If A, B are positive-definite, then their diagonal entries are all positive. Then all the eigenvalues of A are positive, so let λ_1 be the smallest, and let β be the smallest diagonal entry of B. $A - \lambda_1 I$ has nonnegative eigenvalues, so it is positive-semidefinite. Then by the first part, $(A - \lambda_1 I \circ B)$ is positive-semidefinite. Then $0 \leq a^*((A - \lambda_1 I) \circ B)a = a^*(A \circ B)a - a^*(\lambda_1 \circ B)a$, but $a^*(\lambda_1 \circ B)a > 0$ is clear since B is positive-definite. Thus $(A \circ B)$ is positive-definite. ∎

Theorem 1: Let H, K be Hilbert spaces over $\mathbb{C}$. Then the completion of $H \otimes K$ is a Hilbert space over $\mathbb{C}$.

Proof. 1) Since both H and K have a valid inner product, we can use property 1) of their inner products and the result follows directly

$$\begin{aligned}
\overline{\langle h, k\rangle} &= \overline{\sum_{i,j} a_i \overline{b_j} \langle v_{i_1}, w_{j_1}\rangle \langle v_{i_2}, w_{j_2}\rangle} \\
&= \sum_{i,j} \overline{a_i} b_j \overline{\langle v_{i_1}, w_{j_1}\rangle \langle v_{i_2}, w_{j_2}\rangle} \\
&= \sum_{i,j} \overline{a_i} b_j \langle w_{j_1}, v_{i_i}\rangle \langle w_{j_2}, v_{i_2}\rangle \\
&= \langle k, h\rangle
\end{aligned}$$

So we have property 1).
2) Suppose that $g = \sum_{l=1}^{d} c_l u_l \in H \otimes K$, for $u_l = u_{l_1} \otimes u_{l_2}$, and that $x, y \in \mathbb{C}$. Then

$$\begin{aligned}
\langle xh + yg, k\rangle &= \sum_{i,j} x a_i \overline{b_j} \langle v_{i_1}, w_{j_1}\rangle \langle v_{i_2}, w_{j_2}\rangle + \sum_{l,j} y c_l \overline{w_j} \langle u_{l_1}, w_{j_1}\rangle \langle u_{l_2}, w_{j_2}\rangle \\
&= x \sum_{i,j} a_i \overline{b_j} \langle v_{i_1}, w_{j_1}\rangle \langle v_{i_2}, w_{j_2}\rangle + y \sum_{l,j} c_l \overline{w_j} \langle u_{l_1}, w_{j_1}\rangle \langle u_{l_2}, w_{j_2}\rangle \\
&= x\langle h, k\rangle + y\langle g, k\rangle \qquad (2.1)
\end{aligned}$$

So property 2) is satisfied.
3) To prove this property we will make use of the Hadamard product. The larger idea is that the Gram matrix of the inner product on H and K is the Hadamard product of the corresponding Gram matrices on H and K. For $h = \sum_{i=1}^{n} a_i v_i$, where $v_i = v_{i_1} \otimes v_{i_2}$, we may assume without loss of generality that $v_{i_1}, 1 \le i \le n$ and $v_{i_2}, 1 \le i \le n$ are both linearly independent sets of vectors (since if not, we can rearrange the tensors until they are). Then consider the following $n \times n$ Gram matrices:

$$A = \begin{pmatrix} \langle v_{1_1}, v_{1_1}\rangle & \langle v_{1_1}, v_{2_1}\rangle & \cdots & \langle v_{1_1}, v_{n_1}\rangle \\ \langle v_{2_1}, v_{1_1}\rangle & \langle v_{2_1}, v_{2_1}\rangle & \cdots & \langle v_{2_1}, v_{n_1}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_{n_1}, v_{1_1}\rangle & \langle v_{n_1}, v_{2_1}\rangle & \cdots & \langle v_{n_1}, v_{n_1}\rangle \end{pmatrix}$$

$$B = \begin{pmatrix} \langle v_{1_2}, v_{1_2}\rangle & \langle v_{1_2}, v_{2_2}\rangle & \cdots & \langle v_{1_2}, v_{n_2}\rangle \\ \langle v_{2_2}, v_{1_2}\rangle & \langle v_{2_2}, v_{2_2}\rangle & \cdots & \langle v_{2_2}, v_{n_2}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_{n_2}, v_{1_2}\rangle & \langle v_{n_2}, v_{2_2}\rangle & \cdots & \langle v_{n_2}, v_{n_2}\rangle \end{pmatrix}$$

Since we have valid inner products on H, K, these inner products satisfy property 3). This means that, since our vectors are linearly independent, their Gram matrices A and B positive-definite.

Thus their Hadamard product $A \circ B$ is positive-definite by Schur Product Theorem. This Hadamard product is exactly the Gram matrix of the vectors v_i in $H \otimes K$: indeed, let

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

And then $\langle h, h \rangle = \sum_{i,j} a_i \overline{a_j} \langle v_{i_1}, v_{j_1} \rangle \langle v_{i_2}, v_{j_2} \rangle = a^*(A \circ B)a > 0$, if $a \neq 0$, since $A \circ B$ is positive-definite. Thus $\langle h, h \rangle \geq 0$ for all $h \neq 0$, and $\langle 0, 0 \rangle = 0$ is clear. Thus $\langle h, h \rangle \geq 0$ with equality if and only if $h = 0$, so we have 3).

Thus we have a valid inner product on $H \otimes K$, and so its completion with respect to the metric induced by this inner product is a Hilbert space by definition. □

Clearly it follows inductively that the completion of an arbitrary number of tensor products of Hilbert spaces is as well a Hilbert space as well.

Now that we have defined tensor products and Hilbert spaces, we may give the first definition of a Fock space. In this view, we let H be a Hilbert space over $\mathbb{C}$. H represents the quantum state of a single particle in some quantum system, in which different vectors in H represent probabilities of different states. The tensor product of H with itself n times represents the possible quantum states of n identical particles.

Definition: Let H be a Hilbert space over $\mathbb{C}$. Let G be the completion of the space of all elements of the form $v = \bigoplus_{n=0}^{\infty} a_n v_n$, where $v_n \in H^{\otimes n}$, such that $\langle v, v \rangle := \sum_{n=0}^{\infty} n! \langle a_n v_n, a_n v_n \rangle < \infty$. Then we say that G is a *particle Fock space*.

The unit vectors in a Fock space G correspond to possible states of our quantum system. For a given unit vector, the probability that our system has n particles in it is $P_n = n! \langle a_n v_n, a_n v_n \rangle$, with $\sum_{n=0}^{\infty} P_n = 1$ by definition of the induced norm.

That the inner product on G is valid follows directly from the validity of the inner product on a tensor product. In G, for each n, $H^{\otimes n}$ is a subspace of our Fock space, corresponding to the space of n free particles.

G itself is a very large space, but there are two subspaces of particular interesting, its symmetric and antisymmetric parts, which correspond to the bosonic and fermionic Fock spaces, respectively. Symmetric tensors are those that are the same under permutation of their indices, while antisymmetric tensors are those that switch sign under permutation of their indices according to the sign of the permutation.

2.2 Creation and Annihilation Operators on the Particle Fock Space

Two of the most basic operators on the particle Fock space are the creation and annihilation operators. Creation and annihilation operators increase and decrease the number of particles in a space by one, respectively. The creation and annihilation operators are adjoint with each other, so we will define the creation operator on the symmetric Fock space and then find an explicit form for the annihilation operator by taking the adjoint of the creation operator.

Definition: Given a Hilbert space H and the space $H^{\otimes n}$, for a fixed vector $x \in H$ we define the creation operator

$$a^{\dagger}_{n_x} : Sym(H^{\otimes n}) \to Sym(H^{\otimes n+1})$$
$$a^{\dagger}_{n_x}(v_1 \otimes ... \otimes v_n) = Sym(x \otimes v_1 \otimes ... \otimes v_n)$$

That this is a linear operator is clear. Essentially, given a symmetric vector v in H^n, which some n particles in a space, we add an additional particle in the state x to the space, where $x \in H$. We then symmetrize the vector, so the end result is a symmetric vector in H^{n+1}.

We can now derive the annihilation operator

$$a_{n_x} : Sym(H^{n+1}) \to H^n$$

We denote the inner product on the n-particle subspace by $\langle \cdot, \cdot \rangle_n$. The adjoint operator is defined by the relation

$$\langle a^{\dagger}_{n_x} v, w \rangle_{n+1} = \langle v, a_{n_x} w \rangle_n$$

For $v \in H^n$, $w \in H^{n+1}$. So we can calculate $\langle a^{\dagger}_{n_x} v, w \rangle_{n+1}$ directly:

$$\begin{aligned} \langle a^{\dagger}_{n_x} v, w \rangle_{n+1} &= \frac{(n+1)!}{(n+1)!} \sum_{\tau \in S_{n+1}} \langle x, w_{\tau(n+1)} \rangle \prod_{i=1}^{n} \langle v_i, w_{\tau(i)} \rangle \\ &= \sum_{\tau \in S_{n+1}} \langle x, w_{\tau(n+1)} \rangle \prod_{i=1}^{n} \langle v_i, w_{\tau(i)} \rangle \end{aligned}$$

So we sum the inner product of $u \otimes v_1 \otimes ... \otimes v_n$ with all permutation of w.

Then we can define $a_{n_x} w = \sum_{i=1}^{n+1} \langle x, w_i \rangle Sym(\bigotimes_{1 \leq j \leq n+1, j \neq i} w_j)$.

Essentially, we take the inner product of x with each component of w, then multiply the symmetrized tensor product of the remaining n components of w, and sum all of these, along with a factor of $n!$.

That this is the adjoint of $a^\dagger_{n_x}$ is clear, since

$$\begin{aligned}\langle v, a_{n_x} w\rangle_n &= n! \sum_{i=1}^{n+1} \langle x, w_i\rangle \langle v, Sym(\otimes_{1\leq j\leq n+1, j\neq i} w_j)\rangle \\ &= \frac{n!}{n!} \sum_{\tau\in S_{n+1}} \langle x, w_{\tau(n+1)} \prod_{i=1}^{n} \langle v_i, w_{\tau(i)}\rangle \\ &= \langle a^\dagger_{n_x} v, w\rangle_{n+1}\end{aligned}$$

We can now easily extend these to operators on our particle Fock space G. We let $v = \bigoplus_{i=0}^{\infty} v_i \in G$, and then

$$a^\dagger_x v = 0 \bigoplus_{n=1}^{\infty} a^\dagger_{(n-1)_x} v_{n-1}$$

Since the inner product on G is defined by the sum of inner products in each of the n-particle subspaces $H^{\otimes n}$, it's clear that the corresponding annihilation operator on G is

$$a_x = \bigoplus_{n=0}^{\infty} a_{n_x} v_{n+1}$$

Creation operators clearly commute with each other, and thus so do annihilation operators. Creation and annihilation operators, however, do not commute with each other. But for $x, y \in H$, the commutator $a_y a^\dagger_x - a^\dagger_x a_y$ has a nice form. We let $v = \sum_{n=0}^{\infty} v_n \in Sym(G)$, then:

$$\begin{aligned}a_y a^\dagger_x v &= a_y(0 + \bigoplus_{n=1}^{\infty} a^\dagger_{(n-1)x} v_{n-1}) \\ &= \bigoplus_{n=0}^{\infty} a_{n_y} a^\dagger_{n_x} v_n\end{aligned}$$

And

$$\begin{aligned}a^{\dagger x} a_y v &= a^\dagger_x \bigoplus_{n=0}^{\infty} a_{n_x} v_{n+1} \\ &= 0 \bigoplus_{n=1}^{\infty} a^\dagger_{(n-1)_x} a_{(n-1)_x} v_n\end{aligned}$$

Now consider each fixed n. Then

$$\begin{aligned}a_{n_y} a^\dagger_{n_x} v_n &= a_{n_y} Sym(x \otimes v_1 \otimes ... \otimes v_n) \\ &= \langle y, x\rangle Sym(v_n) + \sum_{i=1}^{n} \langle y, v_i\rangle Sym(x \bigotimes_{1\leq j\leq n, j\neq i} v_j)\end{aligned}$$

And

$$\begin{aligned}
a^{\dagger}_{(n-1)x} a_{(n-1)_x} v_n &= a^{\dagger}_{(n-1)x} \sum_{i=1}^{n} \langle y, v_i \rangle Sym(\bigotimes_{1 \le j \le n, j \ne i} v_j) \\
&= Sym(x \otimes \left(\sum_{i=1}^{n} \langle y, v_i \rangle Sym(\bigotimes_{1 \le j \le n, j \ne i} v_j) \right)) \\
&= \sum_{i=1}^{n} \langle y, v_i \rangle Sym(x \bigotimes_{1 \le j \le n, j \ne i} v_j)
\end{aligned}$$

Then we have that

$$(a_y a_x^{\dagger} - a_x^{\dagger} a_y) v = (\bigoplus_{n=0}^{\infty} a_{n_y} a^{\dagger}_{n_x} v_n) - (0 \bigoplus_{n=1}^{\infty} a^{\dagger}_{(n-1)_x} a_{(n-1)_x} v_n)$$

So for a fixed n, we then have

$$\begin{aligned}
a_{n_y} a^{\dagger}_{n_x} v_n - a^{\dagger}_{(n-1)x} a_{(n-1)_x} v_n &= \langle y, x \rangle Sym(v_n) + \sum_{i=1}^{n} \langle y, v_i \rangle Sym(x \bigotimes_{1 \le j \le n, j \ne i} v_j) \\
&- \sum_{i=1}^{n} \langle y, v_i \rangle Sym(x \bigotimes_{1 \le j \le n, j \ne i} v_j) \\
&= \langle y, x \rangle Sym(v_n)
\end{aligned}$$

Which means that, since we take $v \in Sym(G)$,

$$(a_y a_x^{\dagger} - a_x^{\dagger} a_y) v = \langle y, x \rangle v$$

So the commutator is defined by

$$a_y a_x^{\dagger} - a_x^{\dagger} a_y : G \to G$$
$$(a_y a_x^{\dagger} - a_x^{\dagger} a_y) v = \langle y, x \rangle v$$

2.3 The Wave Construction of Fock Spaces

We will now present an alternate definition of Fock spaces, the wave construction. For a Hilbert space H over $\mathbb{C}$, our wave Fock space F will be the set of linear combinations of elements of the form e^v for $v \in H$. We view e^v as a formal object. Elements in our space are then linear combinations of these formal objects.

Definition: Let H be a Hilbert space over $\mathbb{C}$. We define the *wave Fock space* as the set $F = \{\sum_{i=1}^{n} a_i e^{v_i} \mid a_i \in \mathbb{C}, v_i \in H\}$. This defines a vector space over $\mathbb{C}$.

For two vectors $v = \sum_{i=1}^{n} a_i e^{v_i}, w = \sum_{j=1}^{m} b_j e^{w_j} \in F$, we define $p(v, w) = \sum_{i,j} a_i \overline{b_j} e^{\langle v_i, w_j \rangle}$, where $\langle v_i, w_j \rangle$ denotes the inner product in H, where this time $e^{\langle v_1, w_j \rangle}$ is the usual exponential for scalars; this is no longer just formal. This defines a function p on F in somewhat of an analogous manner to how we defined the inner product on tensor products, in that the inner product here is first defined for two elements of the form e^v, e^w and then linearly extended. We will prove shortly that p is a valid inner product on F.

Now let G denote a particle Fock space, both generated by the same Hilbert space H over $\mathbb{C}$. We will denote by $v_i^{\otimes k}$ the kth tensor product of v_i with itself, for $v_i \in H$ (where $v_i^{\otimes 0} = 0$ and $v_i^{\otimes 1} = v$). Let $v \in H$. We want to show that $\sum_{k=0}^{\infty} \dfrac{v^{\otimes k}}{k!}$ is a valid element of G. To show this, let v_n denote its nth partial sum. Assuming without loss of generality that $m > n$,

$$\begin{aligned} ||v_n - v_m||^2 &= \langle \sum_{k=n+1}^{m} \frac{v^{\otimes k}}{k!}, \sum_{k=n+1}^{m} \frac{v^{\otimes k}}{k!} \rangle \\ &= \sum_{k=n+1}^{m} k! \frac{\langle v, v \rangle^k}{(k!)^2} \\ &= \sum_{k=n+1}^{m} \frac{\langle v, v \rangle^k}{k!} \end{aligned}$$

Because the partial sums of $e^{\langle v,v \rangle}$ are a Cauchy sequence in R, it follows that $(v_n)_{n \geq 1}$. is Cauchy in G as well, and thus that it converges. That the limit of v_n is $\sum_{k=0}^{\infty} \dfrac{v^{\otimes k}}{k!}$ is clear, since

$$\lim_{n \to \infty} ||v - v_n||^2 = \lim_{n \to \infty} \sum_{k=n+1}^{\infty} k! \frac{\langle v, v \rangle^k}{(k!)^2} = \lim_{n \to \infty} \sum_{k=n+1}^{\infty} \frac{\langle v, v \rangle^k}{k!} = 0$$

So we have that $\sum_{k=0}^{\infty} \dfrac{v^{\otimes k}}{k!} \in G$.

For $\sum_{i=1}^{n} a_i e^{v_i} \in F$, we will consider the map $f : F \to G$, defined as

$$f\left(\sum_{i=1}^{n} a_i e^{v_i}\right) = \sum_{i=1}^{n} \left(a_i \bigoplus_{k=0}^{\infty} \frac{v_i^{\otimes k}}{k!}\right)$$

.

This map shows that we can think of vectors of the form e^v as being like a Taylor series expansion in terms of tensor products of v. Our ultimate goal is to show that f is an isometric isomorphism between our wave Fock space and the symmetric part of our particle Fock space. The following theorem demonstrates that it is an isometry, which we will use to validate the inner product on F.

Theorem 2: f is a bounded, continuous, injective linear map which preserves p and the inner product on G. Furthermore, p defines a valid inner product on F.

Proof. That f is linear and injective is clear. If we can demonstrate that

$$p(v, w) = \langle f(v), f(w) \rangle$$

Then it follows that p is a valid inner product on F which is preserved by f, and thus that f is bounded and continuous.

First, we will show that p is preserved by f. Suppose that $v, w \in F$, with $v = \sum_{i=1}^{n} a_i v_i$ and $w = \sum_{j=1}^{m} b_i w_j$. Then

$$f(v) = \sum_{i=1}^{n} \left(a_i \bigoplus_{k=0}^{\infty} \frac{v_i^{\otimes k}}{k!} \right)$$

and

$$f(w) = \sum_{j=1}^{m} \left(b_j \bigoplus_{k=0}^{\infty} \frac{w_j^{\otimes k}}{k!} \right)$$

Then we can expand the inner product in F as follows

$$\begin{aligned} \langle f(v), f(w) \rangle &= \langle \sum_{i=1}^{n} \left(a_i \bigoplus_{k=0}^{\infty} \frac{v_i^{\otimes k}}{k!} \right), \sum_{j=1}^{m} \left(b_j \bigoplus_{k=0}^{\infty} \frac{w_j^{\otimes k}}{k!} \right) \rangle \\ &= \sum_{k=0}^{\infty} \sum_{i,j} k! a_i \overline{b_j} \frac{\langle v_i^{\otimes k}, w_j^{\otimes k} \rangle}{(k!)^2} \\ &= \sum_{k=0}^{\infty} \sum_{i,j} a_i \overline{b_j} \frac{\langle v_i, w_j \rangle^k}{k!} \end{aligned}$$

But if we switch the bounds, we can then represent this sum as a sum of Taylor series:

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{i,j} a_i \overline{b_j} \frac{\langle v_i, w_j \rangle^k}{k!} &= \sum_{i,j} \sum_{k=0}^{\infty} a_i \overline{b_j} \frac{\langle v_i, w_j \rangle^k}{k!} \\ &= \sum_{i,j} a_i \overline{b_j} e^{\langle v_i, w_j \rangle} \\ &= p \left(\sum_{i=1}^{n} a_i e^{v_i}, \sum_{j=1}^{m} b_j e^{w_j} \right) \end{aligned}$$

Thus f does in fact preserves p and the inner product on G.

Since $p(f(v), f(w)) = \langle f(v), f(w) \rangle$, and the inner product on G is valid, it is clear that p defines a valid inner product on F. Then because f preserves the inner product, we have that $||v|| = \sqrt{p(v,v)} = \sqrt{\langle f(v), f(v) \rangle} = ||f(v)||$. This implies that $f(v)$ is bounded, and since $f(v)$ is a bounded linear map, it is furthermore continuous. $\square$

Since we have a valid inner product on F, F has an induced metric. From now on we will assume that F is complete with respect to this metric, that is, that $F = \overline{F}$. Since p is a valid inner product, we will also denote $\langle \sum_{i=1}^{n} a_i e^{v_i}, \sum_{j=1}^{m} b_j e^{v_j} \rangle = p(\sum_{i=1}^{n} a_i e^{v_i}, \sum_{j=1}^{m} b_j e^{v_j})$, i.e. we will use the standard inner product notation on F from now on.

Given a vector e^v, for $v \in H$, we imagine e^v as being like a function on H. Given a vector w on H, we can imagine $e^{\langle v,w \rangle}$. v induces a decomposition of H, into $H = span(v) \bigoplus span(v)^{\perp}$. If we imagine a vector w as a variable, then as w grows in the direction of v, $e^{\langle v,w \rangle}$ behaves like e^x. If w grows in a direction contained in $span(v)^{\perp}$, $e^{\langle v,w \rangle}$ stays constant.

Now if we imagine v, w to be real vectors, we can consider e^{iv}. Then $e^{\langle v,w \rangle} = cos(\langle v, w \rangle) + isin(\langle v, w \rangle)$. In this case, as w grows in the direction of v, we move along this wave, while the function still stays constant as w grows in the hyperplane $span(v)^{\perp}$. So e^{iv}, viewed as a function on H, is like a wave with a fixed amplitude in the direction of v.

Given complex-valued v and w, then, e^v is a wave in the direction of v, the amplitude of which is a function of w. We can imagine this by splitting v and w into real and imaginary parts, $e^{\langle v,w \rangle} = e^{\langle Re(v),Re(w) \rangle} * e^{\langle iIm(v),Re(w) \rangle} * e^{\langle Re(v),iIm(w) \rangle} * e^{\langle iIm(v),iIm(w) \rangle}$.

Chapter 3

Maps and Equivalences

3.1 Differentiation in Wave Fock Spaces

In order to prove that $f(F)$ spans the symmetric part of G, we will need to show that it contains its derivatives: for an element $v \in H$, we will show that F contains the derivatives of H-valued functions on $\mathbb{R}$, $t \mapsto e^{tv}$. This will rely upon F being complete.

Let $t, h \in \mathbb{R}$. Then F contains $\dfrac{e^{(t+h)v} - e^{tv}}{h}$ for all t, h. Consider $f\left(\dfrac{e^{(t+h)v} - e^{tv}}{h}\right)$. This is the element $\dfrac{1}{h}\bigoplus_{k=0}^{\infty}\dfrac{((t+h)v)^{\otimes k} - (tv)^{\otimes k}}{k!}$. Now we consider the element $\bigoplus_{k=1}^{\infty}\dfrac{t^{k-1}(v)^{\otimes k}}{(k-1)!}$. We have

$$||\frac{1}{h}\bigoplus_{k=0}^{\infty}\frac{((t+h)v)^{\otimes k} - t^k(v)^{\otimes k}}{k!} - \bigoplus_{k=1}^{\infty}\frac{t^{k-1}(v)^{\otimes k}}{(k-1)!}||^2 \quad = \quad ||\frac{1}{h}\bigoplus_{k=1}^{\infty}\frac{(t+h)^k v^{\otimes k} - t^{k-1}(t+hk)v^{\otimes k}}{k!}||^2$$

When we expand this in terms of the inner product on G, we get

$$\frac{1}{h^2}\sum_{k=1}^{\infty}\frac{((t+h)^{2k} - 2(t+hk)t^{k-1}(t+h)^k + (t+hk)^2 t^{2(k-1)})\langle v, v\rangle^k}{k!}$$

Consider the $k = 1$ term. This is

$$((t+h)^2 - 2(t+h)(t+h) + (t+h)^2)\langle v, v\rangle \quad = \quad 0$$

So the first term of our sum is 0. In every other summand, we can consider only the terms that have a power of h that is at most 2, since the other terms clearly go to 0 as $h \to 0$. Then for all other k, those terms are

$$\langle v, v\rangle^k \frac{2^{2k} + 2kt^{2k-1} + k(2k-1)t^{2k-2}h^2 - 2(t^{2k} + 2kt^{2k-1}h - \left(\frac{k(k-1)}{2} + k^2\right)t^{2k-2}h^2)}{k!}$$
$$+\langle v, v\rangle^k \frac{+t^{2k} + 2kt^{2k-1}h + k^2 t^{2k-2}h^2}{k!} \quad = \quad 0$$

Which simplifies to 0. Since each summand only has terms that include a power of h greater than h^2, the limit of each summand goes to 0 as $h \to 0$. Since each summand decreases with h, and each limit goes to 0 as $h \to 0$, the monotone convergence theorem allows us to take the limit inside the sum, which gives us that

$$\lim_{h\to 0}\frac{1}{h^2}\sum_{k=1}^{\infty}\frac{((t+h)^{2k} - 2(t+hk)t^{k-1}(t+h)^k + (t+hk)^2 t^{2(k-1)})\langle v, v\rangle^k}{k!} = 0$$

And therefore that

$$\lim_{h\to 0}||\frac{1}{h}\bigoplus_{k=0}^{\infty}\frac{((t+h)v)^{\otimes k} - t^k(v)^{\otimes k}}{k!} - \bigoplus_{k=1}^{\infty}\frac{t^{k-1}(v)^{\otimes k}}{(k-1)!}||^2 = 0$$

Since we have convergence in G, we have that $\dfrac{e^{(t+h)v} - e^{tv}}{h}$ converges in F because f preserves the inner product. Clearly, if we set $t = 0$ then its image converges to the element $v \in G$. So, we can take v to be an element in F. In general, we also have that $ve^{tv} \in F$ for a given t, which is legitimized by the differentiation being valid for an arbitrary t.

In fact, we may differentiate iteratively. Suppose we have an element $v^l e^{tv} \in F$, and that

$$f(v^l e^{tv}) = \bigoplus_{k=l}^{\infty} \frac{t^{k-l} v^{\otimes k}}{k - l!}$$

Then we consider

$$||f(\frac{v^l e^{(t+h)v} - v^l e^{tv}}{h}) - \bigoplus_{k=l+1}^{\infty} \frac{t^k v^{\otimes k}}{(k-l-1)!}||^2$$

We may then expand this in terms of the inner product on G, in which case we get

$$\frac{1}{h^2} \sum_{k=l+1}^{\infty} ||\frac{\frac{k!}{(k-l)!}((t+h)^{k-l} v^{\otimes k} - t^{k-l}(v)^{\otimes k}) - \frac{k!}{(k-l-1)!} h t^{k-l-1} v^{\otimes k}}{k!}||_k^2$$

As before, we need only consider the terms for which h is at most 2. That those parts of each term are 0 follows directly from the case when $l = 0$, since the kth term only differs by a factor of $\dfrac{k!}{(k-l)!}$. Thus we may differentiate iteratively. It follows that when we evaluate $v^l e^{tv}$ at $t = 0$, we return v^l, and that $f\left(v^l\right) = v^{\otimes l}$.

3.2 Equivalence of Fock Spaces

Now we have enough to prove that our wave Fock space F is isomorphic to the symmetric portion of our particle Fock space G. Let H be a Hilbert space. Then for each tensor $v \in H^{\otimes n}$, v is a linear combination of pure tensors, so $v = \sum_{i=1}^m a_i v_i$, where v_i is a pure tensor (that is, $v_i = v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_n}$).

Definition: $v \in H^{\otimes n}$ is said to be a *symmetric tensor* if for any $\sigma \in S_n$, where S_n denotes the symmetric group on n elements, we have $v = \sum_{i=1}^m a_i \tau_\sigma(v_i)$, where $\tau_\sigma(v_i) = v_{i_{\sigma(1)}} \otimes v_{i\sigma(2)} \otimes ... \otimes v_{i_{\sigma(n)}}$. If $v \in G$, then we say that v is symmetric if every term of $v = \bigoplus_{i=0}^{\infty} a_i v^{\otimes i}$ is symmetric.

It's clear that the set of symmetric tensors is a closed subspace of $H^{\otimes n}$, since 0 is symmetric and it is closed under addition and scalar multiplication. The symmetric tensors of G likewise form a closed subspace; we will denote the symmetric part of G by $Sym(G)$.

Theorem 3: Given the particle Fock space G and the wave Fock space F, both generated by the same Hilbert space H, we have $F \cong \overline{Sym(G)}$.

Proof. We let $f : F \to G$ be defined as before: $f(\sum_{i=1}^{n} a_i e^{v_i}) = \sum_{i=1}^{n}\left(a_i \bigoplus_{k=0}^{\infty} \frac{v_i^{\otimes k}}{k!}\right)$. Because f preserves the inner product, it must be injective. Thus if we can show that $f(F)$ spans $Sym(G)$, and that for every $v \in F$ we have $f(v) \in Sym(G)$, then we are done - since F is complete, and inner products are preserved, its span must be complete, in which case we will have that $f(F) = \overline{Sym(G)}$.

We know from before that for any $v \in H$, we have an elements of the form $v^l e^{tv} \in F$, for any $t \in \mathbb{R}$, and that $f\left(v^l e^{0v}\right) = v^{\otimes l}$. Thus if this is a basis for $Sym(G)$, then we have shown that $f(F)$ spans $Sym(G)$. Now suppose we have any element $w \in Sym(G)$. Then $w = \bigoplus_{i=0}^{\infty} a_i w_i \in G$, where $w_i \in H^{\otimes i}$, $\sum_{i=0}^{\infty}|a_i|^2\langle w_i, w_i\rangle < \infty$, and each w_i is a symmetric tensor. We will prove that vectors of the form $w^{\otimes i}$ span $H^{\otimes i}$. For a given symmetric w_i, we may assume without loss of generality that $w_i = \sum_{\sigma \in S_i} \tau_\sigma q_i$, for some pure tensor q_i.

We will denote by S_i the set of all binary sequences of length i. For a given $s \in S_i$, we will let s_n denote its nth component. Now consider

$$\frac{1}{2^i}\sum_{s \in B_i}(-1)^{i+\sum_{n=1}^{i} s_n}\left(\sum_{n=1}^{i}(-1)^{s_n} q_{i_n}\right)^{\otimes i} \tag{3.1}$$

This is the sum of the ith tensor product of every permutation of positives and negatives in front of the individual components of q_i, where we alternate sign of the term depending on how many negatives its tensor product contains. Each term of this sum, of which there are 2^i, contributes one copy of each permutation of q_i (we can split each term in our sum up into a tensor product in which each vector component is a single vector component of q_i - then the permutations of q_i, namely these tensor products in which the vector component of q_i never repeats, will always have a positive coefficient).

If we consider a given binary sequence $s \in B_i$, the term corresponding to that sequence is the sum of linear combinations of every pure tensor whose vector components are components of q_i (so for example, if $i = 2$, then each binary sequence corresponds to some linear combination of $q_{i_1} \otimes q_{i_1}, q_{i_1} \otimes q_{i_2}, q_{i_2} \otimes q_{i_1}, q_{i_2} \otimes q_{i_2}$). Any pure tensor which in our sum which is not a permutation of q_i must have a repeated vector component. But the coefficients on all such pure tensors with a given vector component repeated must cancel out: suppose that our repeated vector term is q_{i_1}, and we consider all pure tensors in (3.1) of the form $q_{i_1} \otimes q_{i_1} \otimes q_{i_{j_3}} \otimes ... \otimes q_{i_{j_i}}$ ($q_{i_{j_l}}$ is taken to be some component of q_i, indexed by j_l). Each term, and thus each binary sequence, gives us one pure tensor of the form $q_{i_1} \otimes q_{i_1} \otimes a_3 \otimes ... \otimes a_i$. Given a binary sequence $s \in B_i$, the coefficient of the corresponding pure tensor is $(-1)^{i+\sum_{n=1}^{i} s_i + i - 2 + \sum_{l=3}^{i} s_{j_l}}$. Consider all binary sequences of length i in which $\sum_{l=3}^{i} s_{j_l}$ is even. There are as many of these binary sequences whose sum is odd as there are whose sum is even: there is at least one $2 \le n \le i$ for which $j_l \ne n$, since $3 \le l \le i$ (this is essentially the pigeonhole principle where each bin can store one object). Without loss of generality, we may assume $j_l \neq 2$. Then if $\sum_{l=3}^{i} s_{j_l}$ is even for some s, it corresponds to an opposite-parity sequence s' in which every element is the same, except that $s'_2 = 1$ if $s_2 = 0$ and vice versa. But then the sum of coefficients is $\sum_{s \in B_i, \sum_{l=3}^{i} s_{j_l} \text{is even}} (-1)^{i+\sum_{n=1}^{i} s_n} (\sum_{n=1}^{i} (-1)^{s_n} q_{i_n})^{\otimes i} = 0$, since $\sum_{n=1}^{i} s_n = (\sum_{n=1}^{i} s'_n) \pm 1$ and there is a bijection between s and s'. A similar argument shows that the sum of coefficients for all binary sequences with $\sum_{l=3}^{i} s_{j_l}$ odd is also 0, and that thus every the only terms of (3.1) that remain are the permutations of q_i.

Since each term in (3.1) is of the form $au^{\otimes i}$ for some $u \in H$, we have that tensors of that form are a basis for $H^{\otimes i}$. Since f maps to all such vectors, we have that $f(F)$ spans $\bigoplus_{n=0}^{i-1} 0 \bigoplus H^{\otimes i} \bigoplus_{n=i+1}^{\infty} 0$ for each i. It is clear that $f(v)$ is symmetric for all $v \in F$, by definition of f, so $f(F) \subset Sym(G)$. If $w = \bigoplus_{i=0}^{\infty} a_i w_i \in Sym(G)$, then $w \in G$, so $\sum_{i=0}^{\infty} |a_i|^2 \langle w_i, w_i \rangle < \infty$. Then $\sum_{i=o}^{\infty} |a_i|^2 \langle w_i, w_i \rangle \to 0$ as $k \to \infty$, which means that since $f(F)$ spans $\bigoplus_{n=0}^{i} H^{\otimes n} \bigoplus_{n=i+1}^{\infty} 0$ for all i, $f(F)$ spans $Sym(G)$. Therefore we have $F \cong Sym(G)$, and we are done. □

3.3 Polynomials Over a Hilbert Space

Suppose H is a Hilbert space over $\mathbb{C}$, and let ϕ be a linear functional over H. According to Riesz Representation Theorem, there is a unique element $v \in H$ such that $\phi(x) = \langle v, x \rangle$ for all $x \in H$, and that this is in fact a bijection [1].

We can extend this notion to polynomials over H. We can define lth degree polynomials in one term by fixing some set of elements $v_1, v_2, ..., v_l \in H$, and then letting $P(x) = a \prod_{i=1}^{l} \langle v_i, x \rangle$. If we take finite linear combinations of these monomials, plus any constant, we get polynomial functions over H. We may suppose that $v_i = e_i$ for some basis vector $e_i \in \mathbb{C}^n$, since we can split up the inner products making up P linearly into inner products of these basis vectors. Then $P(x) = a \prod_{i=1}^{n} e_i^{d_i}$ for some exponent v_i. Furthermore, we can associate this polynomial P with an element v_P in G:

$$v_P = \frac{a}{\sqrt{(\sum_{i=1}^{n} d_i)!}} Sym(\bigotimes_{i=1}^{n} \sqrt{d_i!} e_i^{\otimes d_i}) \in Sym(G) \tag{3.2}$$

Which is the symmetric part of the tensor whose vector components are v_i, for $1 \le i \le l$. Extending this linearly, we get a homeomorphic embedding $P \mapsto v_P$ of polynomial functions P into $Sym(G)$.

If we take our Hilbert space to be $\mathbb{C}^n$, then the linear functionals on $\mathbb{C}^n$ take the form $\sum_{i=1}^{n} a_i x_i$, or linear polynomials in n degrees with constant term 0. Thus by forming linear combinations of their products, we get the space of all complex-valued polynomials in n variables. Given two polynomials P, Q over $\mathbb{C}^n$, we can take their inner product in G. Then we get the following result:

Theorem 4: Given two polynomials P, Q in n variables over $\mathbb{C}$, then $\langle v_P, v_Q \rangle = \int \cdots \int_{\mathbb{C}^n} P(z_1, z_2, ..., z_n) \overline{Q(z_1, z_2, ..., z_n)} \frac{1}{\pi^n} e^{-\langle z, z \rangle} d\lambda_n$, where z denotes the coordinate vector in C^n.

Proof. First, it is clear that the integral converges for any polynomial, because $\int_{\mathbb{R}} r^n e^{-r^2} dr$ is convergent for any n (and in any single variable, $P\overline{Q}$ will be bounded outside the unit square by $cr^n e^{-r^2}$ for some c, for some n, so it converges). This also justifies evaluating this integral iteratively.

Every polynomial over $\mathbb{C}^n$ is of the form $\sum_i^m a_i \prod_{j=1}^n z_j^{d_{j_i}}$, for some m. By linearity of the integral and the inner product, we can assume that $m = 1$ without loss of generality. These polynomials correspond to a single pure tensor in $Sym(G)$, where the pure tensor is an element of $H^{\otimes \sum_{j=1}^l d_j}$. Thus we can consider only polynomials of the form

$$P = a \prod_{j=1}^{n} z_j^{d_j}, Q = b \prod_{j=1}^{n} z_j^{d'_j} \tag{3.3}$$

Furthermore, we want to be able to evaluate this integral for polynomials whose exponents are the same for each variable. So suppose that $d_j \neq d'_j$ for some i - without loss of generality, suppose $d_1 \neq d'_1$. Polynomials of the form (3.3) are represented in the form (3.2) as $v_P = \frac{a}{\sqrt{(\sum_{i=1}^n d_i)!}} Sym(\bigotimes_{j=1}^l \sqrt{(d_j!)} e_j^{\otimes d_j})$ and $v_Q = \frac{b}{\sqrt{(\sum_{i=1}^n d_i)!}} Sym(\bigotimes_{j=1}^l \sqrt{d'_j!} e_j^{\otimes d'_j})$. But if $d_j \neq d'_j$, then $\langle v_P, v_Q \rangle = 0$ is clear from the definition of the inner product on G, since e_j are an orthogonal set of vectors in $\mathbb{C}^n$.

Next we can consider the integral

$$\int \cdots \int_{\mathbb{C}^n} P(z_1, z_2, ..., z_n) \overline{Q(z_1, z_2, ..., z_n)} \frac{1}{\pi^n} e^{-\langle z, z \rangle} d\lambda_n$$

If $d_1 \neq d'_1$, then without loss of generality we may assume that $d_1 > d'_1$ (because we taking the integral $d\lambda_n$ rather than dz, we have $\overline{\int f d\lambda} = \int \overline{f} d\lambda$). Then when integrate with respect to $d\lambda_1$ and hold $z_2, z_3, ..., z_n$ constant, our integral is of the form

$$\int_C M(x_1 + iy_1)^{d_1 - d'_1} (x_1^2 + y_1^2)^{d'_1} e^{-(x_1^2 + y_1^2)} d\lambda$$

Where M is taken to the product of all the constant terms (with respect to z_1). Set

$$f(z) = M(x + iy)^{d_1 - d'_1} (x^2 + y^2)^{d'_1} e^{-(x^2 + y^2)}$$

And then

$$f((-1)^{\overline{\frac{1}{d_1 - d'_1}}} z) = -f(z)$$

(when we rotate z, $(x^2 + y^2)$ remains constant, but $(-1)^{\overline{\frac{1}{d_1 - d'_1}}}$ (i.e., the $d_1 - d'_1$th root of (-1)) can be pulled out of the $(x + iy)^{d_1 - d'_1}$ term). Since f switches sign through rotation (similar to integrating an odd function over an interval of the form [-R, R]), it follows that

$$\int_C M(x_1 + iy_1)^{d_1 - d'_1} (x_1^2 + y_1^2)^{d'_1} e^{-(x_1^2 + y_1^2)} d\lambda = 0$$

And thus that

$$\int \cdots \int_{\mathbb{C}^n} P(z_1, z_2, ..., z_n)\overline{Q(z_1, z_2, ..., z_n)}e^{-\langle z,z\rangle} d\lambda_n = 0$$

Thus we may assume that $d_i = d_i'$ for all i. Then we have that

$$P(z)\overline{Q(z)} = a\bar{b}\prod_{j=1}^{l}(x_j^2 + y_j^2)^{d_j}$$

So, we consider

$$\frac{a\bar{b}}{\pi^n}\int \cdots \int_{\mathbb{C}^{n-1}} \int_{\mathbb{C}} \prod_{j=1}^{l}(x_j^2 + y_j^2)^{d_j} e^{-(\sum_{j=1}^{l} x_j^2+y_j^2)} d\lambda_n$$

Integrating with respect to each coordinate is legitimate by Tonelli's theorem, since this function is nonnegative. So we can integrate with respect to z_1. Most of this function is constant with respect to a given coordinate, so we can pull out the constants with respect to z_1 and switch to polar coordinates, which will allow us to easily evaluate the integral

$$\begin{aligned}
& \frac{a\bar{b}}{\pi^n}\textstyle\int \cdots \int_{\mathbb{C}^{n-1}} \int_{\mathbb{C}} \prod_{j=1}^{l}(x_j^2 + y_j^2)^{d_j} e^{-(\sum_{j=1}^{l} x_j^2+y_j^2)} d\lambda_n \\
= & \frac{a\bar{b}}{\pi^n}\textstyle\int \cdots \int_{\mathbb{C}^{n-1}} \prod_{j=2}^{l}(x_j^2 + y_j^2)^{d_j} e^{-(\sum_{j=2}^{l} x_j^2+y_j^2)} \int_{\mathbb{C}}(x_1^2 + y_1^2)^{d_1} e^{-(x_1^2+y_1^2)} d\lambda_n \\
= & \frac{a\bar{b}}{\pi^n}\textstyle\int \cdots \int_{\mathbb{C}^{n-1}} \prod_{j=2}^{l}(x_j^2 + y_j^2)^{d_j} e^{-(\sum_{j=2}^{l} x_j^2+y_j^2)} \int_0^{2\pi}\int_0^{\infty} r^{2d_1+1} e^{-r^2} dr d\theta d\lambda_{n-1}
\end{aligned}$$

Now we perform the substitution $u = r^2$. It's easy to see that the function is constant along each annulus. Then with this substitution, we get

$$\begin{aligned}
& \frac{a\bar{b}}{\pi^n}\textstyle\int \cdots \int_{\mathbb{C}^{n-1}} \prod_{j=2}^{l}(x_j^2 + y_j^2)^{d_j} e^{-(\sum_{j=2}^{l} x_j^2+y_j^2)} \int_0^{2\pi}\int_0^{\infty} r^{2d_1+1} e^{-r^2} dr d\theta d\lambda_{n-1} \\
= & \frac{a\bar{b}}{\pi^n}\textstyle\int \cdots \int_{\mathbb{C}^{n-1}} \prod_{j=2}^{l}(x_j^2 + y_j^2)^{d_j} e^{-(\sum_{j=2}^{l} x_j^2+y_j^2)} \int_0^{2\pi}\int_0^{\infty} \dfrac{1}{2} u^{d_1} e^{-u} du d\theta d\lambda_{n-1}
\end{aligned}$$

The integrals in each of other variables are evaluated exactly the same way, by switching to polar coordinates as we evaluate over each component of $\mathbb{C}^n$, as follows

$$\begin{aligned}
& \frac{a\bar{b}}{\pi^n}\int\cdots\int_{\mathbb{C}^{n-1}}\prod_{j=2}^{l}(x_j^2+y_j^2)^{d_j}e^{-(\sum_{j=2}^{l}x_j^2+y_j^2)}\int_0^{2\pi}\int_0^{\infty}\frac{1}{2}u^{d_1}e^{-u}dud\theta d\lambda_{n-1} \\
= & \frac{a\bar{b}}{\pi^n}\int\cdots\int_{\mathbb{C}^{n-1}}\prod_{j=2}^{l}(x_j^2+y_j^2)^{d_j}e^{-(\sum_{j=2}^{d_1}x_j^2+y_j^2)}\int_0^{2\pi}(-u^{d_1}e^{-u}|_{\infty}+u^{d_1}e^{-u}|_0+\int_0^{\infty}d_1u^{d_1-1}e^{-u}du)d\theta d\lambda_{n-1} \\
= & \frac{a\bar{b}}{2\pi^n}\int\cdots\int_{\mathbb{C}^{n-1}}\prod_{j=2}^{l}(x_j^2+y_j^2)^{d_j}e^{-(\sum_{j=2}^{l}x_j^2+y_j^2)}\int_0^{2\pi}(0+\int_0^{\infty}d_1u^{d_1-1}e^{-u}du)d\theta d\lambda_{n-1} \\
= & \cdots \\
= & \frac{a\bar{b}}{2\pi^n}\int\cdots\int_{\mathbb{C}^{n-1}}\prod_{j=2}^{l}(x_j^2+y_j^2)^{d_j}e^{-(\sum_{j=2}^{l}x_j^2+y_j^2)}\int_0^{2\pi}(a\int_0^{\infty}d_1!e^{-u}du)d\theta d\lambda_{n-1} \\
= & \frac{a\bar{b}}{2\pi^n}\int\cdots\int_{\mathbb{C}^{n-1}}\prod_{j=2}^{l}(x_j^2+y_j^2)^{d_j}e^{-(\sum_{j=2}^{l}x_j^2+y_j^2)}\int_0^{2\pi}d_1!d\theta d\lambda_{n-1} \\
= & \frac{a\bar{b}}{\pi^{n-1}}\int\cdots\int_{\mathbb{C}^{n-1}}d_1!\prod_{j=2}^{l}(x_j^2+y_j^2)^{d_j}e^{-(\sum_{j=2}^{l}x_j^2+y_j^2)}d\lambda_{n-1} \\
= & \cdots \\
= & a\bar{b}\prod_{j=1}^{n}d_i!
\end{aligned}$$

Now we can evaluate the inner product of v_P and v_Q, and we find

$$\begin{aligned}
\langle v_P, v_Q\rangle &= \left(\sum_{i=1}^{n}d_i\right)!\frac{a}{\sqrt{(\sum_{i=1}^{n}d_i)!}}\frac{\bar{b}}{\sqrt{(\sum_{i=1}^{n}d_i)!}}\prod_{i=1}^{n}\sqrt{d_i!}\sqrt{d_i!}\langle e_i, e_i\rangle^{d_i} \\
&= a\bar{b}\prod_{j=1}^{n}d_i! \\
&= \int\cdots\int_{\mathbb{C}}^{n}P(z)\overline{Q(z)}\frac{1}{\pi^n}e^{-\langle z,z\rangle}d\lambda_n
\end{aligned}$$

And thus we are done; the identity holds in all cases through linearity. $\square$

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Academic Vitae

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