Analysis of the Double Parton Distribution Functions in Quantum Chromodynamics

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Abstract

We demonstrate the mathematical formalism for the construction of consistent initial conditions for the double parton distribution functions in the collinear approximation. The initial conditions within this framework have an important property that they exactly and simultaneously satisfy both the momentum sum rule and the quark number sum rule. Furthermore, in this formalism, the double parton distribution’s functional behavior is related to the single parton distribution functions. We find that this condition imposes certain relations on the large and small $x$ behavior of both single and double parton distribution functions. By making use of the Mellin transformation we analytically solve the evolution equation at leading logarithmic order for the gluon channel double parton distribution function. Furthermore, we illustrate the double parton correlations for the gluon channel and show how they change with the evolution scale.
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Chapter 1

Introduction
1.1 Strong Interactions

There exist four fundamental forces of nature: the gravitational force, the electromagnetic force, the weak nuclear force, and the strong nuclear force. The strong nuclear force is responsible for interactions between quarks and gluons which make up hadrons such as the proton and neutron. The strong interaction is described by a quantum field theory of quantum chromodynamics (QCD).

The theory of QCD contains three types of charge, referred to as color charge. The particles that carry color charge, fermionic quarks and bosonic gluons, are the particles that participate in strong interactions. Quarks carry three charges; gluons, however, carry \(3^2 - 1 = 8\) charges, as they are in the adjoint representation. In addition to color charge quarks also carry flavor. There are six flavors of quarks; they are listed below with some of their properties in Table 1.1 [1].

\[
\text{Table 1.1. Quarks in QCD}
\]

<table>
<thead>
<tr>
<th>Quark Name</th>
<th>Quark</th>
<th>Charge</th>
<th>Mass</th>
<th>Baryon Number</th>
<th>Isospin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>(u)</td>
<td>(\frac{2}{3})</td>
<td>(\sim 4) MeV</td>
<td>1/3</td>
<td>+1/2</td>
</tr>
<tr>
<td>Down</td>
<td>(d)</td>
<td>(-\frac{1}{3})</td>
<td>(\sim 7) MeV</td>
<td>1/3</td>
<td>-1/2</td>
</tr>
<tr>
<td>Charm</td>
<td>(c)</td>
<td>(\frac{2}{3})</td>
<td>(\sim 1.5) GeV</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>Strange</td>
<td>(s)</td>
<td>(-\frac{1}{3})</td>
<td>(\sim 135) MeV</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>Top</td>
<td>(t)</td>
<td>(\frac{2}{3})</td>
<td>(\sim 175) GeV</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>Bottom</td>
<td>(b)</td>
<td>(-\frac{1}{3})</td>
<td>(\sim 5) GeV</td>
<td>1/3</td>
<td>0</td>
</tr>
</tbody>
</table>

In addition to these six types of quarks, there is also a corresponding antiquark associated with each flavor. Furthermore, there are also gluons to consider. Gluons are the gauge bosons of QCD, acting as the exchange particles for the strong force between quarks, analogous to photons in the theory of quantum electrodynamics (QED). The gluon, however, carries color charge. Therefore gluons not only mediate the strong interaction, but they also interact with themselves.

The Lagrangian for QCD, without incorporating gauge constraints or ghost terms, is given by [1].

\[
\mathcal{L}_{\text{classical}} = -\frac{1}{4} F_{A}^{\alpha \beta} F_{A}^{\alpha \beta} + \sum_{\text{flavors}} \bar{q}_a (i \slashed{D} - m)_{ab} q_b, \tag{1.1.1}
\]

where \(q_b\) represents the field for the quark with color index \(b\). The indices \(a, b = 1, \ldots, N_c\), where \(N_c = 3\) is the number of colors. Furthermore, \(\slashed{D}\) is shorthand notation for \(\gamma_\mu D^\mu\). \(\gamma_\mu\) are the gamma matrices and \(D^\mu\) is the covariant derivative for a non-Abelian gauge theory. The spinor indices of \(\gamma_\mu\) and \(q_a\) have been suppressed. The rightmost terms above describe the interaction of spin 1/2 quarks of mass \(m\) and massless spin 1 gluons.

Furthermore, \(F_{A}^{\alpha \beta}\) is the field strength tensor. It is constructed from the gluon field \(A_\alpha^A\) as follows...
\[ F_{\alpha\beta}^A = \left[ \partial_\alpha A_\beta^A - \partial_\beta A_\alpha^A - g f^{ABC} A_\alpha^B A_\beta^C \right]. \] (1.1.2)

The coupling constant \( g \) is defined in a manner analogous to QED. In particular, \( \alpha_s = g^2/4\pi \), where \( \alpha_s \) is the strong coupling constant. \( f^{ABC} \) are the structure constants of \( SU(3) \). The structure constants of a Lie group determine the commutation relations between its generators in the associated Lie algebra. The indices \( A, B, C \) represent the eight color degrees of freedom of the gluon field, \( A \). The third, non-Abelian term on the right hand side of Eq. (1.1.2) is the distinguishing feature between QED and QCD. As a result of this term, cubic and quartic gluon self interactions are present in the theory of QCD. The Lagrangian above gives rise to four fundamental couplings of the strong interaction, seen below in Figure 1.1. It is these cubic and quartic interactions that ultimately give rise to asymptotic freedom.

\[ Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \beta(\alpha_s). \] (1.1.3)

In QCD, the \( \beta \) function can be written as a perturbative expansion in \( \alpha_s \) as follows [1]
\[ \beta(\alpha_s) = -b\alpha_s^2 \left( 1 + b'\alpha_s + \mathcal{O} (\alpha_s^2) \right), \]  
(1.1.4)

where the parameters \( b \) and \( b' \) are given by

\[ b = \frac{11C_a - 2N_f}{12\pi}, \]  
(1.1.5)  

\[ b' = \frac{17C_a^2 - 5C_a N_f - 3C_f N_f}{2\pi (11C_a - 2N_f)}, \]  
(1.1.6)

where \( N_f \) is the number of active flavors. Furthermore, \( C_a \) and \( C_f \) are defined as follows

\[ C_a = N_c, \]  
(1.1.7)  

\[ C_f = \frac{N_c^2 - 1}{2N_c}. \]  
(1.1.8)

For the specific case of \( SU(3) \) we have that \( C_a = 3 \) and \( C_f = 4/3 \). We then have that

\[ b = \frac{33 - 2N_f}{12\pi}, \]  
(1.1.9)  

\[ b' = \frac{153 - 19N_f}{2\pi (33 - 2N_f)}. \]  
(1.1.10)

Using the previous relations, one can say that

\[ Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = -b\alpha_s^2(Q^2) \left[ 1 + b'\alpha_s(Q^2) + \mathcal{O} (\alpha_s^2(Q^2)) \right]. \]  
(1.1.11)

Assuming that at the initial scale, \( Q_0^2 \), \( \alpha_s \) is known and neglecting the \( b' \) and higher order terms on the right hand side of Eq. (1.1.11), we have that at leading order

\[ \alpha_s(Q^2) = \frac{\alpha_s(Q_0^2)}{1 + b\alpha_s(Q_0^2) \log \frac{Q^2}{Q_0^2}}, \]  
(1.1.12)

If both \( Q_0^2 \) and \( Q^2 \) are in the perturbative region, then Eq. (1.1.12) gives a relationship between \( \alpha_s(Q_0^2) \) and \( \alpha_s(Q^2) \). Furthermore, we see that the strong coupling constant goes to zero as energy tends towards infinity; this is what is known as asymptotic freedom.
Furthermore, from the expression for the beta function given in Eq. (1.1.12), we see that the strength of the strong coupling constant becomes large at small $Q^2$. This is known as confinement. Confinement is a property of QCD stating that the force between quarks does not diminish as the distance between them increases. As a result of this, when one separates a quark from other quarks, at some point it becomes more energetically favorable for a quark-antiquark pair to be spontaneously created out of the vacuum. To be more specific, color charged particles cannot be isolated singularly; quarks attract each other to form color neutral composite particles known as hadrons. As quarks are often ejected in high energy collisions that occur at hadron colliders, this phenomenon is of profound importance. In particular, it tells us that we will not detect ejected, free quarks, but rather we will detect the jets of hadrons that were created as a result of confinement.

1.2 Hadron Collisions

Data gathered by observing particle collisions can be used to experimentally verify QCD. To experimentally probe these scattering events in an effort test and further develop QCD, one makes use of high energy particle colliders. The ideal place to test QCD, then, is at the Large Hadron Collider (LHC) where protons are accelerated to reach energies of $8 \text{ TeV}$. In Figure 1.2 below we see a depiction of a dijet event as seen in ATLAS [2]. This event display shows a process where two jets of particles are produced with very large energy. On the microscopic level we can understand this process as a production of two quarks with high transverse momentum. As a result of asymptotic freedom, these quarks are behaving as quasi-free objects. In the detector, however, one observes color neutral hadrons. This is a result of the quarks being confined, hence they undergo the hadronization process before they can be detected.

\footnote{The LHC’s collision energy will soon be increased to $13 \text{ TeV}$.}
Figure 1.2. Here we see a dijet event as measured by the ATLAS detector at the LHC [2]. This dijet event shows that hard processes are occurring in high energy proton-proton collisions. It also shows that asymptotic freedom is present as quarks are ejected from the hadrons during these collisions. Finally, the fact that we observe hadrons in the detector indicates that hadronization, and therefore confinement, is observed.

Furthermore, as a result of asymptotic freedom one can work in a scheme such that one is able to separate, or factor, the long- and short-distance contributions to any physical cross section involving large momentum transfers [1, 3, 4]. This approximation is applied to nearly every theoretical calculation as the nature of the strong interaction is not sufficiently understood to facilitate more exact calculations. As a result of this factorization approximation we can represent single parton scattering diagrammatically as seen below in Figure 1.3. Furthermore, a dijet event as seen above in Figure 1.2 can be represented as a double parton scattering diagram as seen below in Figure 1.4. Single parton scattering occurs when there is one hard collision per proton-proton interaction. Specifically, this can be pictured diagrammatically as follows:
The object $\hat{\sigma}_{f f'}(Q^2)$ is the short-distance cross section for the scattering of partons of types $f$ and $f'$. This partonic cross section can be computed in QCD as an expansion in terms of the strong coupling constant, $\alpha_s$. The function $D_1^f(x, Q^2)$ is known as a single parton distribution function. This parton distribution function describes the probability density for finding a parton of flavor $f$ with a longitudinal momentum fraction $x$ at a scale $Q^2$. As a result of the non-perturbative nature of partons, parton distribution functions cannot be obtained by utilizing perturbative QCD.

Analogous to single parton scattering, double parton scattering is when two hard collisions occur per proton-proton interaction. Similarly, we can also represent double parton scattering diagrammatically as follows:
Figure 1.4. Here we see one possible diagram depicting double parton scattering.

Similarly to the single parton case, the functions $D_{2}^{f_{1}f_{2}}(x_{1}, x_{2}, Q^{2})$ are called double parton distribution functions. As one might expect, they describe the probability density for finding two partons of flavor $f_{1}$ and $f_{2}$ with longitudinal momentum fractions $x_{1}$ and $x_{2}$ respectively at a scale $Q^{2}$. There has been ample experimental evidence for double parton scattering [5–10]. These double parton distribution functions are the primary focus of this analysis.
Chapter 2

Parton Distributions
2.1 Single Parton Distributions

2.1.1 Evolution Equations for Single Parton Distribution Functions

In this section we discuss evolution equations for the single parton distributions. The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations can be written down as a $2N_f + 1$ dimensional matrix equation, where $N_f$ is the number of flavors, as follows [11–13].

$$\frac{d}{d \log Q^2} \left( q_i \left( x, Q^2 \right) g \left( x, Q^2 \right) \right) = \frac{\alpha_s(Q^2)}{2\pi} \sum_{q_j,q_j} \int_x^1 \frac{d \xi}{\xi} \begin{pmatrix} P_{q_j q_j} \left( \frac{x}{\xi}, \alpha_s \right) & P_{q_j g} \left( \frac{x}{\xi}, \alpha_s \right) \\ P_{g q_j} \left( \frac{x}{\xi}, \alpha_s \right) & P_{g g} \left( \frac{x}{\xi}, \alpha_s \right) \end{pmatrix} \begin{pmatrix} q_j \left( \xi, Q^2 \right) \\ g \left( \xi, Q^2 \right) \end{pmatrix}. \quad (2.1.1)$$

The functions $P_{q q}, P_{q g}, P_{g q},$ and $P_{g g}$ are called splitting functions. Using the invariance of QCD under charge conjugation and SU($N_f$) flavor symmetry we can say that [1]

$$P_{q j, q_j} = P_{q_j q_j}, \quad (2.1.2)$$

$$P_{q j, q_j} = P_{q_j q_j}, \quad (2.1.3)$$

$$P_{q j, g_j} = P_{q_j g_j} \equiv P_{q g}, \quad (2.1.4)$$

$$P_{g q, i} = P_{g q_i} \equiv P_{g g}. \quad (2.1.5)$$

The above equations say that the splitting functions $P_{q g}$ and $P_{g q}$ are independent of the quark flavor and that splitting functions are the same for quarks and antiquarks. Neglecting the different masses of the quarks and expanding the matrix equation given by Eq. (2.1.1) into its quark and gluon components, the evolution equations are given by the following [14]

$$\frac{dq_i(x, Q^2)}{d \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{d \xi}{\xi} \left( q_i \left( \xi, Q^2 \right) P_{q q} \left( \frac{x}{\xi} \right) + g \left( \xi, Q^2 \right) P_{q g} \left( \frac{x}{\xi} \right) \right), \quad (2.1.6)$$

$$\frac{dg(x, Q^2)}{d \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{d \xi}{\xi} \left( \sum_{i=1}^{2N_f} q_i \left( \xi, Q^2 \right) P_{q q} \left( \frac{x}{\xi} \right) + g \left( \xi, Q^2 \right) P_{g g} \left( \frac{x}{\xi} \right) \right). \quad (2.1.7)$$

The first term in Eq. (2.1.6) mathematically represents the fact that a quark with momentum fraction $x$ can be produced from a parent quark with a larger momentum fraction $y > x$ which has radiated a gluon. The second term considers the possibility that a quark with momentum fraction $x$ is the result of quark antiquark pair creation by a parent gluon with momentum fraction $y > x$. In a similar fashion, the first term in Eq. (2.1.7) describes the instance where a gluon with momentum fraction $x$ is radiated by a quark with momentum fraction $y$. In this term, the sum running from 1 to $2N_f$ represents that this gluon could have been radiated by any of the quark flavors as well as their corresponding antiflavors. The second term represents a gluon of momentum fraction $y$ radiating a gluon of momentum fraction $x$. 
2.1.2 Sum Rules

Single Parton distribution functions are number densities and therefore must be bound by certain physical constraints. The first notable constraint that must be satisfied is momentum conservation. Mathematically, this is

\[ 1 = \int_0^1 dx \left[ g(x, Q^2) + \sum_{i=1}^{N_f} q_i(x, Q^2) \right]. \] (2.1.8)

The physical meaning of this expression is that all produced partons’ momenta must add up to the total momentum present in the initial state. The second constraint is quark number conservation. This is mathematically expressed as

\[ N_i = \int_0^1 dx \left( q_i(x, Q^2) - \bar{q}_i(x, Q^2) \right). \] (2.1.9)

Here, the physical meaning is that the difference between the number of quarks and antiquarks of a particular flavor \( i \) is conserved and is equal to the number of quarks of flavor \( i, N_i \), present in the initial state.

The momentum sum rule and quark number sum rule impose conditions on the form of the splitting functions. Furthermore, each splitting function can be written as a power series in \( \alpha_s \),

\[ P_{q_i q_j}(z, \alpha_s) = \delta_{ij} P_{qq}^{(0)}(z) + \frac{\alpha_s}{2\pi} P_{q_i q_j}^{(1)}(z) + \cdots, \] (2.1.10)

\[ P_{q g}(z, \alpha_s) = P_{q g}^{(0)}(z) + \frac{\alpha_s}{2\pi} P_{q g}^{(1)}(z) + \cdots, \] (2.1.11)

\[ P_{g q}(z, \alpha_s) = P_{g q}^{(0)}(z) + \frac{\alpha_s}{2\pi} P_{g q}^{(1)}(z) + \cdots, \] (2.1.12)

\[ P_{g g}(z, \alpha_s) = P_{g g}^{(0)}(z) + \frac{\alpha_s}{2\pi} P_{g g}^{(1)}(z) + \cdots. \] (2.1.13)

At leading order, these splitting functions \( P_{ab}^{(0)} \) have the physical interpretation as the probability of finding a parton of type \( a \) in a parton of type \( b \) with a fraction \( x \) of the longitudinal momentum of the parent parton and a transverse momentum much less than \( Q^2 \). Furthermore, these splitting functions satisfy

\[ \int_0^1 dx \, P_{q q}(x) = 0, \] (2.1.14)

\[ \int_0^1 dx \, [P_{q q}(x) + P_{g g}(x)] = 0, \] (2.1.15)

\[ \int_0^1 dx \, [P_{q g}(x) + P_{g q}(x)] = 0, \] (2.1.16)
which correspond to quark number conservation and momentum conservation in the splittings of quarks and gluons respectively [1]. These relations also hold order by order in perturbation theory.

The leading order splitting functions are given by [1]

\[
P_{qq}^{(0)}(x) = C_f \left[ \frac{1 + x^2}{(1 - x)_+} + \frac{3}{2} \delta(1 - x) \right], \quad (2.1.17)
\]

\[
P_{qg}^{(0)}(x) = T_R \left[ x^2 + (1 - x)^2 \right], \quad (2.1.18)
\]

\[
P_{gq}^{(0)}(x) = C_f \left[ \frac{1 + (1 - x)^2}{x} \right], \quad (2.1.19)
\]

\[
P_{gg}^{(0)}(x) = C_a \left[ \frac{x}{(1 - x)_+} + \frac{1 - x}{x} + x(1 - x) + \delta(1 - x) \frac{11C_a - 4N_fT_R}{6} \right], \quad (2.1.20)
\]

where the constants are defined as

\[
C_a = 3, \quad (2.1.21)
\]

\[
C_f = 4/3, \quad (2.1.22)
\]

\[
T_R = 1/2. \quad (2.1.23)
\]

Furthermore, the plus-prescription is defined such that its integral with any sufficiently smooth distribution \(f\) is

\[
\int_0^1 dx \frac{f(x)}{(1 - x)_+} = \int_0^1 dx \frac{f(x) - f(1)}{1 - x} \quad (2.1.24)
\]

and

\[
\frac{1}{(1 - x)_+} = \frac{1}{1 - x} \quad \text{for} \ 0 \leq x < 1. \quad (2.1.25)
\]

### 2.2 Double Parton Distributions

#### 2.2.1 Evolution Equations for Double Parton Distribution Functions

In this section we introduce the evolution equations for double parton distribution functions. It is first important to note that we can rewrite the evolution equation for single parton distribution functions, Eq. (2.1.1), as [15]
\[ \frac{\partial}{\partial t} D_1^f(x,t) = \sum_{f'} \int_0^1 du \, K_{f'f}(x,u,t) D_1^{f'}(u,t), \]  

(2.2.1)

with the evolution parameter

\[ t = \log \left( \frac{\log \left[ \frac{Q^2}{\Lambda^2_{QCD}} \right]}{\log \left[ \frac{Q_0^2}{\Lambda^2_{QCD}} \right]} \right). \]  

(2.2.2)

Note that when \( Q^2 = Q_0^2 \) we have that \( t = 0 \). Furthermore, the integral kernels, \( K_{f'f}(x,u,t) \), describe the real and virtual parton emissions

\[ K_{f'f}(x,u,t) = K_{f'f}^R(x,u,t) - \delta(u-x) \delta_{f'f} K_{f}^V(x,t), \]  

(2.2.3)

where the superscript \( R \) and \( V \) denote real and virtual parts respectively. The real part is given by

\[ K_{f'f}^R(x,u,t) = \frac{1}{2\pi b} \frac{1}{u P_{f'f}^{R} \left( \frac{x}{u},t \right)} \theta(u-x) \]  

(2.2.4)

with \( b = \frac{33-2N_f}{12\pi} \). The virtual part, \( K_{f}^V(x,t) \), is calculated from the momentum sum rule [13],

\[ 1 = \sum_f \int_0^1 dx \, xD_1^f(x,t), \]  

(2.2.5)

or is computed directly from diagrams. The functions \( P_{f'f}^R \) are the splitting functions described by Eq. (2.1.10)-Eq. (2.1.13) with leading order terms given by Eq. (2.1.17) and Eq. (2.1.20).

The evolution equations for double parton distribution functions are known only in the leading logarithm approximation [16, 17] and are given by

\[ \frac{\partial}{\partial t} D_{2}^{f_1f_2}(x_1, x_2, t) = \sum_{f'} \int_0^{1-x_2} du \, K_{f_1f'f_2}(x_1, u, t) D_{2}^{f_2f}(u, x_2, t) \]  

(2.2.6)

\[ + \sum_{f'} \int_0^{1-x_1} du \, K_{f_2f'u}(x_2, u, t) D_{2}^{f_1f'}(x_1, u, t) + \sum_{f'} K_{f'f_1f_2}^{R}(x_1, x_1 + x_2, t) D_{1}^{f'}(x_1 + x_2, t). \]

The two integrals on the right hand side above describe the evolution of a single parton with the second parton treated as a spectator. Hence, the upper integration limits resulting from the fact that
\[ x_1 + x_2 \leq 1. \] The third term on the right-hand side describes the splitting of a single parton into two partons. For this reason it contains the single parton distribution function. Therefore, Eq. (2.2.1) and Eq. (2.2.6) form a coupled set of equations for which initial conditions for both the single and double parton distribution functions must be specified.

Using delta functions and Heaviside step functions, we can rewrite Eq. (2.2.6) in a more useful, but perhaps less transparent manner

\[
\frac{\partial}{\partial t} D^{f_1 f_2}_{2}(x_1, x_2, t) = \frac{1}{2\pi b} \sum_{f_1} \int_0^1 dy \int_0^1 dz \delta(x_1 - yz) \theta(1 - z - x_2) P^{(0)}_{f_1 f_1}(y) D^{f_1 f_2}_{2}(z, x_2, t) + \\
+ \frac{1}{2\pi b} \sum_{f_2} \int_0^1 dy \int_0^1 dz \delta(x_2 - yz) \theta(1 - x_1 - z) P^{(0)}_{f_2 f_2}(y) D^{f_1 f_2}_{2}(x_1, z, t) + \\
+ \frac{1}{2\pi b} \sum_{f'} \frac{1}{x_1 + x_2} P^{(0)}_{f' \rightarrow f_1 f_2} \left( \frac{x_1}{x_1 + x_2} \right) D^{f'}_{1}(x_1 + x_2, t). \tag{2.2.7}
\]

This form of the evolution equations will be more useful when solving the evolution equations.

### 2.2.2 Sum rules

Just as we discussed the momentum sum rule for single parton distribution functions, given in Eq. (2.1.8) and again more compactly in Eq. (2.2.5), we must now discuss sum rules for double parton distribution functions. The analogous momentum sum rule for double parton distribution functions, \( D^{f_1 f_2}_{2}(x_1, x_2, t) \), is given by [15]

\[
\sum_{f_1} \int_0^{1-x_2} dx_1 x_1 \frac{D^{f_1 f_2}_{2}(x_1, x_2, t)}{D^{f_2}_{1}(x_2, t)} = 1 - x_2. \tag{2.2.8}
\]

The physical interpretation of this statement is understood by treating the ratio of parton distributions in the integral, \( \frac{D^{f_1 f_2}_{2}(x_1, x_2, t)}{D^{f_2}_{1}(x_2, t)} \), as the conditional probability of finding parton \( f_1 \) with momentum fraction \( x_1 \) while treating parton flavor \( f_2 \) and momentum fraction \( x_2 \) as fixed. Clearly then, the total momentum fraction carried by the partons \( f_1 \) is \( 1 - x_2 \). The reason that the integral only runs from zero to \( 1 - x_2 \) is because \( D^{f_1 f_2}_{2}(x_1, x_2, t) \) vanishes for \( x_1 + x_2 > 1 \). We note that Eq. (2.2.8) is symmetric both in the interchange of parton flavors \( f_1 \) and \( f_2 \) and in interchanging \( x_1 \) with \( x_2 \) as it must be. Note that conservation of momentum allows us to relate double and single parton distribution functions for any value of \( Q^2 \).

For double parton distribution functions, the valence quark number sum rule is given by [15].

\[
\int_0^{1-x_2} dx_1 \left( D^{q_1 f_2}_{2}(x_1, x_2, t) - D^{q_2 f_2}_{2}(x_1, x_2, t) \right) = (N_i - \delta_{f_2 q_i} + \delta_{f_2 \bar{q}_i}) D^{f_2}_{1}(x_2, t). \tag{2.2.9}
\]
Note that the valence quark sum rule for two partons depends on the flavor of the second parton. Furthermore, note that while $f_2$ refers to any parton, $q_i$ and $\bar{q}_i$ only refers to quarks. These sum rules are preserved by the evolution equations, Eq. (2.2.6), given in the previous section, therefore the problem is reduced to finding initial conditions that satisfy these sum rules and then solving the evolution equations.
Chapter 3

Initial Conditions for Double Parton Distributions
3.1 Mellin Transform

One of our most powerful tools in analyzing parton distribution functions is that of the Mellin transformation. The Mellin transform of a function \( f(x) \) is given by

\[
\tilde{f}(n) = \int_0^1 dx \, x^{n-1} f(x),
\]

(3.1.1)

where \( n \) is a complex number. Similarly, the double Mellin transform of a function \( f(x_1, x_2) \) is given by

\[
\tilde{f}(n_1, n_2) = \int_0^1 dx_1 \int_0^1 dx_2 \, x_1^{n_1-1} x_2^{n_2-1} f(x_1, x_2) \theta(1 - x_1 - x_2),
\]

(3.1.2)

where \( n_1 \) and \( n_2 \) are complex numbers. Under a Mellin transform convolution becomes multiplication, hence the simplifications that arise when applying the Mellin transform to our parton distribution evolution equations.

The inverse Mellin transform is given by

\[
f(x) = \int_C \frac{dn}{2\pi i} x^{-n} \tilde{f}(n),
\]

(3.1.3)

where the integration contour \( C \) is taken in the complex \( n \) plane and lies to the right of all singularities present in \( \tilde{f}(n) \). Similarly, the inverse double Mellin transform is given by

\[
f(x_1, x_2) = \int_{C_1} \frac{dn_1}{2\pi i} \int_{C_2} \frac{dn_2}{2\pi i} \, x_1^{-n_1} x_2^{-n_2} \tilde{f}(n_1, n_2),
\]

(3.1.4)

where once again the contours \( C_1 \) and \( C_2 \) are taken in the complex \( n_1 \) and \( n_2 \) planes respectively and lie to the right of all singularities present in \( \tilde{f}(n_1, n_2) \). Applying the Mellin transform to our single parton distribution evolution equations one has that at leading order

\[
\frac{\partial}{\partial t} \tilde{D}_1^f(n, t) = \sum_{f'} \gamma_{ff'}(n) \tilde{D}_1^{f'}(n, t),
\]

(3.1.5)

where \( \gamma \) is as defined below and \( t \) is as defined in Eq. (2.2.2). Similarly, applying a double Mellin transformation to the double parton distribution evolution equation, Eq. (2.2.7), and making some simplifications, we have that
\[ \frac{\partial}{\partial t} \tilde{D}^{f_1 f_2}_2 (n_1, n_2, t) = \sum_{f_1'} \gamma_{f_1' f_1} (n_1) \tilde{D}^{f_1' f_2}_2 (n_1, n_2, t) + \sum_{f_2'} \gamma_{f_2' f_2} (n_2) \tilde{D}^{f_2' f_2}_2 (n_1, n_2, t) + \]
\[ + \sum_{f'} \tilde{\gamma}_{f' f_1 f_2} (n_1, n_2) \tilde{D}^{f'}_2 (n_1 + n_2, t), \]
(3.1.6)

where we define \( \gamma \) and \( \tilde{\gamma} \) as follows

\[ \gamma_{f_1 f_2} (n) = \frac{1}{2\pi b} \int_0^1 dy \, y^n P_{f_1 f_2}^{(0)} (y), \]
(3.1.7)

\[ \tilde{\gamma}_{f' f_1 f_2} (n_1, n_2) = \frac{1}{2\pi b} \int_0^1 dy \, y^{n_1} (1 - y)^{n_2} P_{f' f_1 f_2}^{(0)} (y). \]
(3.1.8)

We can also write the momentum and valence quark number sum rules for single parton distribution functions in terms of Mellin moments. Taking the Mellin transform of both sides of Eq. (2.1.8) and Eq. (2.1.9) we have that

\[ 1 = \sum_f \tilde{D}_1^f (2, t) \]
(3.1.9)

\[ N_i = \tilde{D}_1^{q_i} (1, t) - \tilde{D}_1^{\bar{q}_i} (1, t). \]
(3.1.10)

Furthermore, we can write down both the momentum and valence quark number sum rules for double parton distribution functions in the Mellin moment space. By taking the Mellin transform of both sides of Eq. (2.2.8) we have that

\[ \sum_{f_1} \tilde{D}^{f_1 f_2}_2 (2, n_2, t) = \tilde{D}^{f_2}_1 (n_2, t) - \tilde{D}^{f_2}_1 (n_2 + 1, t). \]
(3.1.11)

Similarly, by applying the Mellin transform to the valence quark number sum rule for double parton distribution functions, we have that

\[ \tilde{D}^{q_i f_2}_2 (1, n_2, t) - \tilde{D}^{\bar{q}_i f_2}_2 (1, n_2, t) = (N_i - \delta_{f_2 q_i} + \delta_{f_2 \bar{q}_i}) \tilde{D}^{f_2}_1 (n_2, t). \]
(3.1.12)

Analogous sum rules hold for the second parton as well

\[ \sum_{f_2} \tilde{D}^{f_1 f_2}_2 (n_1, 2, t) = \tilde{D}_1^{f_1} (n_1, t) - \tilde{D}_1^{f_1} (n_1 + 1, t), \]
(3.1.13)

\[ \tilde{D}^{f_1 q_i}_2 (n_1, 1, t) - \tilde{D}^{f_1 \bar{q}_i}_2 (n_1, 1, t) = (N_i - \delta_{f_1 q_i} + \delta_{f_1 \bar{q}_i}) \tilde{D}_1^{f_1} (n_1, t). \]
(3.1.14)
3.2 Initial Conditions for the Single Channel

In this section we construct an ansatz for the single gluon channel that satisfies the momentum sum rule. We start by postulating that the single parton distribution at the initial scale $Q_0^2 (t = 0)$ is of the form

$$D_g^1 (x, t = 0) = N_1 x^{-\alpha} (1 - x)^\beta,$$  \hspace{1cm} (3.2.1)

where $N_1$ is the normalization which can be fixed from the momentum sum rule for single parton distribution functions,

$$1 = \int_0^1 dx x D_g^1 (x, t = 0).$$  \hspace{1cm} (3.2.2)

This gives us that

$$N_1 = \frac{1}{B(2 - \alpha, 1 + \beta)},$$  \hspace{1cm} (3.2.3)

where $B$ is the Euler Beta Function. Then we have that

$$\tilde{D}_g^1 (n, t = 0) = \frac{B(n - \alpha, \beta + 1)}{B(2 - \alpha, \beta + 1)}.$$  \hspace{1cm} (3.2.4)

Furthermore, we take as our ansatz for the double parton distribution

$$D_{gg}^2 (x_1, x_2, t = 0) = N_2 x_1^{\tilde{\alpha}_1} x_2^{\tilde{\alpha}_2} (1 - x_1 - x_2)^{\tilde{\beta}} \theta(1 - x_1 - x_2).$$  \hspace{1cm} (3.2.5)

Taking the double Mellin transform of our ansatz we have that

$$\tilde{D}_{gg}^2 (n_1, n_2, t = 0) = N_2 \frac{\Gamma(n_1 - \tilde{\alpha})\Gamma(n_2 - \tilde{\alpha})\Gamma(1 + \tilde{\beta})}{\Gamma(n_1 + n_2 + 1 + \tilde{\beta} - 2\tilde{\alpha})},$$  \hspace{1cm} (3.2.6)

where $\Gamma$ is the Gamma function. Now for the case of only the gluon channel, the momentum sum rules in the Mellin moment space reads
\[ \tilde{D}^{gg}_{2}(n_1, 2, t) = \tilde{D}^{g}_{1}(n_1, t) - \tilde{D}^{g}_{1}(n_1 + 1, t), \quad (3.2.7) \]
\[ \tilde{D}^{gg}_{2}(2, n_2, t) = \tilde{D}^{g}_{1}(n_2, t) - \tilde{D}^{g}_{1}(n_2 + 1, t). \quad (3.2.8) \]

Setting \( n_2 = 2 \) in Eq. (3.2.6) we have that

\[ \tilde{D}^{gg}_{2}(n_1, 2, t = 0) = N_2 \frac{\Gamma(n_1 - \tilde{\alpha})\Gamma(2 - \tilde{\alpha})\Gamma(1 + \tilde{\beta})}{\Gamma(n_1 + 3 + \tilde{\beta} - 2\tilde{\alpha})}. \quad (3.2.9) \]

This gives us the left hand side of Eq. (3.1.13). The right hand side of Eq. (3.1.13), written in terms of the Mellin moments of single parton distributions is

\[ \tilde{D}^{g}_{1}(n_1, t = 0) - \tilde{D}^{g}_{1}(n_1 + 1, t = 0) = \frac{1}{B(2 - \alpha, \beta + 1)} \left( B(n_1 - \alpha, \beta + 1) - B(n_1 + 1 - \alpha, \beta + 1) \right) \]
\[ = \frac{B(n_1 - \alpha, \beta + 2)}{B(2 - \alpha, \beta + 1)} \]
\[ = \frac{1}{B(2 - \alpha, \beta + 1)} \Gamma(n_1 - \alpha)\Gamma(2 + \beta) \Gamma(2 + \beta + n_1 - \alpha), \quad (3.2.10) \]

where we have used the fact that

\[ B(x, y) = B(x + 1, y) + B(x, y + 1). \quad (3.2.11) \]

We then require that Eq. (3.2.9) be equivalent to Eq. (3.2.10). This is achieved by simultaneously requiring

\[ \tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta + \alpha - 1 \quad (3.2.12) \]

and setting the normalization constant equal to

\[ N_2 = \frac{1}{B(2 - \alpha, \alpha + \beta)B(2 - \alpha, \beta + 1)} = \frac{1}{B(2 - \alpha, \beta + 1)B(2 - \alpha, \beta + 1)}. \quad (3.2.13) \]

The first observation is that the exponents which govern small \( x \) behavior are the same in the double parton distribution as in the single parton distribution, which is reflected by the fact that \( \tilde{\alpha} = \alpha \). This is physically understandable, since we expect that momentum correlation will not
affect partons with very small $x$. The second comment is that the exponents $\tilde{\beta}$ and $\beta$ differ by $\alpha$. That is, the growth of partons at small $x$ is actually correlated with the power at large $x$. The final comment is that these results are obviously symmetric with respect to the interchange of $n_1$ and $n_2$. This is required physically and is immediately apparent by looking at the property of the Beta function given in Eq. (3.2.11).

We now generalize this example by assuming that we can expand the single parton distribution in a series of beta distributions with different exponents $\alpha_k, \beta_k$ and different weights $a_k$. That is, in the momentum space, we would have that

$$D_1^g (x, t = 0) = N_1 \sum_{k=1}^{K} a_k x^{-\alpha_k} (1 - x)^{\beta_k}. \quad (3.2.14)$$

Without loss of generality we can set $a_1 = 1$. In principle there is no constraint on the value of $K$; the above sum could have an infinite number of terms. Let us assume that we know the coefficients $a_k$ in this distribution, for example from the fit to one of the existing parametrizations of the single parton distribution functions. In the Mellin moment space we then have

$$\tilde{D}_1^g (n, t = 0) = N_1 \sum_{k=1}^{K} a_k B(n - \alpha_k, \beta_k + 1). \quad (3.2.15)$$

The momentum sum rule for the single parton distribution functions fixes the normalization constant for $N_1$ to be

$$N_1 = \left[ \sum_{k=1}^{K} a_k B(2 - \alpha_k, \beta_k + 1) \right]^{-1}. \quad (3.2.16)$$

The difference of the single parton distribution functions appearing on the right hand side of Eq. (3.1.13) is then

$$\tilde{D}_1^g (n, t = 0) - \tilde{D}_1^g (n + 1, t = 0) = N_1 \sum_{k=1}^{K} a_k B(n - \alpha_k, \beta_k + 2) \quad (3.2.17)$$

where we have again used Eq. (3.2.11). We now take the ansatz for the double parton distribution function in the Mellin moment space to be

$$\tilde{D}_{2}^{gg} (n_1, n_2, t = 0) = N_2 \sum_{k=1}^{K} c_k \frac{\Gamma(n_1 - \tilde{\alpha}_k)\Gamma(n_2 - \tilde{\alpha}_k)\Gamma(1 + \tilde{\beta}_k)}{\Gamma(n_1 + n_2 + 1 + \beta_k - 2\tilde{\alpha}_k)}, \quad (3.2.18)$$
where we can take $c_1 = 1$. This, of course, corresponds to a series of functions of the type given in Eq. (3.2.6) but now with different exponents $\tilde{\alpha}_k$ and $\tilde{\beta}_k$. It is important to note that the number of terms in the above sum is the same as the number of terms in the expansion of the single parton distribution function. Upon setting $n_2 = 2$, we have that

$$
\hat{D}_{2g}^{ag}(n_1, 2, t = 0) = N_2 \sum_{k=1}^{K} c_k \frac{\Gamma(n_1 - \tilde{\alpha}_k)\Gamma(n_2 - \tilde{\alpha}_k)\Gamma(1 + \tilde{\beta}_k)}{\Gamma(n_1 + n_2 + 1 + \tilde{\beta}_k - 2\tilde{\alpha}_k)}.
$$

(3.2.19)

Using the momentum sum rule in the Mellin moment space, we can find the normalization $N_2$ and all of the coefficients $c_k$. In the single channel case the identification of the terms on the left and right hand sides of the momentum sum rule, Eq. (3.2.7), is simple: we just identify the k-th term in each series, i.e.

$$
N_1 a_k B(n - \alpha_k, \beta_k + 2) = N_2 c_k \frac{\Gamma(n_1 - \tilde{\alpha}_k)\Gamma(2 - \tilde{\alpha}_k)\Gamma(1 + \tilde{\beta}_k)}{\Gamma(n_1 + 3 + \tilde{\beta}_k - 2\tilde{\alpha}_k)},
$$

(3.2.20)

which gives us

$$
\tilde{\alpha}_k = \alpha_k, \quad \tilde{\beta}_k = \beta_k + \alpha_k - 1.
$$

(3.2.21)

The conditions on the coefficients $c_k$ are

$$
N_2 c_k = N_1 a_k \frac{1}{B(\beta_k + \alpha_k, 2 - \alpha_k)}.
$$

(3.2.22)

For $k = 1$ we simply have that

$$
N_2 = N_1 \frac{1}{B(\alpha_1 + \beta_1, 2 - \alpha_1)}
$$

(3.2.23)

and hence we have that

$$
c_k = a_k \frac{B(\alpha_1 + \beta_1, 2 - \alpha_1)}{B(\beta_k + \alpha_k, 2 - \alpha_k)}.
$$

(3.2.24)

Therefore, in principle, if we can get the initial conditions parametrized in the form of Eq. (3.2.14), or equivalently Eq. (3.2.15), then we have an ansatz for the double parton distribution function that satisfies the sum rule and is symmetric.
Chapter 3. Initial Conditions for Double Parton Distributions

3.3 Initial Conditions for the Gluons and Quarks

In this section we shall generalize the approach presented in the previous section to include quarks and to check if the momentum and quark number sum rules can be simultaneously satisfied for this ansatz. In the instance where we have quarks with different flavors in the evolution equations, the momentum sum rules, Eq. (3.1.11) and Eq. (3.1.13), read

\[ \sum_{f_1} \tilde{D}_{f_1 f_2}^f (2, n_2, t) = \tilde{D}_{f_1}^f (n_2, t) - \tilde{D}_{f_1}^f (n_2 + 1, t), \]  
\[ \sum_{f_2} \tilde{D}_{f_1 f_2}^f (n_1, 2, t) = \tilde{D}_{f_1}^f (n_1, t) - \tilde{D}_{f_1}^f (n_1 + 1, t). \]  

We also have the quark number sum rules, Eq. (3.1.12) and Eq. (3.1.14)

\[ \tilde{D}_{q_i}^f (1, n_2, t) - \tilde{D}_{\bar{q}_i}^f (1, n_2, t) = A_{if_2} \tilde{D}_{f_2}^f (n_2, t), \]  
\[ \tilde{D}_{f_1 q_i}^f (n_1, 1, t) - \tilde{D}_{f_1 \bar{q}_i}^f (n_1, 1, t) = A_{if_1} \tilde{D}_{f_1}^f (n_1, t), \]  

where \( A_{if_1} = N_i - \delta_{f_1 q_i} + \delta_{f_1 \bar{q}_i} \) and \( N_i \) is the number of valence quarks of flavor \( i \). Essentially, we must generalize the expansion in terms of the beta distributions to include quark degrees of freedom. We do assume, however, that we have the same number of terms in the expansion for each flavor and for gluons. That is, the sums over \( k \) contain the same number of terms. The corresponding expansions, Eq. (3.2.15) and Eq. (3.2.19), must be considered for each flavor

\[ \tilde{D}_{f_1}^f (n, t = 0) = \sum_{k=1} N_{f_1}^k B(n - \alpha_{f_1}^k, \beta_{f_1}^k + 1), \]  
\[ \tilde{D}_{f_1 f_2}^f (n_1, n_2, t = 0) = \sum_{k=1} N_{f_1 f_2}^k \frac{\Gamma(n_1 - \tilde{\alpha}_{f_1}^k)\Gamma(n_2 - \tilde{\alpha}_{f_2}^k)\Gamma(1 + \tilde{\beta}_{f_1 f_2}^k)}{\Gamma(n_1 + n_2 + 1 + \tilde{\beta}_{f_1 f_2}^k - \tilde{\alpha}_{f_1}^k - \tilde{\alpha}_{f_2}^k)}. \]  

The momentum sum rule for single parton distribution functions will put one constraint equation on the normalizations \( N_{f_1}^k \). Now, we can write the momentum sum rule, Eq. (3.3.1), using the above expansion and compare the \( k \)-th term on each side. We obtain the generalization of Eq. (3.2.20)

\[ \sum_{f_1} N_{f_1 f_2}^k \frac{\Gamma(2 - \tilde{\alpha}_{f_1}^k)\Gamma(n_2 - \tilde{\alpha}_{f_2}^k)\Gamma(1 + \tilde{\beta}_{f_1 f_2}^k)}{\Gamma(n_2 + 3 + \tilde{\beta}_{f_1 f_2}^k - \tilde{\alpha}_{f_1}^k - \tilde{\alpha}_{f_2}^k)} = N_{f_2}^k B(n_2 - \alpha_{f_2}^k, \beta_{f_2}^k + 2). \]  

Again, we obtain the conditions which read:
\[ \tilde{\alpha}^{f_2} = \alpha^{f_2}, \quad \tilde{\beta}^{f_1f_2} = \beta^{f_2} + \tilde{\alpha}^{f_1} - 1 = \beta^{f_2} + \alpha^{f_1} - 1. \]  

(3.3.8)

Furthermore, from the symmetry required under interchanging \( n_1 \) with \( n_2 \) and \( f_1 \) with \( f_2 \), we have that

\[ \tilde{\beta}^{f_1f_2} = \tilde{\beta}^{f_2f_1}, \]  

(3.3.9)

which by Eq. (3.3.8) means that

\[ \beta^{f_2} + \alpha^{f_1} = \beta^{f_1} + \alpha^{f_2}. \]  

(3.3.10)

This is nothing but the constraint on the powers of the single parton distribution functions. If there are \( N_f \) quark flavors, then we will have \( 2N_f \) constraints. There are \( 2(2N_f + 1) \) parameters \( \alpha, \beta \) for each value of \( k \), thus only \( 2N_f + 2 \) will be free.

Now, by choosing these conditions, we can write the equations for the normalizations \( N_k^{f_1f_2} \), which read

\[ \sum_{f_1} N_k^{f_1f_2} B(2 - \alpha_k^{f_1}, \alpha_k^{f_1} + \beta^{f_2}) = N_k^{f_2}. \]  

(3.3.11)

Now let us try to see what the quark number sum rule gives, and whether the momentum sum rule and quark number sum rules can be simultaneously satisfied. We use our ansatz, Eq. (3.3.6), in Eq. (3.3.3) and compare terms with the same \( k \) to obtain:

\[ N_k^{q_1f_2} \frac{\Gamma(1 - \tilde{\alpha}_k^q) \Gamma(n_2 - \tilde{\alpha}_k^{f_2}) \Gamma(1 + \tilde{\beta}_k^{q_f})}{\Gamma(n_2 + 2 + \tilde{\beta}_k^{q_f} - \tilde{\alpha}_k^q - \tilde{\alpha}_k^{f_2})} - N_k^{q_2f_2} \frac{\Gamma(1 - \tilde{\alpha}_k^q) \Gamma(n_2 - \tilde{\alpha}_k^{f_2}) \Gamma(1 + \tilde{\beta}_k^{q_f})}{\Gamma(n_2 + 2 + \tilde{\beta}_k^{q_f} - \tilde{\alpha}_k^q - \tilde{\alpha}_k^{f_2})} =
\]

\[ = A_{lqf_2} N_k^{f_2} B(n_2 - \alpha_k^{f_2}, 1 + \beta_k^{f_2}). \]  

(3.3.12)

Now, let us compare the above Eq. (3.3.12) with the similar condition Eq. (3.3.7) coming from the momentum sum rule. We observe that one of the beta function arguments on the right hand side of these equations is shifted by 1. On the left hand side of these relations, we notice that one of the arguments of the multinomial beta function is also shifted by 1. This means that we get consistent constraints on \( \tilde{\alpha}, \tilde{\beta} \) coefficients from both momentum sum and quark number sum rules. From Eq. (3.3.12) we get that

\[ \tilde{\alpha}_k^{f_2} = \alpha_k^{f_2}, \quad \tilde{\beta}_k^{q_1f_2} = \beta_k^{f_2} + \alpha_k^{q_1} - 1 \]  

(3.3.13)
and

\[ \tilde{\beta}^f_{k2} = \beta^f_k + \alpha^q_k - 1, \quad (3.3.14) \]

which are identical conditions to Eq. (3.3.8). Thus we can use these conditions and write additional

\[ N^q_{k1} B(1 - \alpha^q_k, \alpha^q_k + \beta^f_k) - N^{\bar{q}}_{k1} B(1 - \alpha^{\bar{q}}_k, \alpha^{\bar{q}}_k + \beta^f_k) = A_{1f2} N^f_{k2}. \quad (3.3.15) \]

Eq. (3.3.11) and Eq. (3.3.15) are two equations for unknowns \( N^f_{k2} \); they have to be solved for
each \( k \) separately. There are \( 2N_f + 1 \) equations in Eq. (3.3.11) and \( N_f (2N_f + 1) \) equations
in Eq. (3.3.15) which gives \( (2N_f + 1)(N_f + 1) \) constraints on \( (2N_f + 1)(2N_f + 1 + 1)/2 \)
\( (2N_f + 1)(N_f + 1) \) normalization constants of double parton distributions. However, due to the
flavor symmetry of double parton distribution functions,

\[ D^f_{1f2} (n_1, n_2, Q^2) = D^f_{2f1} (n_2, n_1, Q^2), \]

and also because of the sum rules for the single parton distributions, there are \( N_f \) less constraints
and therefore \( N_f \) normalization constants are free. In order to see this let us take the momentum
sum rule and evaluate it at \( n_2 = 1 \) for \( f_1 = q_i \) and \( f_2 = \bar{q}_i \). This of course needs to be considered
with caution since the quantity \( D^q_{1i} (1, t = 0) \) is the quark number. For this reason, this quantity
may be divergent; hence we assume some kind of regularization. Formally, we then have

\[ \sum_{f_1} \tilde{D}^f_{2f1} (2, 1, t = 0) = \tilde{D}^q_{1} (1, t = 0) - \tilde{D}^{\bar{q}}_{1} (2, t = 0), \quad (3.3.16) \]

\[ \sum_{f_1} \tilde{D}^f_{2f1} (2, 1, t = 0) = \tilde{D}^{q}_1 (1, Qt = 0) - \tilde{D}^{\bar{q}}_1 (2, t = 0). \quad (3.3.17) \]

Furthermore, let us take the quark number sum rule and evaluate it at \( n_2 = 2, t = 0 \):

\[ \tilde{D}^q_{2f2} (1, 2, t = 0) - \tilde{D}^{\bar{q}}_{2f2} (1, 2, t = 0) = (N_i - \delta_{q_f2} + \delta_{\bar{q}_f2}) \tilde{D}^{f2}_{1} (2, t = 0). \quad (3.3.18) \]

Therefore we can write
\[
\sum_{f_1} \tilde{D}_2^{g_1} (2, 1, t = 0) = \sum_{f_1} \tilde{D}_2^{f_1} (1, 2, t = 0) = \\
= \sum_{f_1} \tilde{D}_2^{g_1} (1, 2, t = 0) - \sum_{f_1} (N_i - \delta_{q_f} + \delta_{\bar{q}_f}) \tilde{D}_1^{f_1} (2, t = 0) \\
= \sum_{f_1} \tilde{D}_2^{g_1} (1, 2, t = 0) - N_i \sum_{f_1} \tilde{D}_1^{f_1} (2, t = 0) + \tilde{D}_1^{q_1} (2, t = 0) \\
- \tilde{D}_1^{\bar{q}_1} (2, t = 0) \\
= \sum_{f_1} \tilde{D}_2^{g_1} (2, 1, t = 0) - N_i + \tilde{D}_1^{q_1} (2, t = 0) - \tilde{D}_1^{\bar{q}_1} (2, t = 0). \tag{3.3.19}
\]

Then using Eq. (3.3.16) we have that

\[
\sum_{f_1} \tilde{D}_2^{g_1} (2, 1, t = 0) - N_i + \tilde{D}_1^{q_1} (2, t = 0) - \tilde{D}_1^{\bar{q}_1} (2, t = 0) = \tilde{D}_1^{q_1} (1, t = 0) - \tilde{D}_1^{\bar{q}_1} (2, t = 0), \tag{3.3.20}
\]

which is Eq. (3.3.17). Note that we have used sum rule conditions on single parton distributions given by Eq. (3.1.9) and Eq. (3.1.10). Hence, the corresponding $N_f$ equations for the normalizations, $N_k^{g_1,f_2}$, are also linearly dependent.
Chapter 4

Solutions for Double Parton Distributions
4.1 Analytical Solution for the Single Channel in Mellin Space

In this section we solve the evolution equation for double parton distribution functions while only considering the gluon channel. In particular, we explicitly evaluate the Mellin transformations of Eq. (2.2.7) for the single channel case and obtain a more specific case of the evolution equation given by Eq. (3.1.6). When only considering the gluon channel, Eq. (2.2.7) reads

\[
\frac{\partial}{\partial t} D_{2g}^{gg}(x_1, x_2, t) = \frac{1}{2\pi b} \int_0^1 dy \int_0^1 dz \delta(x_1 - yz) \theta(1 - z - x_2) P_{gg}^{(0)}(y) D_{2g}^{gg}(z, x_2, t) + \\
+ \frac{1}{2\pi b} \int_0^1 dy \int_0^1 dz \delta(x_2 - yz) \theta(1 - x_1 - z) P_{gg}^{(0)}(y) D_{2g}^{gg}(x_1, z, t) + \\
+ \frac{1}{2\pi b x_1 + x_2} P_{gg}^{(0)} \left( \frac{x_1}{x_1 + x_2} \right) D_1^{g}(x_1 + x_2, t) .
\]  

(4.1.1)

where \( t \) is given by Eq. (2.2.2). We first apply the double Mellin transform on both sides of Eq. (4.1.1). On the left hand side we have that

\[
\int_0^1 dx_1 \int_0^1 dx_2 x_1^{n_1-1} x_2^{n_2-1} \theta(1 - x_1 - x_2) D_{2g}^{gg}(x_1, x_2, t) = \tilde{D}_{2g}^{gg}(n_1, n_2, t).
\]  

(4.1.2)

Therefore, on the left hand side we simply have \( \frac{\partial}{\partial t} \tilde{D}_{2g}^{gg}(n_1, n_2, t) \). Considering the double Mellin transform of the first term on the right hand side of Eq. (4.1.1), we have

\[
\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dy \int_0^1 dz \delta(x_1 - yz) \theta(1 - x_1 - x_2) \theta(1 - x_1 - x_2) x_1^{n_1-1} x_2^{n_2-1} P_{gg}^{(0)}(y) D_{2g}^{gg}(z, x_2, t).
\]  

(4.1.3)

Evaluating the \( x_1 \) integral by using the delta function, we have

\[
\int_0^1 dy \int_0^1 dz \left[ \int_0^1 dz_2 \theta(1 - z - x_2) \theta(1 - yz - x_2) (yz)^{n_1-1} x_2^{n_2-1} P_{gg}^{(0)}(y) D_{2g}^{gg}(z, x_2, t) \right].
\]  

(4.1.4)

Examining the theta functions, we see that \( \theta(1 - z - x_2) \theta(1 - yz - x_2) = \theta(1 - z - x_2) \). This is so because \( y, z \in [0, 1] \), therefore \( yz \leq z \). We then have

\[
\left[ \int_0^1 dy y^{n_1-1} P_{gg}^{(0)}(y) \right] \left[ \int_0^1 dz \int_0^1 dx_2 z^{n_1-1} x_2^{n_2-1} \theta(1 - z - x_2) D_{2g}^{gg}(z, x_2, t) \right] = \gamma_{gg}(n_1) \tilde{D}_{2g}^{gg}(n_1, n_2, t),
\]  

(4.1.5)
where we have used the definition of the double Mellin transform and Eq. (3.1.7). Following the exact same procedure for the second term on the right hand side of Eq. (4.1.1), we get

\[ \gamma_{gg}(n_2) \tilde{D}_2^{gg}(n_1, n_2, t). \]  

(4.1.6)

We now apply the double Mellin transform to the third term on the right hand side of Eq. (4.1.1)

\[ \int_0^1 dx_1 \int_0^1 dx_2 \theta(1 - x_1 - x_2) x_1^{n_1-1} x_2^{n_2-1} \frac{1}{x_1 + x_2} P^{(0)}_{gg} \left( \frac{x_1}{x_1 + x_2} \right) D_1^q(x_1 + x_2, t). \]  

(4.1.7)

We make the change of variables \( x_1 \rightarrow z = x_1 + x_2 \). We then have

\[ \int_0^1 dx_1 x_1^{n_1} \int_{x_1}^{1-x_2} dz \theta(1-z)(z-x_1)^{n_2} \frac{1}{z} P^{(0)}_{gg} \left( \frac{x_1}{z} \right) D_1^q(z, t) \]

\[ = \int_0^1 dx_1 x_1 \int_{x_1}^1 dz \frac{1}{z} P^{(0)}_{gg} \left( \frac{x_1}{z} \right) D_1^q(z, t) (z-x_1)^{n_2}, \]

(4.1.8)

where we have changed the upper bound on the \( z \) integral to 1 because of the \( \theta(1 - z) \) term. This expression can be rewritten as follows

\[ \int_0^1 dx_1 x_1 \int_0^1 dy \int_0^1 dz \delta(x_1 - yz) P^{(0)}_{gg}(y) D_1^q(z, t)(z-x_1)^{n_2} \]

(4.1.9)

where we have used the fact that \( P^{(0)}_{gg}(x) = 0 \) for \( x \geq 1 \) to change the lower bound of the \( z \) integral. Using the delta function to evaluate the \( x_1 \) integral, we have

\[ \int_0^1 dy \int_0^1 dz (yz)^{n_1} (y - yz)^{n_2} P^{(0)}_{gg}(y) D_1^q(z, t) \]

\[ = \left[ \int_0^1 dy y^{n_1} (1 - y)^{n_2} P^{(0)}_{gg}(y) \right] \left[ \int_0^1 dz z^{n_1 + n_2} D_1^q(z, t) \right]. \]

Using Eq. (3.1.8), we have then for the third term on the right hand side of Eq. (4.1.1)

\[ \tilde{\gamma}_{gg}(n_1, n_2) \tilde{D}_1^q(n_1 + n_2, t) \]

(4.1.10)

Putting Eq. (4.1.5), Eq. (4.1.6), and Eq. (4.1.10) together, we have that
\[ \frac{\partial}{\partial t} \tilde{D}_{gg}^{2} (n_1, n_2, t) = \gamma_{gg}(n_1) \tilde{D}_{gg}^{2} (n_1, n_2, t) + \gamma_{gg}(n_2) \tilde{D}_{gg}^{2} (n_1, n_2, t) + \tilde{\gamma}_{gg}(n_1, n_2) \tilde{D}_{l}^{g}(n_1 + n_2, t). \] (4.1.11)

Our goal is now to solve this differential equation. We first consider the homogeneous equation without the third term. Then we have that
\[ \frac{\partial}{\partial t} \tilde{D}_{gg}^{2} (n_1, n_2, t) = \gamma_{gg}(n_1) \tilde{D}_{gg}^{2} (n_1, n_2, t) + \gamma_{gg}(n_2) \tilde{D}_{gg}^{2} (n_1, n_2, t). \] (4.1.12)

The solution to this homogeneous part of the evolution equation is now trivial. In particular, we have that
\[ \tilde{D}_{gg}^{2} (n_1, n_2, t) = e^{\gamma_{gg}(n_1)t} e^{\gamma_{gg}(n_2)t} \tilde{D}_{gg}^{2} (n_1, n_2, t = 0). \] (4.1.13)

To solve the entirety of the differential equation, we propose an ansatz solution, namely
\[ \tilde{D}_{gg}^{2} (n_1, n_2, t) = e^{\gamma_{gg}(n_1)t} e^{\gamma_{gg}(n_2)t} F_{gg}^{2} (n_1, n_2, t). \] (4.1.14)

Taking the derivative of both sides with respect to \( t \) we have that
\[ \frac{\partial}{\partial t} F_{gg}^{2} (n_1, n_2, t) = e^{-\gamma_{gg}(n_1)t} e^{-\gamma_{gg}(n_2)t} \tilde{\gamma}_{gg}(n_1, n_2) \tilde{D}_{l}^{g}(n_1 + n_2, t) \] (4.1.15)

The first line above is obtained by directly differentiating Eq. (4.1.14) while the second line is obtained from Eq. (4.1.11). Notice that in both lines above the first two terms are identical. Therefore we have that
\[ \frac{\partial}{\partial t} F_{gg}^{2} (n_1, n_2, t) = e^{-\gamma_{gg}(n_1)t} e^{-\gamma_{gg}(n_2)t} \tilde{\gamma}_{gg}(n_1, n_2) \tilde{D}_{l}^{g}(n_1 + n_2, t). \] (4.1.16)

The solution to this equation is given by
\[ F_{gg}^{2} (n_1, n_2, t) = \int_{0}^{t} dt' e^{-\gamma_{gg}(n_1)t'} e^{-\gamma_{gg}(n_2)t'} \tilde{\gamma}_{gg}(n_1, n_2) \tilde{D}_{l}^{g}(n_1 + n_2, t) + \tilde{D}_{gg}^{2} (n_1, n_2, t = 0). \] (4.1.17)
Using this result in Eq. (4.1.14) we have that

\[
\tilde{D}_{gg}^2(n_1, n_2, t) = \int_0^t dt' e^{\gamma_{gg}(n_1)(t-t')} e^{\gamma_{gg}(n_2)(t-t')} \tilde{\gamma}_{gg}(n_1, n_2) \tilde{D}_{1}(n_1 + n_2, t) + e^{\gamma_{gg}(n_1)t} e^{\gamma_{gg}(n_2)t} \tilde{D}_{gg}^2(n_1, n_2, t = 0). \tag{4.1.18}
\]

When only considering the gluon channel, the single parton distribution function, \( \tilde{D}_{1}(n) \), is given by

\[
\tilde{D}_{1}(n) = e^{\gamma_{gg}(n)t} \tilde{D}_{1}(n, t = 0). \tag{4.1.19}
\]

We then immediately see that we have an analytic form for the double parton distribution function, \( \tilde{D}_{gg}^2(n_1, n_2, t) \). We simply need to provide initial conditions for \( \tilde{D}_{gg}^2 \) and \( \tilde{D}_{1} \).

### 4.2 Numerical Results

Recall that the evolution parameter, \( t \), is given by

\[
t = \log \left( \frac{\log \left[ Q_0^2/\Lambda_{QCD}^2 \right]}{\log \left[ Q_0^2/\Lambda_{QCD}^2 \right]} \right), \tag{4.2.1}
\]

where \( Q_0^2 = 1 \text{ GeV}^2 \) and \( \Lambda_{QCD}^2 \) is taken from MSTW 2008 fit parameters [18]. Its value is given by \( \Lambda_{QCD}^2 = 0.187 \text{ GeV}^2 \).

Furthermore, \( \gamma(n) \) and \( \tilde{\gamma}(n_1, n_2) \) are given by

\[
\gamma(n) = \frac{1}{2\pi b} \left[ 2C_a \left( \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n} + \frac{1}{n-1} \right) - 2C_a \left( \psi(n+1) + \gamma_E \right) + \frac{11C_a}{6} - \frac{4N_f T_R}{6} \right], \tag{4.2.2}
\]

\[
\tilde{\gamma}(n_1, n_2) = \frac{2C_a}{2\pi b} \left[ \frac{\Gamma(n_1+1)\Gamma(n_2-1)}{\Gamma(n_1+n_2)} + \frac{\Gamma(n_1-1)\Gamma(n_2+1)}{\Gamma(n_1+n_2)} + \frac{\Gamma(n_1+1)\Gamma(n_2+1)}{\Gamma(n_1+n_2+2)} \right], \tag{4.2.3}
\]

where \( \Gamma \) is the gamma function, \( \psi \) is the polygamma function, and \( \gamma_E \) is the Euler-Mascheroni constant. \( C_a \) and \( N_f \) are as defined in Chapter 1. Furthermore, the constants above are defined as follows:
\[ N_f = 0, \]
\[ C_a = 3, \]
\[ T_R = 1/2, \]
\[ b = \frac{33 - 2N_f}{12\pi}. \] (4.2.4)

We now assume that the single parton distribution function for gluons has the form obtained from the leading order fits given by [18].

\[ xD_{1g}^g(x, Q^2) = N_1 x^{-\delta_g} (1 - x)^{\eta_g} \left( 1 + \epsilon_g \sqrt{x} + \gamma_g x \right), \] (4.2.5)
\[ \delta_g = 0.83657, \] (4.2.6)
\[ \eta_g = 2.3882, \] (4.2.7)
\[ \epsilon_g = -38.997, \] (4.2.8)
\[ \gamma_g = 1445.5, \] (4.2.9)
\[ N_1 = \left[ \int_0^1 dx x^{-\delta_g} (1 - x)^{\eta_g} \left( 1 + \epsilon_g \sqrt{x} + \gamma_g x \right) \right]^{-1} \approx 0.00337. \] (4.2.10)

We note that in the MSTW 2008 fits presented in [18] the normalization is different. Here we normalize the distribution such that the momentum sum rule is satisfied. Similarly, \( \tilde{D}_2(n_1, n_2, t = 0) \) is also a specified initial condition. In momentum space it is given by the ansatz given in Eq. (3.2.5). To be precise and more easily make use of our previously developed formalism, we take the ansatz for the initial condition of the single distribution to be of the form:

\[ D_{1g}^g(x) = N_1 \left[ a_1 x^{-\alpha_1} (1 - x)^{\beta_1} + a_2 x^{-\alpha_2} (1 - x)^{\beta_2} + a_3 x^{-\alpha_3} (1 - x)^{\beta_3} \right], \] (4.2.11)
where the parameters are given by

\[ a_1 = 1, \quad a_2 = \epsilon_g, \quad a_3 = \gamma_g, \] (4.2.12)
\[ \alpha_1 = \delta_g + 1, \quad \alpha_2 = \delta_g + 1/2, \quad \alpha_3 = \delta_g, \] (4.2.13)
\[ \beta_1 = \beta_2 = \beta_3 = \eta_g. \] (4.2.14)

According to the formalism described in Chapter 3, we take the double parton distribution to be of the form

\[ D_{2g}^g(x_1, x_2, Q_0^2) = N_2 \left[ c_1 x_1^{-\tilde{\alpha}_1} x_2^{-\tilde{\alpha}_1} (1 - x_1 - x_2)^{\tilde{\beta}_1} + c_2 x_1^{-\tilde{\alpha}_2} x_2^{-\tilde{\alpha}_2} (1 - x_1 - x_2)^{\tilde{\beta}_2} + c_3 x_1^{-\tilde{\alpha}_3} x_2^{-\tilde{\alpha}_3} (1 - x_1 - x_2)^{\tilde{\beta}_3} \right]. \] (4.2.15)
Also, due to the formalism given in chapter 3, the parameters must obey the resulting conditions:

\[ \tilde{\alpha}_k = \alpha_k, \quad \tilde{\beta}_k = \beta_k - 1 + \alpha_k. \]  

(4.2.16)

Furthermore, we also have the conditions on the coefficients \( c_k \) and \( N_2 \):

\[ N_2 = \frac{1}{B(\alpha_1 + \beta_1, 2 - \alpha_1)}, \]  

(4.2.17)

\[ c_k = \frac{a_k}{B(\beta_k + \alpha_k, 2 - \alpha_k)}, \]  

(4.2.18)

with \( c_1 = a_1 = 1 \).

Using the initial distribution given by Eq. (4.2.11) and the parameters given above we see below in Figure 4.1 the form of our single parton distribution function at the initial scale \( Q^2_0 \). For all plots shown, parton indices are omitted.

**Figure 4.1.** Single parton distribution, Eq. (4.2.11), given at the initial scale \( Q^2_0 \).

Similarly, using the initial distribution given by Eq. (4.2.15) and the parameters above, we see below in Figure 4.2 the form of the double parton distribution function given at the initial scale, \( Q^2_0 \) and \( x_2 = 10^{-2} \).
Double Parton Distribution Function at Initial Scale

\[ x_2 = 1 \times 10^{-2}, Q^2 = 1 \text{ GeV}^2 \]

\[ x_1 x_2 D_2 (x_1, x_2, Q^2) \]

\[ 10^{-5} \quad 10^{-4} \quad 10^{-3} \quad 0.001 \quad 0.01 \quad 0.1 \quad 1 \]

\[ 0 \quad 50 \quad 100 \quad 150 \]

\[ x_1 \]

Figure 4.2. The double parton distribution function, Eq. (4.2.15), given at the initial scale \( Q_0^2 \) and \( x_2 = 10^{-2} \).

In the Mellin moment space, our ansatz for the single distribution is given by Eq. (3.2.15) and our ansatz for the double distribution is given by Eq. (3.2.18). Using these initial conditions in Eq. (4.1.18) we then have the evolved double parton distribution function at an arbitrary scale, \( Q^2 \).

We can numerically invert the analytic solutions obtained in Eq. (4.1.19) and Eq. (4.1.18); in particular we have that

\[
D_1^g (x, Q^2) = \int_C \frac{dn}{2\pi i} x^{-n} \tilde{D}_1^g (n_1, Q^2),
\]

\[
D_2^{gg} (x_1, x_2, Q^2) = \int_{C_1} \frac{dn_1}{2\pi i} \int_{C_2} \frac{dn_2}{2\pi i} x_1^{-n_1} x_2^{-n_2} \tilde{D}_2^{gg} (n_1, n_2, Q^2).
\]

where the contour \( C \) is taken to the right of all poles present in \( \tilde{D}_1^g (n_1, Q^2) \) and the contours \( C_1 \) and \( C_2 \) are such that they lie to the right of all poles present in \( \tilde{D}_2^{gg} (n_1, n_2, Q^2) \).

Using Eq. (4.1.19) and numerically performing the integral in Eq. (4.2.19) we have the form of the single parton distribution function evolved up to scales \( Q^2 = 25 \text{ GeV}^2 \) and \( Q^2 = 100 \text{ GeV}^2 \), seen below in Figure 4.3.
In a similar fashion, using Eq. (4.1.18) and numerically performing the double integral in Eq. (4.2.20) we obtain a slice of the double parton distribution function for fixed $x_2 = 10^{-2}$ and scales $Q^2 = 25 \text{ GeV}^2$ and $Q^2 = 100 \text{ GeV}^2$. These can be seen below in Figure 4.4.

**Figure 4.3.** Single parton distribution function evolved to two different scales, namely $Q^2 = 25 \text{ GeV}^2$ and $Q^2 = 100 \text{ GeV}^2$. 

In a similar fashion, using Eq. (4.1.18) and numerically performing the double integral in Eq. (4.2.20) we obtain a slice of the double parton distribution function for fixed $x_2 = 10^{-2}$ and scales $Q^2 = 25 \text{ GeV}^2$ and $Q^2 = 100 \text{ GeV}^2$. These can be seen below in Figure 4.4.
Figure 4.4. Double parton distribution function evolved to two different scales, $Q^2 = 25$ GeV$^2$ and $Q^2 = 100$ GeV$^2$, with $x_2 = 10^{-2}$.

It should be noted that after the evolution up to some scale $Q^2$ that the momentum sum rule given by Eq. (2.2.8) is exactly satisfied by the parton distributions shown above.

Finally, in an effort to determine how well double parton distribution functions factorize, that is, how good of an approximation it is to say that $D_{gg}^{gg}(x_1, x_2, Q^2) = D_g^{gg}(x_1, Q^2) D_g^{gg} (x_2, Q^2)$, we plot the following quantity:

$$R^{gg} (x_1, x_2, Q^2) = \frac{D_{gg}^{gg} (x_1, x_2, Q^2)}{D_g^{gg} (x_1, Q^2) D_g^{gg} (x_2, Q^2)}.$$  \hfill (4.2.21)
This quantity can be seen below at the initial scale, $Q_0^2$, with $x_2 = 10^{-2}$ in Figure 4.5.

**Figure 4.5.** The ratio of double parton distribution function to the product of single parton distribution functions at the initial scale $Q_0^2 = 1 \text{ GeV}^2$ and $x_2 = 10^{-2}$.

We see that at this scale, this ratio is not at all near unity. Hence, one cannot factorize a double parton distribution function into a product of single parton distribution functions. In Figure 4.6 below, we see the ratio of the evolved parton distribution functions at scales $Q^2 = 25 \text{ GeV}^2$ and $Q^2 = 100 \text{ GeV}^2$ with $x_2 = 10^{-2}$.
Figure 4.6. The ratio of double parton distribution function to a product of single parton distributions functions at two different scales, $Q^2 = 25 \text{ GeV}^2$ and $Q^2 = 100 \text{ GeV}^2$, with $x_2 = 10^{-2}$.

Additionally, in Figure 4.7 below we see the quantity given in Eq. (4.2.21) plotted at the initial scale $Q_0^2$ and with a momentum fraction $x_2 = 0.3$.

Figure 4.7. The ratio of double parton distribution function to the product of single parton distribution functions at the initial scale $Q_0^2 = 1 \text{ GeV}^2$ and $x_2 = 0.3$.

We see that at this scale the double parton distribution function does not factorize well. This is
because the momentum fraction $x_2$ is much larger, therefore we expect there to be correlation between the two partons, and hence factorization not to be a good approximation. In Figure 4.8 below, we see the ratio of the evolved parton distribution functions at scales $Q^2 = 25 \text{ GeV}^2$ and $Q^2 = 100 \text{ GeV}^2$.

![Figure 4.8](image)

**Figure 4.8.** The ratio of double parton distribution function to a product of single parton distributions functions at two different scales, $Q^2 = 25 \text{ GeV}^2$ and $Q^2 = 100 \text{ GeV}^2$, with $x_2 = .3$.

It is observed that the factorization of the double parton distribution function into a product of single parton distribution functions holds at small values of the momentum fraction $x$. This is
understandable since partons which carry very small values of momenta should not experience strong kinematical correlations. On the other hand, we observe that factorization is broken for \( x > 10^{-1} \). Here, strong kinematical correlations exist and are enforced by the momentum sum rule, Eq. (2.2.8).
Chapter 5

Conclusions
Double parton distribution functions are of profound importance for calculating scattering cross sections in high energy collisions that occur in particle accelerators. For this reason, an accurate model of parton distribution functions is necessary. Unfortunately, these parton distribution functions cannot be obtained via perturbative QCD; due to limitations in lattice QCD calculations, single parton distribution functions are obtained by fitting observables to experimental data.

We have demonstrated the mathematical formalism for the construction of consistent initial conditions for the double parton distribution functions in the collinear approximation. These initial conditions have the property that they exactly and simultaneously satisfy the momentum sum and quark number sum rules. Furthermore, in this formalism, the double parton distribution’s functional behavior is related to the single parton distribution functions. We have shown that this condition imposes certain relations on the large and small $x$ behavior of both the single and double parton distribution functions.

We analytically solved the evolution equation at leading logarithmic order for the gluon channel double parton distribution function by making use of the Mellin transformation. We have illustrated that at small $x$ the factorization of a double parton distribution function into a product of single parton distribution functions approximately holds. At large $x$ however, this factorization breaks down as a result of kinematical correlations.

This is the first formalism that exactly and simultaneously satisfies both the momentum sum rule and quark number sum rule. Future outlook is to apply this formalism to the case where quarks are considered and to study the nature of the double parton distribution functions when both gluons and quarks are considered in the evolution equations.
Bibliography


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Education

The Pennsylvania State University, University Park, PA 16802.
- Physics and Mathematics double major
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Experience

Undergraduate Research Assistant, The Pennsylvania State University, University Park, PA 16802.
- Under the direction of Dr. Anna Staśto I am focusing on the phenomenology of high energy collisions of protons relevant for the Large Hadron Collider. In particular I am analyzing the multi-parton interactions from a theoretical and phenomenological standpoint. The goal is to compute the cross section of the Drell-Yan process in double parton interactions and to make predictions for the Large Hadron Collider.

Laboratory Teaching Assistant, The Pennsylvania State University, University Park, PA 16802.
- I assisted students with introductory electromagnetism laboratories and graded their weekly lab reports.

Undergraduate Research Assistant, The Pennsylvania State University, University Park, PA 16802.
- Under the direction of Dr. Douglas Cowen I simulated atmospheric neutrino backgrounds at the IceCube Neutrino Observatory in an attempt to be able to distinguish these background signals from those predicted to be exhibited by proton decay.

Learning Assistant, The Pennsylvania State University, University Park, PA 16802.
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Tutor, Penn State Harrisburg Learning Center, Middletown, PA 17057.
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Memberships and Scholarly Societies

- Society of Physics Students
- National Society of Collegiate Scholars
- Golden Key International Honour Society
- Sigma Pi Sigma Physics Honor Society
Scholarships and Awards

- **2014**
  - Evan Pugh Scholar Junior Award.
  - Leonard Euler Memorial Scholarship.

- **2013**
  - President Sparks Award.

- **2012**
  - President’s Freshman Award.

Computer skills

- **Basic**
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  - OpenOffice, C++

- **Advanced**
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Languages

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References

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