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# Problems in Partition Thoery

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# Abstract

Chapter 1: Partitions without sequences of consecutive integers as parts have been studied recently by many authors, including Andrews, Holroyd, Liggett, and Romik, among others. Their results include a description of combinatorial properties, hypergeometric representations for the generating functions, and asymptotic formulas for the enumeration functions. We complete a similar investigation of partitions into distinct parts without sequences, which are of particular interest due to their relationship with the Rogers-Ramanujan identities. Our main results include a double series representation for the generating function, an asymptotic formula for the enumeration function, and several combinatorial inequalities.

Chapter 2: We use the idea of index invariance under the Franklin mapping to prove higher power generalizations of two results discovered by M. V. Subbarao. We then apply similar ideas to a two-variable generalization of the Rogers-Ramanujan identities due to G. E. Andrews.

# Table of Contents

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# Chapter 1

# Distinct Parts Partitions Without Sequences

## 1.1 Introduction and statement of results

For $k \geq 2$, a *$k$-sequence* in an integer partition is any $k$ consecutive integers that all occur as parts (a standard general reference for integer partitions is [2]). Note that the case $k = 1$ is excluded because any part in a nonempty partition trivially forms a "1-sequence". The study of partitions *without* sequences was introduced by MacMahon in Chapter IV of [20]. Let $p_k(n)$ be the number of partitions of $n$ with no $k$-sequences, and let $p_k(m,n)$ be the number of such partitions with $m$ parts. Since the presence of a $k$-sequence in a partition also implies the presence of a $(k-1)$-sequence, MacMahon's results on page 53 of [20] can be stated as the following generating function for $p_2(m,n)$,

$$G_2(z;q) := \sum_{n,m\geq 0} p_2(m,n) z^m q^n = 1 + \sum_{n\geq 1} \frac{z^n q^n \left(q^6;q^6\right)_{n-1}}{(1-q^n)\left(q^2;q^2\right)_{n-1}\left(q^3;q^3\right)_{n-1}}, \tag{1.1.1}$$

where the $q$-Pochhammer symbol is defined by $(a)_n = (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$.

Partitions without $k$-sequences for arbitrary $k \geq 2$ arose more recently in the work of Holroyd, Liggett, and Romik on probabilistic bootstrap percolation models [14]. These partitions were also studied by Andrews [3], who found a (double) $q$-hypergeometric series expansion for the generating function,

$$\begin{aligned} G_k(z;q) &:= \sum_{n,m\geq 0} p_k(m,n) z^m q^n \qquad (1.1.2)\\ &= \frac{1}{(zq;q)_\infty} \sum_{r,s\geq 0} \frac{(-1)^r z^{kr+(k+1)s} q^{\frac{(k+1)k(r+s)^2}{2} + \frac{(k+1)(s+1)s}{2}}}{\left(q^k;q^k\right)_r \left(q^{k+1};q^{k+1}\right)_s}. \end{aligned}$$

Andrews' proof of this expression followed from the theory of $q$-difference equations. The first two authors and Lovejoy provided an alternative bijective proof [9], as well as some additional combinatorial insight into Andrews' $q$-difference equations. It should also be noted that Andrews gave another separate treatment of the case $k = 2$ in [3, Theorem 4], where he transformed MacMahon's expression (1.1.1) in order to write $G_2(1;q)$ in terms of one of Ramanujan's famous mock theta functions [25] (see [5, 10, 19] for a sampling of other recent results on the role of mock modular forms in hypergeometric $q$-series).

In addition to the combinatorial results described above, it is also of great interest to determine the asymptotic behavior of partitions; such study dates back to Hardy and Ramanujan's famous formula ((1.41) in [13]), which states that as $n \to \infty$,

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}. \tag{1.1.3}$$

In fact, such formulas for partitions without $k$-sequences were particularly important in [14], as the *metastability threshold* of the $k$-cross bootstrap percolation model is intimately related to asymptotic estimates of $\log(p_k(n))$. These approximations were subsequently refined in [3], [7], and [11], with the most recent progress due to Kane and Rhoades [17, Theorem 1.8], who proved the asymptotic formula

$$p_k(n) \sim \frac{1}{2k}\left(\frac{1}{6}\left(1-\frac{2}{k(k+1)}\right)\right)^{\frac{1}{4}} \frac{1}{n^{\frac{3}{4}}} \exp\left(\pi\sqrt{\frac{2}{3}\left(1-\frac{2}{k(k+1)}\right)n}\right). \tag{1.1.4}$$

*Remark.* The exponent in this formula was first determined by Holroyd, Liggett, and Romik [14], who showed that

$$\log\left(p_k(n)\right) \sim 2\sqrt{\left(\lambda_1 - \lambda_k\right) n}, \tag{1.1.5}$$

where $\lambda_k := \pi^2/(3k(k+1))$. Note that as $k$ becomes large this expression approaches $2\sqrt{\lambda_1} = \pi\sqrt{2n}/3$, which is the same exponent for $\log(p(n))$ seen in (1.1.3). However, the convergence regime is more intricate for the full enumeration functions, as it is not true that (1.1.4) approaches (1.1.3) as $k \to \infty$, even though $p_k(n) = p(n)$ for sufficiently large $k$.

We note further that the value of $\lambda_k$ was derived in [14] by way of the very interesting auxiliary function $f_k : [0,1] \to [0,1]$, which is defined as the unique decreasing, positive solution to the functional equation $f^k - f^{k+1} = x^k - x^{k+1}$. Theorem 1 of [14] gives the evaluation

$$\lambda_k = -\int_0^1 \log\left(f_k(x)\right) \frac{dx}{x}.$$

An alternative proof of the above evaluation is given in [4], which proceeds by rewriting $\lambda_k$ as a double integral and then making a change of variables that essentially gives the integral representation of the dilogarithm function [26].

In this paper we consider a natural variant of MacMahon's partitions by restricting to those partitions with no $k$-sequences that only have distinct parts. Following the spirit of the results mentioned above, we provide expressions for generating functions, describe their combinatorial properties, and determine asymptotic formulas. Let $Q_k(n)$ be the number of partitions of $n$ with no $k$-sequences or repeated parts, and define the refined enumeration function $Q_k(m,n)$ to be the number of such partitions with $m$ parts. Furthermore, denote the generating function by

$$\mathcal{C}_k\left(z;q\right) := \sum_{m,n\geq 0} Q_k(m,n) z^m q^n.$$

If the parts are not counted, i.e. $z = 1$, then we also write $\mathcal{C}_k(q) := \mathcal{C}_k(1;q)$.

We begin by considering the combinatorics of the case $k = 2$, which corresponds to those partitions into distinct parts with no sequences. This case is analogous to MacMahon's original study of partitions with no sequences, and a similar argument (using partition conjugation) leads to a generating function much like (1.1.1). In fact, this case was also considered by MacMahon, as on page 5 of [20] he studied the combinatorics of partitions in which each part differs by at least 2. Noting that $Q_2(n)$ also counts the number of such partitions of $n$, the formula at the bottom of page 6 in [20] implies that

$$\mathcal{C}_2(z;q) = \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n}.$$

In Chapter III of [20], MacMahon further observed that the distinct parts partitions with no sequences naturally arise as the combinatorial interpretation Rogers and Ramanujan's famous analytic identities [23]. In particular, these imply that $\mathcal{C}_2(z;q)$ specializes to the corresponding products (equations (10) and (11) of [23])

$$\mathcal{C}_2(1;q) = \frac{1}{\left(q,q^4;q^5\right)_\infty},$$
$$\mathcal{C}_2(q;q) = \frac{1}{\left(q^2,q^3;q^5\right)_\infty}.$$

*Remark.* The Rogers-Ramanujan identities have inspired an incredible amount of work across divergent areas of mathematics ever since their introduction more than a century ago. For a small (and by no means exhaustive!) collection of recent work, refer to [6, 10, 12, 16].

Our first result gives double hypergeometric $q$-series expressions for our new partition functions that are analogous to (1.1.2).

**Theorem 1.1.1.** *For $k \geq 2$, we have*

$$\mathcal{C}_k\left(z;q\right) = \sum_{j,r\geq 0} \frac{(-1)^j z^{kj+r} q^{\frac{(r+kj)(r+kj+1)}{2}+k\frac{j(j-1)}{2}}}{\left(q^k;q^k\right)_j \left(q;q\right)_r}.$$

We give two proofs of this theorem; the first uses $q$-difference equations as in [3] and [9], while the second follows the bijective arguments of [9].

*Remark.* In fact, the statement of Theorem 1.1.1 and equation (1.1.2) also hold for the trivial case $k=1$; here the $q$-series identities are true with $G_1(q) = \mathcal{C}_1(q) = 1$.

We next turn to the asymptotic study of partitions without $k$-sequences or repeated parts. As in [8], we use the Constant Term Method and a Saddle Point analysis in order to determine the asymptotic behavior of $\mathcal{C}_k(q)$ near $q=1$, and then apply Ingham's Tauberian Theorem to obtain an asymptotic formula for the coefficients. Before stating our results, we introduce two auxiliary functions (see Section 1.2 for the definition of the dilogarithm), namely

$$\begin{aligned} g_k(u) &:= -2\pi^2u^2 + \mathrm{Li}_2\left(e^{2\pi i u}\right) - \frac{1}{k}\mathrm{Li}_2\left(e^{2\pi i k u}\right), \\ h_k(x) &:= x^{k+1} - 2x + 1. \end{aligned}$$

We show in Proposition 1.2.1 that $h_k$ has a unique root $w_k \in (0,1)$, and we let $v_k$ be the point on the positive imaginary axis such that $w_k = e^{2\pi i v_k}$. In other words, $v_k := i\log(w_k^{-1})/(2\pi)$.

**Theorem 1.1.2.** *Using the notation above, as $n \to \infty$, we have*

$$Q_k(n) \sim \frac{\sqrt{\pi} g_k(v_k)^{\frac{1}{4}}}{\sqrt{-g_k''(v_k)} n^{\frac{3}{4}}} e^{2\sqrt{g_k(v_k)n}}.$$

*Remark.* The exponent for this result can be written in a form similar to (1.1.5). In particular,

$$\log\left(Q_k(n)\right) \sim 2\sqrt{\left(\gamma_1 - \gamma_k\right)n},$$

where $\gamma_k := -\int_0^{\frac{1}{2}} \log\left(f_k(x)\right) dx/(x(1-x))$.

We do not present the proof of this alternative expression for the exponent, as it follows directly from the arguments in Section 3 of [14] (with probability $(1+q^j)^{-1}$ for the analogous event $C_j$). Furthermore, the values of $\gamma_k$ do not simplify as cleanly as the $\lambda_k$, as the integral does not reduce to a dilogarithm evaluation. However, it is true that $\gamma_k$ decreases monotonically to 0 as $k$ increases, since the $f_k$ are decreasing in $k$. Additionally, a short calculation shows that $\gamma_1 = \pi^2/12$, which is again compatible with the exponent of Hardy and Ramanujan's asymptotic formula for partitions into distinct parts. The corresponding enumeration function was denoted by $q(n)$ in [13], where they showed that

$$q(n) \sim \frac{1}{4\cdot 3^{\frac{1}{4}} n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{3}}}.$$

*Remark.* In the case $k = 2$ we find that $w_2 = \phi^{-1}$, where $\phi := (1+\sqrt{5})/2$ is the golden ratio. Furthermore, the first and third special values on page 7 of [26] give the evaluation

$$\begin{aligned} g_2(v_2) &= \frac{1}{2}(\log\phi)^2 + \mathrm{Li}_2\left(\phi^{-1}\right) - \frac{1}{2}\mathrm{Li}_2\left(\phi^{-2}\right) \\ &= \frac{1}{2}(\log\phi)^2 + \frac{\pi^2}{10} - (\log\phi)^2 - \frac{\pi^2}{30} + \frac{1}{2}(\log\phi)^2 = \frac{\pi^2}{15}. \end{aligned}$$

Plugging in to the theorem statement, this gives

$$c_2(n) \sim \frac{\sqrt{\phi}}{2\cdot 3^{\frac{1}{4}}\sqrt{5}n^{\frac{3}{4}}} e^{2\pi\sqrt{\frac{n}{15}}},$$

which was previously proven by Lehner in his study of the Rogers-Ramanujan products in [18].

The remainder of the paper is structured as follows. In Section 1.2 we give many basic identities for hypergeometric $q$-series and determine the critical points of the auxiliary functions $g_k$ and $h_k$. Section 1.3 contains analytic and combinatorial proofs of the double series representation from Theorem 1.1.1, and also presents several combinatorial observations. We conclude with Section 1.4, where we use the Constant Term Method and a Saddle Point analysis to prove the asymptotic formula from Theorem 1.1.2.

## 1.2 Hypergeometric series and auxiliary functions

In this section we recall several standard facts from the theory of hypergeometric $q$-series, including useful identities for special functions and modular transformations.

### 1.2.1 Definitions and identities for $q$-series

The *dilogarithm* function [26, p. 5] is defined for complex $|x| < 1$ by

$$\mathrm{Li}_2(x) := \sum_{n\geq 0} \frac{x^n}{n^2}.$$

This function has a natural $q$-deformation that is known as the *quantum dilogarithm* [26, p. 28], which is given by ($|x|, |q| < 1$)

$$\mathrm{Li}_2(x;q) := -\log(x;q)_\infty = \sum_{n\geq 1} \frac{x^n}{n(1-q^n)}.$$

Moreover, an easy calculation shows that its Laurent expansion begins with the terms

$$\mathrm{Li}_2\left(x;e^{-\varepsilon}\right) = \frac{1}{\varepsilon}\mathrm{Li}_2(x) - \tfrac{1}{2}\log(1-x) + O(\varepsilon), \tag{1.2.1}$$

where the series converges uniformly in $x$ as $\varepsilon \to 0^+$.

Next, we recall two identities due to Euler, which state that [2, equations (2.2.5) and (2.2.6)]

$$\frac{1}{(x;q)_\infty} = \sum_{n\geq 0} \frac{x^n}{(q;q)_n}, \tag{1.2.2}$$

$$(x;q)_\infty = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\frac{n(n-1)}{2}}}{(q;q)_n}. \tag{1.2.3}$$

Finally, Jacobi's *theta function* is defined by

$$\theta(q;x) := \sum_{n\in\mathbb{Z}} q^{n^2} x^n. \tag{1.2.4}$$

In order to determine the asymptotic behavior near $q = 1$, we use for $\varepsilon > 0$ the modular inversion formula (cf. [24, p. 290]),

$$\theta\left(e^{-\varepsilon}; e^{2\pi i u}\right) = \sqrt{\frac{\pi}{\varepsilon}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi^2(n+u)^2}{\varepsilon}}. \tag{1.2.5}$$

### 1.2.2 Auxiliary functions

We now prove several useful facts about the auxiliary functions $h_k$ and $g_k$.

**Proposition 1.2.1.** *Adopt the above notation.*

$(i)$ *There is a unique root* $w_k \in (0,1)$ *of* $h_k(x)$.

$(ii)$ *The unique critical point of* $g_k$ *on the positive real axis is given by* $v_k$ *such that* $e^{2\pi i v_k} = w_k$. *Furthermore,* $g''(v_k) < 0$.

*Proof.* $(i)$ Descartes' Rule of Signs implies that $h_k$ has either zero or two positive real roots. It is immediate to verify that $h_k(0) = 1$, $h_k(1) = 0$ and $h_k(3/4) < 0$ for $k \geq 2$, so the second root must lie in $(0,1)$ as claimed.

$(ii)$ Next, to identify the critical points of $g_k$, we calculate its derivative

$$g_k'(u) = -4\pi^2 u - \log\left(1 - e^{2\pi i u}\right) 2\pi i + \log\left(1 - e^{2\pi i k u}\right) 2\pi i.$$

This vanishes precisely when

$$2\pi i u + \log\left(\frac{1 - e^{2\pi i k u}}{1 - e^{2\pi i u}}\right) = 0.$$

Exponentiating and writing $x := e^{2\pi i u}$ then shows that the critical points of $g_k(u)$ correspond to the roots of $h_k(x)$.

Finally, we calculate the second derivative (again writing $x = e^{2\pi i u}$) of $g_k$

$$g_k''(u) = -4\pi^2 + \frac{(2\pi i)^2 x}{1-x} - \frac{(2\pi i)^2 k x^k}{1-x^k} = -4\pi^2 \left(\frac{1}{1-x} - \frac{k x^k}{1-x^k}\right).$$

At the critical point this further simplifies, since $1 - w_k^k = w_k^{-1}(1 - w_k)$, which gives

$$g_k''(v_k) = \frac{-4\pi^2 \left(1 - k w_k^{k+1}\right)}{1 - w_k}.$$

We claim that at the critical point $1 - k w_k^{k+1} > 0$. Indeed, the derivative of $h_k$ is

$$h_k'(x) = (k+1)x^k - 2, \tag{1.2.6}$$

and at the root $w_k$, we have $h_k'(w_k) < 0$. Plugging in $w_k$ to (1.2.6), multiplying by $w_k$ and substituting $w_k^k = 2w_k - 1$ then implies that

$$0 > (k+1)w_k^{k+1} - 2w_k = k w_k^{k+1} - 1.$$

This completes the proof of $(ii)$. □

## 1.3 Proof of Theorem 1.1.1

### 1.3.1 Analytic proof

We follow Andrews' proof of Theorem 2 in [3]. First we observe that $\mathcal{C}_k$ satisfies the $q$-difference equation

$$\mathcal{C}_k(z;q) = \sum_{j=0}^{k-1} z^j q^{\frac{j(j+1)}{2}} \mathcal{C}_k\left(zq^{j+1};q\right). \tag{1.3.1}$$

The terms on the right result from conditioning on the length of the sequence that begins with 1. The $j = 0$ term corresponds to the case where there is no 1, and thus the smallest part is at least 2; the other terms correspond to the case that there is a run $1, 2, \ldots, j$, and no $j + 1$, so the next part is at least $j + 2$. Applying (1.3.1) twice, we obtain the relation

$$\mathcal{C}_k(z;q) - zq\mathcal{C}_k(zq;q) = \mathcal{C}_k(zq;q) - z^k q^{\frac{k(k+1)}{2}} \mathcal{C}_k\left(zq^{k+1};q\right). \tag{1.3.2}$$

Now consider the double series

$$F_k(z;q) := \sum_{j,r\geq 0} \frac{(-1)^j z^{kj+r} q^{\frac{(r+kj)(r+kj+1)}{2} + k\frac{j(j-1)}{2}}}{\left(q^k;q^k\right)_j (q;q)_r}.$$

Expanding this as a series in $z$, so that $F_k(z;q) =: \sum_{n\geq 0} \gamma_n(q) z^n$, we therefore have

$$\gamma_n = \sum_{kj+r=n} \frac{(-1)^j q^{\frac{r(r+1)}{2} + krj + \frac{k(k+1)j^2}{2}}}{\left(q^k;q^k\right)_j (q;q)_r}.$$

Now we calculate

$$\begin{aligned}(1-q^n)\,\gamma_n &= \sum_{kj+r=n} \frac{(-1)^j q^{\frac{r(r+1)}{2} + krj + \frac{k(k+1)}{2}j^2}}{\left(q^k;q^k\right)_j (q;q)_r} \Big( (1-q^r) + q^r\left(1-q^{kj}\right) \Big) \\ &= q^n \gamma_{n-1} - q^{(k+1)(n-k) + \frac{k(k+1)}{2}} \gamma_{n-k},\end{aligned}$$

where the first term follows from the shift $r \mapsto r+1$, and the second term from $j \mapsto j+1$. Multiplying by $z^n$ and summing over $n$ finally gives the $q$-difference equation

$$F_k(z;q) = (1+zq)F_k(zq;q) - z^k q^{\frac{k(k+1)}{2}} F_k\left(zq^{k+1};q\right).$$

As this is equivalent to (1.3.2), we therefore conclude (cf. [1] and the uniqueness of solutions to $q$-difference equations) that $\mathcal{C}_k = F_k$, completing the proof of Theorem 1.1.1.

### 1.3.2 Combinatorial proof

In this section we follow the approach from Section 3.2 of [9], using a combinatorial decomposition of partitions into simple components that essentially split the double summation in Theorem 1.1.1. Denote the size of a partition $\lambda$ by $|\lambda|$ and write $\ell(\lambda)$ for the number of parts, or *length*. Let $\mathcal{D}_k$ be the set of partitions without $k$-sequences or repeated parts, and note that with this notation we have

$$\mathcal{C}_k(z;q) = \sum_{\lambda \in \mathcal{D}_k} z^{\ell(\lambda)} q^{|\lambda|}.$$

If $\lambda \in \mathcal{D}_k$ and $\ell(\lambda) = m$, so that $\lambda = \lambda_1 + \cdots + \lambda_m$ in nonincreasing order, then define $\lambda'$ by removing a triangular partition $(m-1)+(m-2)+\cdots+1$, so that the new parts are

$$\lambda'_j := \lambda_j - (m-j), \qquad 1 \leq j \leq m.$$

The definition of $\mathcal{D}_k$ implies that $\lambda'$ is a partition in which each part occurs at most $k-1$ times, so

$$\sum_{\lambda \in \mathcal{D}_k} z^{\ell(\lambda')} q^{|\lambda'|} = \prod_{n \geq 1} \left(1 + zq^n + z^2q^{2n} + \cdots + z^{k-1}q^{n(k-1)}\right) = \frac{\left(z^kq^k;q^k\right)_\infty}{(zq;q)_\infty}.$$

Euler's summation formulas (Corollary 2.2 in [2]) then imply the double series

$$\sum_{\lambda \in \mathcal{D}_k} z^{\ell(\lambda')} q^{|\lambda'|} = \sum_{j,r \geq 0} \frac{(-1)^j z^{kj} q^{\frac{kj(j+1)}{2}}}{\left(q^k;q^k\right)_j} \frac{z^r q^r}{(q;q)_r}. \tag{1.3.3}$$

To complete the proof, observe that

$$\begin{aligned} \ell(\lambda) &= \ell(\lambda'), \\ |\lambda| &= |\lambda'| + \frac{\ell(\lambda)(\ell(\lambda)+1)}{2}. \end{aligned}$$

Plugging in to (1.3.3), we obtain

$$\mathcal{C}_k(z;q) = \sum_{j,r \geq 0} \frac{(-1)^j z^{kj+r} q^{\frac{(kj+r)(kj+r-1)}{2} + \frac{kj(j+1)}{2} + r}}{\left(q^k;q^k\right)_j (q;q)_r}.$$

Theorem 1.1.1 follows upon simplifying the exponent of $q$.

*Remark.* For example, if $k=3$ and $\lambda = 15+12+11+9+8+4+2+1$, then the associated $\lambda'$ is $8+6+6+5+5+2+1+1$, which consists of parts that are repeated at most twice.

### 1.3.3 Monotonicity

We close with several additional combinatorial observations on the monotonicity of the enumeration functions.

**Proposition 1.3.1.** *For $m, n \geq 0$ and $k \geq 2$, we have*

$(i)$ $Q_k(m,n) \leq Q_{k+1}(m,n)$,

$(ii)$ $Q_k(m,n) \leq Q_k(m,n+1)$.

*Proof.* As mentioned in the introduction, $(i)$ follows immediately from the definition. For $(ii)$, note that if $\lambda \in \mathcal{D}_k$ is a partition of $n$ with $m$ parts, then

$$(\lambda_1 + 1) + \lambda_2 + \cdots + \lambda_m$$

is a partition of $n+1$ with $m$ parts. Furthermore, this new partition remains in $\mathcal{D}_k$ since $\lambda_1 + 1 > \lambda_1 > \lambda_2 > \cdots > \lambda_m$. □

*Remark.* Part $(ii)$ has the important consequence that

$$Q_k(n) \leq Q_k(n+1). \tag{1.3.4}$$

A similar results hold for partitions without $k$-sequences: Lemma 10 in [14] states that $p_k(n) \leq p_k(n+1)$ (compare to (1.1.5), noting that $\lambda_k$ are increasing in $k$). However, the analog to part $(ii)$ is false in this case, as in general

$$p_k(m,n) \not\leq p_k(m,n+1).$$

For example, the partitions without sequences of 2 are $\{2, 1+1\}$, while the partitions of 3 are $\{3, 1+1+1\}$, so that $p_2(2,2) = 1$ and $p_2(2,3) = 0$.

## 1.4 Asymptotic Formulas

In this section, we study the asymptotic behavior of partitions without sequences or repeated parts, proving Theorem 1.1.2. We first determine the asymptotic behavior of the generating function $\mathcal{C}_k(q)$, and then deduce the asymptotic formula for its coefficients by applying Ingham's Tauberian Theorem.

### 1.4.1 Constant Term Method and Saddle Point analysis

We determine the asymptotic behavior of $\mathcal{C}_k(q)$ near $q = 1$ by using the Constant Term Method and a Saddle Point analysis. Throughout we restrict to real $q = e^{-\varepsilon}$ with $\varepsilon > 0$. The use of the Constant Term Method in the analytic study of $q$-series traces back to Meinardus [21], and Nahm, Recknagel, and Terhoeven introduced the additional tools of asymptotic expansions and Saddle Point analysis [22]. In order to apply these techniques to double summation $q$-series, we follow the work of the first two authors in [8].

The main technical result of this analysis is an asymptotic formula for $\mathcal{C}_k(q)$.

**Proposition 1.4.1.** *If $q = e^{-\varepsilon}$, then as $\varepsilon \to 0^+$ we have*

$$\mathcal{C}_k(q) = \frac{2\pi}{\sqrt{-g_k''(v_k)}}\left(1 + O\left(\varepsilon^{\frac{1}{2}}\right)\right)\exp\left(\frac{g_k(v_k)}{\varepsilon}\right).$$

*Proof.* We begin by rewriting the double series from Theorem 1.1.1 in the case $z = 1$ as

$$\begin{aligned}\mathcal{C}_k(q) &= q^{-\frac{1}{8}}\sum_{j,r\geq 0}\frac{(-1)^j q^{\frac{1}{2}\left(r+kj+\frac{1}{2}\right)^2+\frac{kj(j-1)}{2}}}{\left(q^k;q^k\right)_j (q;q)_r}\\ &= \text{coeff}\left[x^0\right]\left(q^{-\frac{1}{8}}\sum_{n\in\mathbb{Z}}x^{-n}q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\sum_{j\geq 0}\frac{(-1)^j x^{kj}q^{\frac{kj(j-1)}{2}}}{\left(q^k;q^k\right)_j}\sum_{r\geq 0}\frac{x^r}{(q;q)_r}\right).\end{aligned}$$

The sums on $n, j$, and $r$ can be expressed in terms of well-known functions using (1.2.4), (1.2.3), and (1.2.2), respectively. Plugging in the above definitions and applying Cauchy's Theorem, we obtain the integral representation

$$\begin{aligned}\mathcal{C}_k(q) &= \text{coeff}\left[x^0\right]\left(\theta\left(q^{\frac{1}{2}};x^{-1}q^{\frac{1}{2}}\right)\exp\left(-\mathrm{Li}_2\left(x^k;q^k\right)+\mathrm{Li}_2\left(x;q\right)\right)\right)\\ &= \int_{[0,1]+ic}\theta\left(q^{\frac{1}{2}};x^{-1}q^{\frac{1}{2}}\right)\exp\left(-\mathrm{Li}_2\left(x^k;q^k\right)+\mathrm{Li}_2\left(x;q\right)\right)du,\end{aligned}$$

where $c > 0$ is a constant that will be specified shortly. Letting $\varepsilon \mapsto \varepsilon/2$ and $u \mapsto -u + i\varepsilon/(4\pi)$ in (1.2.5), we obtain

$$\begin{aligned}\mathcal{C}_k(q) &= \sqrt{\frac{2\pi}{\varepsilon}}\sum_{n\in\mathbb{Z}}\int_{[0,1]+ic}\exp\left(-\frac{2\pi^2}{\varepsilon}\left(n-u+\frac{i\varepsilon}{4\pi}\right)^2-\mathrm{Li}_2\left(x^k;q^k\right)+\mathrm{Li}_2\left(x;q\right)\right)du\\ &= \sqrt{\frac{2\pi}{\varepsilon}}\int_{\mathbb{R}+ic}\exp\left(-\frac{2\pi^2}{\varepsilon}\left(u-\frac{i\varepsilon}{4\pi}\right)^2-\mathrm{Li}_2\left(x^k;q^k\right)+\mathrm{Li}_2\left(x;q\right)\right)du. \qquad (1.4.1)\end{aligned}$$

By (1.2.1), we have the asymptotic expansion

$$\mathrm{Li}_2\left(x;q\right)-\mathrm{Li}_2\left(x^k;q^k\right) = \frac{1}{\varepsilon}\left(\mathrm{Li}_2(x)-\frac{1}{k}\mathrm{Li}_2\left(x^k\right)\right)-\frac{1}{2}\log(1-x)+\frac{1}{2}\log\left(1-x^k\right)+O(\varepsilon).$$

In order to perform a Saddle Point analysis, the leading $1/\varepsilon$ term from the exponent in (1.4.1) must be isolated, which gives the definition of the auxiliary function $g_k$. The overall exponent in the integrand can then be written as

$$\exp\left(\frac{g_k(u)}{\varepsilon}+\pi iu+\frac{1}{2}\log\left(1-x^k\right)-\frac{1}{2}\log(1-x)+O(\varepsilon)\right). \qquad (1.4.2)$$

Proposition 1.2.1 implies that the asymptotic expansion of the integral is dominated by the critical point $v_k$, and the natural choice for the integration path is to set $c := \log(w_k^{-1})/(2\pi)$.

To conclude, we follow the standard argument by expanding the Taylor series around $v_k$ in (1.4.2), using the change of variables $u = v_k + \sqrt{\varepsilon}z$. We thereby obtain

$$\sqrt{\frac{1-w_k^k}{1-w_k}}\sqrt{w_k}\exp\left(\frac{g_k(v_k)}{\varepsilon}+\frac{g_k''(v_k)}{2}z^2+O\left(\varepsilon^{\frac{1}{2}}\right)\right).$$

The terms outside of the exponential in the above expression simplify to 1, so with the change of variables taken into account the integral becomes

$$\begin{aligned}\mathcal{C}_k(q) &= \sqrt{\frac{2\pi}{\varepsilon}}\left(1+O\left(\varepsilon^{\frac{1}{2}}\right)\right)\exp\left(\frac{g_k(v_k)}{\varepsilon}\right)\sqrt{\varepsilon}\int_{\mathbb{R}} e^{\frac{g_k''(v_k)}{2}z^2}dz \\ &= \frac{2\pi}{\sqrt{-g_k''(v_k)}}\left(1+O\left(\varepsilon^{\frac{1}{2}}\right)\right)\exp\left(\frac{g_k(v_k)}{\varepsilon}\right).\end{aligned}$$

The final equality follows from the Gaussian integral evaluation, which concludes the proof. □

### 1.4.2 Ingham's Tauberian Theorem

The asymptotic formula for partitions into distinct parts without sequences is now a consequence of the following Tauberian Theorem from [15]. This result describes the asymptotic behavior of the coefficients of a power series using its analytic behavior near the radius of convergence.

**Theorem 1.4.2** (Ingham)**.** *Let* $f(q) = \sum_{n\geq 0} a(n)q^n$ *be a power series with weakly increasing nonnegative coefficients and radius of convergence equal to* $1$*. If there are constants* $A > 0$ *and* $\lambda, \alpha \in \mathbb{R}$ *such that as* $\varepsilon \to 0^+$ *we have*

$$f\left(e^{-\varepsilon}\right) \sim \lambda\varepsilon^{\alpha}\exp\left(\frac{A}{\varepsilon}\right),$$

*then, as* $n \to \infty$*,*

$$a(n) \sim \frac{\lambda}{2\sqrt{\pi}}\frac{A^{\frac{\alpha}{2}+\frac{1}{4}}}{n^{\frac{\alpha}{2}+\frac{3}{4}}}\exp\left(2\sqrt{An}\right).$$

*Proof of Theorem 1.1.2.* Proposition 1.3.1 part $(ii)$ implies that the coefficients are monotonically increasing (see (1.3.4)). We can therefore apply Theorem 1.4.2 to the asymptotic formula from Proposition 1.4.1 and directly obtain the stated asymptotic formula for $Q_k(n)$. □

# Chapter 2

# Further Multivariate Generalizations of Euler's Pentagonal Number Theorem and the Rogers-Ramanujan Identities

## 2.1 Introduction

In 1970, M. V. Subbarao published a paper [31] providing combinatorial proofs of the following two generalizations of Euler's celebrated pentagonal number theorem:

$$\sum_{k=1}^{\infty} a^k(-aq^k)(aq;q)_{k-1} = \sum_{k=1}^{\infty}(-1)^k\big(a^{3k-1}q^{k(3k-1)/2} + a^{3k}q^{k(3k+1)/2}\big); \tag{2.1.1}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k a^{2k} q^{k(k+1)/2}}{(aq;q)_k} = \sum_{k=1}^{\infty}(-1)^k\big(a^{3k-1}q^{k(3k-1)/2} + a^{3k}q^{k(3k+1)/2}\big), \tag{2.1.2}$$

where $(a;q)_n$ is the $q$-shifted factorial defined as follows for $n \geq 0$:

$$(a;q)_n = \prod_{k=0}^{n-1}\big(1 - aq^k\big). \tag{2.1.3}$$

Both L. J. Rogers and N. J. Fine had proven before Subbarao that these three series are equal [27]. However, these proofs are both analytic. Subbarao's fundamental observation was that the Franklin mapping leaves the sum of the largest part and number of parts in the Ferrers diagram unchanged. (We refer the reader to [30] for complete details of Franklin's proof of the pentagonal number theorem). Then upon noticing that the exponent of $a$ in (2.1.1) records this quantity, which we will henceforth refer to as the *index*, we immediately see that the left-hand side equals

$$\sum_{k,r\geq 1}\big(S_{\mathrm{e}}(k,r) - S_{\mathrm{o}}(k,r)\big)a^r q^k. \tag{2.1.4}$$

Following Subbarao, $S_{\mathrm{e}}(n,m)$ $\big(S_{\mathrm{o}}(n,m)\big)$ denotes the number of partitions of $n$ with index $m$ into an even (odd) number of parts. After calculating the index for certain partitions of the pentagonal numbers for which the mapping fails, the desired result quickly follows.

At the end of his paper, Subbarao noted that invariance of the index under the Franklin mapping implies the invariance of real-valued functions of the index (such as the square of the index). He proceeded to ask whether or not one could obtain identities like (2.1.1) and (2.1.2) by taking advantage of the invariance of this more general quantity. We answer this question in the affirmative by rewriting the left-hand sides of (2.1.1) and (2.1.2), and using Franklin's combinatorial methods to deduce the following pair of identities:

$$\sum_{r,k\geq 1}(-1)^r q^{r(r-1)/2+k} a_{k+r}\begin{bmatrix}k-1\\ r-1\end{bmatrix} = \sum_{k=1}^{\infty}(-1)^k\big(a_{3k-1}q^{k(3k-1)/2} + a_{3k}q^{k(3k+1)/2}\big); \tag{2.1.5}$$

$$\sum_{r,k,j\geq 1}(-1)^k q^{rk+k(k-1)/2+j-1} a_{2k+r+j-2}\begin{bmatrix}k+j-3\\ j-1\end{bmatrix} = \sum_{k=1}^{\infty}(-1)^k\big(a_{3k-1}q^{k(3k-1)/2} + a_{3k}q^{k(3k+1)/2}\big). \tag{2.1.6}$$

Here the $a_n$ are indeterminates and the $\begin{bmatrix}n\\ k\end{bmatrix}$ are $q$-binomial coefficients defined as

$$\begin{bmatrix}n\\ k\end{bmatrix} = \begin{bmatrix}n\\ k\end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \tag{2.1.7}$$

for $0 \leq k \leq n$, and 0 otherwise. Equation (2.1.6) is a rewriting of (2.1.5) so that it is revealed as a companion to (2.1.2). Equations (2.1.1) and (2.1.2) follow from (2.1.5) and (2.1.6) upon setting $a_n = a^n$ and invoking the $q$-binomial theorem.

We note in passing an interesting identity that arises upon setting $a_n = n$ in (2.1.5) and using Zagier's identity [33, Theorem 2] to rewrite the subsequent right-hand side:

$$\sum_{r,k\geq 1} (-1)^r q^{r(r-1)/2+k}(k+r)\begin{bmatrix} k-1 \\ r-1 \end{bmatrix} = \sum_{n=0}^{\infty}[(q;q)_\infty - (q;q)_n] - (q;q)_\infty \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}. \tag{2.1.8}$$

Zagier introduced the series

$$F_1(q) := \sum_{n=1}^{\infty} n(q;q)_{n-1}q^n, \tag{2.1.9}$$

which he showed is equal [33] to

$$F_2(q) := \sum_{n=0}^{\infty}[(q;q)_n - (q;q)_\infty]. \tag{2.1.10}$$

As a power series in $\zeta - q$ for any root of unity $\zeta$, $F_1(q)$ equals the Kontsevich function

$$F(q) := \sum_{n=0}^{\infty}(q)_n, \tag{2.1.11}$$

and Zagier used this fact to describe the expansion of $F(q)$ for $q$ near roots of unity. (Indeed, $F(q)$ only makes sense as a complex function of $q$ at roots of unity, for the series does not converge anywhere else in $\mathbb{C}$). $F(q)$ was further studied in [34] as an example of a quantum modular form.

The key to proving (2.1.5) and (2.1.6) is nothing more than a few simple combinatorial observations. However, their relevance lies in the fact that similar techniques allow us to generalize Andrews' analytic version of Schur's combinatorial proof of the Rogers-Ramanujan identities [28]. Thus the second objective of this paper is to prove the following two theorems, where the $D_n$ are the classical Schur polynomials which admit simple closed form expressions as given in [28]:

**Theorem 2.1.1.** *If $D_{-1} = D_0 = 1$, $D_n = D_{n-1} + q^n D_{n-2}$ for $n > 0$ and $y_n$ is a sequence of indeterminates, then*

$$\sum_{k,n\geq 1} (-1)^k q^{\binom{k}{2}+2n} D_{n-2}\, y_{k+2n} \left( \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{2-k} \begin{bmatrix} n \\ k-2 \end{bmatrix} \right)$$
$$= \sum_{m=1}^{\infty} (-1)^m \big( y_{5m-2} q^{m(5m-1)/2} + y_{5m-1} q^{m(5m+1)/2} \big). \tag{2.1.12}$$

**Theorem 2.1.2.** *If $D^*_{-1} = 0$, $D^*_0 = 1$, $D^*_n = D^*_{n-1} + q^n D^*_{n-2}$ for $n > 0$ and $y_n$ is a sequence of indeterminates, then*

$$\sum_{k,n\geq 1} (-1)^k q^{\binom{k}{2}+2n} D_{n-2}\, y_{k+2n} \left( \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{2-k} \begin{bmatrix} n \\ k-2 \end{bmatrix} \right)$$
$$= \sum_{m=1}^{\infty} (-1)^m \big( y_{5m} q^{m(5m+3))/2} - y_{5m+2} q^{(m+1)(5m+1)/2} \big). \tag{2.1.13}$$

In Section 2 we will deduce (2.1.5) and (2.1.6) using Franklin's combinatorial methods and prove these results using recurrence-based arguments in Section 3. In Section 4 we will prove Theorems 1 and 2 and obtain the Rogers-Ramanujan identities as special cases. In Section 5 we will make some concluding remarks.

## 2.2 Combinatorial Arguments for (2.1.5) and (2.1.6)

We begin by studying the summand on the left-hand side of (2.1.1). For a particular $k$, the coefficient of $q^n$ is a polynomial in $a$ where each term is of the form $(-1)^r a^{k+r}$. Here, $r$ designates the number of parts in a particular partition of $n$. Let us now fix a particular $r = r_0$ and extract, from each polynomial coefficient of each $q^n$ that arises in the summand, the terms of the form $(-1)^{r_0} a^{k+r_0}$. If we do this for each $r$ from 1 to $k$, then we may reformulate the summand as

$$\sum_{r=1}^{k} p_{k,r}(q)(-1)^r a^{k+r}, \tag{2.2.1}$$

where $p_{k,r}(q)$ is the generating function for partitions into $r$ distinct parts with largest part exactly $k$. Hence $p_{k,r}(q)$ is simply the coefficient of $z^r$ in

$$zq^k(-zq;q)_{k-1} = \sum_{r=0}^{k-1} \begin{bmatrix} k-1 \\ r \end{bmatrix} z^{r+1} q^{r(r+1)/2+k}. \tag{2.2.2}$$

By the $q$-binomial theorem [29, Eq. (3.3.6)],

$$(z;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}, \tag{2.2.3}$$

this coefficient is found to be

$$\begin{bmatrix} k-1 \\ r-1 \end{bmatrix} q^{\binom{r}{2}+k}. \tag{2.2.4}$$

Now, because $a$ is a parameter that explicitly records the index in this generating function, we may invoke the invariance of the square of the index under the Franklin mapping to see that (2.1.5) holds in the case $a_n = a^{n^2}$. The argument in fact reveals that (2.1.5) holds generally for $a_n = a^{n^u}$, where $u$ is any real number, and so we may replace $a^{k+r}$ in (2.2.1) with any real-valued function of $k+r$, from which (2.1.5) easily follows. Note that the combinatorial argument is essential in passing from $a^{k+r}$ to $a_{k+r}$. As mentioned before, (2.1.5) is trivially true in the case $a_{k+r} = a^{k+r}$, but without knowing a priori that the identity resulting from comparing coefficients of $a^n$ holds (which we analytically deduce in the next section), one cannot say immediately that the statement is true for general $a_n$.

For (2.1.6), we first prove the following lemma. The argument is similar to the one offered by Subbarao for (2.1.2), using the same notation and paraphrasing in a little more detail.

**Lemma 2.2.1.** *For* $|a|, |q| < 1$,

$$\sum_{r,k\geq 1} \frac{(-1)^r a^{k+2r-1} q^{rk+\binom{r}{2}}}{(aq;q)_{r-1}} = \sum_{k=1}^{\infty} (-1)^k \left(a^{3k-1} q^{k(3k-1)/2} + a^{3k} q^{k(3k+1)/2}\right). \tag{2.2.5}$$

*Proof.* We note that a typical term in the summand of the left-hand side of (2.2.4) is of the form $u(n)q^n$, where

$$u(n)q^n = (-1)^r a^{k+2r-1} q^{rk+\binom{r}{2}} (aq)^{b_1} (aq^2)^{b_2} \cdots (aq^{r-1})^{b_{r-1}} \tag{2.2.6}$$

with $b_i \geq 1$. Clearly the exponent on $a$ is $k + 2r - 1 + b_1 + \cdots + b_{r-1}$, and $n$ equals

$$rk + \binom{r}{2} + \sum_{i=1}^{r-1} i b_i. \tag{2.2.7}$$

We rewrite (2.2.7) as a sum of $r$ distinct positive integers $c_1 + \cdots + c_r$, with $c_j = k + j + b_{r-1} + \cdots + b_{r-j}$ for $1 \leq j \leq r-1$ and $c_r = k$. The desired result then follows upon invoking the Franklin mapping. □

Alternatively, one may note that the sum indexed by $k$ is a geometric series whose sum equals $aq^r/(1 - aq^r)$. Thus the left-hand side of (2.2.4) equals the left-hand side of (2.1.2), which in turn equals the right-hand side of (2.2.4). Hence the lemma is proven.

The $q$-binomial theorem in the following form [29, Eq. (3.3.7)]:

$$\frac{1}{(z;q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j \tag{2.2.8}$$

applied to $\frac{1}{(aq;q)_{r-1}}$ then results in (2.1.6), after employing the same logic used to prove (2.1.5).

## 2.3 Analytic Proofs of (2.1.5) and (2.1.6)

The fact that the $a$ terms are isolated on either side of (2.1.5) and (2.1.6) suggests that we can prove the identities by comparing coefficients. We comment that our arguments here are alternatives to the classical proofs of (2.1.1) and (2.1.2), which employ the Rogers-Fine identity. After shifting $k \to k - r$ in the left-hand side of (2.1.5) and interchanging the order of summation, we obtain the following identity upon comparing coefficients:

$$q^{k-1} \sum_{r=1}^{k-1} (-1)^r q^{r(r-3)/2+1} \begin{bmatrix} k-r-1 \\ r-1 \end{bmatrix} = \begin{cases} (-1)^k q^{k(3k+1)/2} & \text{if } k \equiv 0 \pmod 3, \\ 0 & \text{if } k \equiv 1 \pmod 3, \\ (-1)^k q^{k(3k-1)/2} & \text{if } k \equiv -1 \pmod 3. \end{cases} \tag{2.3.1}$$

Dividing both sides by $q^{k-1}$, shifting $r \to r + 1$, and substituting $k - 2$ for $n$, (2.3.1) becomes

$$\sum_{r=0}^{n} (-1)^r q^{r(r-1)/2} \begin{bmatrix} n-r \\ r \end{bmatrix} = \begin{cases} (-1)^k q^{k(3k-1)/2} & \text{if } n = 3k, \\ (-1)^k q^{k(3k+1)/2} & \text{if } n = 3k+1, \\ 0 & \text{if } n = 3k+2. \end{cases} \tag{2.3.2}$$

Let $S(n)$ denote the left-hand side of (2.3.2), $R(n)$ the right-hand side, and

$$T(n) = \sum_{r=0}^{n} (-1)^r q^{r(r+1)/2} \begin{bmatrix} n-r \\ r \end{bmatrix}. \tag{2.3.3}$$

Then

$$\begin{aligned} T(n)-T(n-1) &= \sum_{r=0}^{n}(-1)^r q^{r(r+1)/2}\left(\begin{bmatrix} n-r \\ r \end{bmatrix} - \begin{bmatrix} n-r-1 \\ r \end{bmatrix}\right) \\ &= \sum_{r=0}^{n-1}(-1)^r q^{r(r+1)/2+n-2r}\begin{bmatrix} n-r-1 \\ r-1 \end{bmatrix} \\ &= -q^{n-1}\sum_{r=0}^{n-2}(-1)^r q^{r(r-1)/2}\begin{bmatrix} n-r-2 \\ r \end{bmatrix} \\ &= -q^{n-1}S(n-2). \end{aligned} \tag{2.3.4}$$

Similarly we can prove $T(n-1)-T(n-2)=S(n)$, which implies that $S(n)=-q^{n-2}S(n-3)$. A quick check that $R(n)$ satisfies this same recurrence proves (2.1.5). Warnaar presents a different proof of (2.3.2) in [32].

In (2.1.6), we first shift $j \to j-2r+2$ and interchange the $j$ and $r$ sums to obtain

$$\sum_{k,j,r\geq 1}(-1)^r q^{kr+r(r-1)/2+j-2r+1} a_{k+j}\begin{bmatrix} j-r-1 \\ r-2 \end{bmatrix}. \tag{2.3.5}$$

Now we shift $j \to j-k$ and interchange the $j$ and $k$ sums to get

$$\sum_{j=1}^{\infty} a_j \sum_{k=1}^{j-1} \sum_{r=1}^{\lfloor (j-k+1)/2 \rfloor} (-1)^r q^{kr+r(r-1)/2+j-k-2r+1}\begin{bmatrix} j-k-r-1 \\ r-2 \end{bmatrix}. \tag{2.3.6}$$

We study the inner two sums:

$$\begin{aligned} &\sum_{k=1}^{j-1} \sum_{r=1}^{\lfloor (j-k+1)/2 \rfloor} (-1)^r q^{kr+r(r-1)/2+j-k-2r+1}\begin{bmatrix} j-k-r-1 \\ r-2 \end{bmatrix} \\ &\qquad = \sum_{r=1}^{\lfloor j/2 \rfloor}(-1)^r q^{r(r-1)/2+(j-2r+1)r}\sum_{k=0}^{j-2r} q^{k(1-r)}\begin{bmatrix} k+r-2 \\ k \end{bmatrix} \\ &\qquad = \sum_{r=1}^{j-1}(-1)^r q^{r(r-1)/2+j-r}\begin{bmatrix} j-r-1 \\ r-1 \end{bmatrix}, \end{aligned} \tag{2.3.7}$$

where the first step involves an interchange of sums followed by the substitution $k \to j-2r+1-k$, and the second uses the identity

$$\sum_{k=0}^{K} q^{km}\begin{bmatrix} k-m-1 \\ k \end{bmatrix} = q^{Km}\begin{bmatrix} K+m \\ K \end{bmatrix}, \tag{2.3.8}$$

which can be proven by induction. The result then follows upon comparison with (2.3.1).

## 2.4 Proofs of Theorems 2.1.12 and 2.1.13

Andrews' original left-hand side (as presented in Theorems 1 and 2 of [28]) associated with our Theorem 1 is

$$-\sum_{n=1}^{\infty}(yq;q)_{n-1}y^{2n+1}q^{2n}D_{n-2}-\sum_{n=1}^{\infty}(yq;q)_{n}y^{2n+2}q^{2n+1}D_{n-2}. \tag{2.4.1}$$

The proof proceeds in much the same way as that of (2.1.5), the only difference being that we now extract terms of the form $(-1)^{k_0}y^{2n+k_0}$ and $(-1)^{k_0}y^{2n+1+k_0}$. (See [28] for details as to why $2n+k_0$ and $2n+1+k_0$ are the invariant quantities under Schur's transformations). One obtains two simplified series that have the same form as (2.2.1), namely they each involve $p_{n,k}(q)$ summed against a polynomial in $y$. Substituting in (2.2.4) for $p_{n,k}(q)$, with $k$ replaced by $n$, yields Theorem 1. Theorem 2 is just the identity that results from the same procedure applied to

$$-\sum_{n=1}^{\infty}(yq;q)_{n-1}y^{2n+1}q^{2n}D^{*}_{n-2}-\sum_{n=1}^{\infty}(yq;q)_{n}y^{2n+2}q^{2n+1}D^{*}_{n-2}. \tag{2.4.2}$$

We now deduce the first of the Rogers-Ramanujan identities [29] from Theorem 1, which states that

$$(q;q)_\infty\lim_{n\to\infty}D_n=(q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=1+\sum_{n=1}^{\infty}(-1)^n\big(q^{n(5n-1)/2}+q^{n(5n+1)/2}\big), \tag{2.4.3}$$

and leave out a proof for the second as the treatment would be nearly identical. Adding 1 to the left-hand side of (2.1.12) and setting $y_n$ to be the constant sequence 1 yields

$$1+\sum_{n,k\geq 1}(-1)^k q^{k(k-1)/2+2n}D_{n-2}\left(\begin{bmatrix}n-1\\k-1\end{bmatrix}+q\begin{bmatrix}n\\k-1\end{bmatrix}\right). \tag{2.4.4}$$

Using (2.2.3), (2.4.4) becomes

$$\begin{aligned}1-\sum_{n=1}^{\infty}q^nD_{n-2}\big(q^n(q;q)_{n-1}+q^{n+1}(q;q)_n\big)\\ =(q;q)_\infty\lim_{n\to\infty}D_n,\end{aligned}$$

by [28, Lemma 1]. Clearly 1 added to the the right-hand side of (2.1.12) equals

$$1+\sum_{n=1}^{\infty}(-1)^n\big(q^{n(5n-1)/2}+q^{n(5n+1)/2}\big), \tag{2.4.5}$$

and (2.4.3) is proven.

## 2.5 Concluding Remarks

It would be highly desirable to obtain simple $q$-hypergeometric proofs of Theorems 1 and 2 in this paper or the corresponding theorems in Andrews' paper. A shift from $k \to k-2n$ on the left-hand side of (2.1.12), followed by an interchange of sums, yields the following identity after comparing coefficients of $y_k$:

$$S(k) = \sum_{n=1}^{\lfloor (k-1)/2 \rfloor} (-1)^k q^{(k-2n-1)(k-2n-2)+2n+1} D_{n-2} \times \left( q^{k-2n-2} \begin{bmatrix} n-1 \\ k-2n-1 \end{bmatrix} - \begin{bmatrix} n \\ k-2n-2 \end{bmatrix} \right) := T(k), \tag{2.5.1}$$

where

$$S(k) = \begin{cases} (-1)^k q^{k(5k-1)/2} & \text{if } k \equiv 3 \pmod 5, \\ (-1)^k q^{k(5k+1)/2} & \text{if } k \equiv 4 \pmod 5, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.2}$$

Christoph Koutschan has used his package HolonomicFunctions to provide a computer proof of this equality. His package generates the simple recurrence relation $T(k+5) = -q^{k+4}T(k)$, and it is not difficult to check that $S(k)$ satisfies this as well.

Also interesting would be further analytic or partition theoretic generalizations of this type.

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# ACADEMIC VITA


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### RESEARCH INTERESTS

- Partition Theory, Basic hypergeometric series with applications to Bailey Transform

### EDUCATION

**2016 B.S. Mathematics, Penn State University, State College, PA** (expected)

- Junior undergraduate student with adviser George Andrews

### PROGRAMS ATTENDED

- **Summer 2014, Invitational Visit to University of Cologne:** Worked with Kathrin Bringmann and Larry Rolen for two weeks on (i) distinct parts partitions avoiding $k$ sequences and (ii) quantum modularity of two-variable partial theta functions
- **Summer 2014, Penn State REU in Lie Groups and Lie Algebras:** Took up two projects, one research and the other expository; Expository paper: *A Detailed Approach to the Macdonald Identities*; Research talk: *Analytic Identities Arising from Gordon's Combinatorial Generalization of the Rogers-Ramanujan Identities*
- **Summer 2013, Oakland University REU in Discrete Math:** Worked on problems in (i) Constraint programming and (ii) Matching preclusion in hypercube and star graphs

### PUBLICATIONS

- *Generalized Matching Preclusions*; with Zachary Wheeler, Eddie Cheng, Dana Ferranti, and Laszlo Liptak; submitted to *Discrete Applied Mathematics*
- *Further Multivariate Generalizations of Euler's Pentagonal Number Theorem and the Rogers-Ramanujan Identities*; submitted to *Integers*
- *Distinct Parts Partitions Without Sequences*; with Kathrin Bringmann and Karl Mahlburg; submitted to *The Electronic Journal of Combinatorics*