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MINIMUM VARIANCE FRONTIER SPANNING TEST

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ABSTRACT

Portfolio managers of all kinds are faced with the challenging task of building the most efficiently diversified portfolio by selecting investment instruments from wide array of risky and risk-free investment opportunities. The concept of diversification has been around for centuries, but it was not until 1952 when Harry Markowitz brought about the Modern Portfolio Theory that diversification optimized through the minimum variance frontier. The minimum variance frontier minimizes the level of volatility for a given level of expected return for a portfolio. This paper aims to discover what different compositional changes made to the portfolio will result in the best return-risk dynamic of the minimum variance frontier. Results will bring about better understanding of the relationship between specific key financial characteristics and qualities of investment instruments and performance of the minimum variance frontier.

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INTRODUCTION

The Markowitz portfolio theory or modern portfolio theory dictates that with a portfolio that is composed of risky assets such equities and fixed income securities, one is able to extract the minimum variance frontier through minimizing variance or standard deviation of portfolio for every level of expected return. This combination of minimum variance for every given level of expected return plotted on a graph represents the minimum variance frontier, which highlights the most efficient and achievable investment boundary for the given portfolio of risky assets.

The minimum variance frontier is a critical investment toolset to determining optimal portfolio diversification. Using such toolset, an investor should be able to achieve the best-diversified portfolio by picking portfolios that offer lowest risk level for any given level of expected return. The modern portfolio theory essentially strives to construct the most efficiently diversified portfolio through interplaying the inherent nature of risk and reward expressed in every financial asset.

The implication of the minimum variance frontier is of significant use today as fund managers of prominent investment management firms and hedge funds rely on this metric to assess their wide array of investment opportunities as well as plot out the most efficient portfolio compositions. Because investors and fund managers alike face having to pick only a limited number of risky assets, finding the minimum variance frontier of any given portfolio becomes a crucially important portfolio analytical toolset to finding the optimally risky portfolio that aligns with investment criteria and objectives.

This research paper aims to focus on analyzing the various compositional effects adding investment instruments of different asset classes with varying financial characteristics would have on the portfolio in terms of its minimum variance frontier. This investigation will help us understand what kind of financial assets, if added onto a portfolio, would have had historically generated the largest shift in the minimum variance frontier, resulting in the lowest level of risk and highest level of return.

LITERATURE REVIEW & RELEVANT STUDIES

Many studies and research papers pertaining to the minimum variance frontier published in the last several decades since the feat of Harry Markowitz in 1952. Because minimum variance frontier is utilized as a function of generating the most diversified portfolio, many studies have been conducted in the realms of international markets. Diversifying a portfolio to include not just domestic but also international securities prove to be statistically significant in its attempt to optimize diversification (Kan, 2012). In other words, international investments improve the variance of the minimum variance portfolio. However, this pertains to merely the minimum variance portfolio, which is the portfolio with the lowest level of variance on the minimum variance frontier. In regards to other investment opportunity sets on the minimum variance frontier, there is not enough data to conclude whether or not international investment is beneficial in diversification.

According to the William Sharpe (Sharpe, 1964), the most optimal risky portfolio on the minimum variance frontier is the tangential point of the capital allocation line, which represents every possible investment opportunity set for risky and risk-free assets. This tangential point describes to have the highest Sharpe ratio or return-risk trade off. Theoretically, the tangential point is the opportunity set on the minimum variance frontier that generates highest profitability. However, according to new empirical data, the global minimum variance portfolio serves to be more attractive investment opportunities (Jorion, 1991). Historically, the minimum variance portfolios outperformed any other investment set on the minimum variance frontier (DeMiguel & Nogales, 2007).

It is important to note such variability of performance of the modern portfolio theory, because as time goes by, the investment world changes with it. The market continuously adapts to changes of expectation in lieu of innovation and technological advancements, globalization of a financial market that promotes interconnectivity, and regulatory and political dynamics. Because the market is bound to change, the investment opportunity sets and their return-risk profiles will change dramatically as well. Thus, it is paramount to analyze compositional changes of portfolio and its varying effect on the minimum variance frontier not only from an objective viewpoint, but also from a subjective viewpoint. Subjective viewpoint promotes rationality of investment decisions that supersedes quantitative modeling or research.

Research conducted by Andrey Ukhov (2005) entitled "Expanding the Frontier One Asset at a Time" demonstrates to appease discrepancies reflected between the theoretical and actual return-risk profiles historically based on the modern portfolio theory. The study first looks to establish *n* assets in its portfolio. Then, it analyzes its compositional changes from creating (n+1) assets. The research found that the covariance carries heavy importance in dictating compositional changes resulting shifts in the minimum variance frontier. The following is his concluding remark regarding covariance (2005):

> Another new result in our paper is the three mutual fund theorem that shows the connection between the classic mean-variance two mutual fund spanning result and the role of co-variances. When the new asset is uncorrelated with all existing assets, the frontier is spanned by the new asset and by two mutual funds located on the old frontier. When the new asset is correlated with any of the existing assets, the picture changes significantly. The new frontier can no longer be spanned by the investment in the new asset alone (and the two mutual fund containing only old assets) (p. 13).

Despite the long-established fact that covariance and correlation of security dictate the directional changes in the return and risk of the portfolio, Andrey Ukhov acknowledges that the minimum variance frontier is a static framework. Because the inputs to the function of the modern portfolio theory are static, the outputs may not always be the most accurate reflection of investment reality. He points out that the opportunity sets are always changing because of continuous changes in the market. For example, new assets are introduced to the market through several channels such as initial public offering (IPO), privatization program, and financial market liberalization. He points out that because new additional assets are introduced to the market on a daily basis, investment opportunities sets change. This changes the way the minimum variance frontier behaves in an ever-evolving world. Therefore, the premise of his research explores how the frontier of modern portfolio theory changes with different and newer inputs.

Uhkov perceives the staggering rate at which financial innovation have grown to keep up with industrial and technological innovation. In addition, risk-sharing opportunities have grown significantly, enabling investors to hedge risks better. However, research conducted by Oh (1996) shows that financial innovation has had no effect on the risk pricing of minimum variance frontier in a fixed economic state. More precisely, the spread between the price of risky assets and price of bonds was unaffected despite progressive financial innovation.

Based on a research conducted by Northern Trust entitled *"Minimum Variance Portfolios: Challenging Traditional Concepts of Risk and Reward"* (2012), the minimum variance portfolio outperformed the Russell 3000 Index from January 31st 1979 to June 30th 2011. The minimum variance portfolio had generated 14.55%, while the Index generated 11.69%, resulting in an outperformance of 2.86% in the last 32 years. At the same time, the portfolio delivered such return with lower level of volatility than the Index. Such outperformance had also occurred in other countries such as Switzerland whose equity market is less established, but has displayed far greater volatility and return than the U.S. equity market on average. Similarly, the minimum variance portfolio in Switzerland generated greater risk-adjusted returns than value weighted benchmark in the last 20 years while maintain lower volatility due to higher weight contributed to value stocks, which tend to be highly correlated with one another (Bork, 2011). In fact, according to Northern Trust, weight allocation of the minimum variance portfolio may explain such outperformance, because a portfolio can get concentrated in only handful equities or sectors. This may cause portfolios to exhibit low systemic risk, but high idiosyncratic risk. Having such concentrated positions can entail assuming implicit risks that may be resulting in outperformance relative to the benchmark. This could be a worrisome trend, because investing in the minimum variance portfolio could mean taking on high levels of idiosyncratic risk, liquidity risk, and turnover rate risk without realizing their implications.

SAMPLE DATA

Sample Selection

In order to carry out my research, I need to establish a standard or base portfolio that can compare to compositional changes made to the original portfolio through adding on various financial assets. The base portfolio will be composed of ten equities, but additional assets can be either equity or fixed income for research purposes. In order to pick out ten random stocks for the portfolio, I utilized the NASDAQ automated data processing website. The additional assets of equities and fixed income Exchange Traded Funds have come from NASDAQ, S&P 500, Dow Jones Industrial Average, and various investment fund indexes.

The research will look at eight different categorical assets that are added to the base portfolio to analyze their effects on the minimum variance frontier. The categories include dividend yield, price-to-earnings ratio, beta, and market capitalization, international Exchange Traded Fund, long-term zero-coupon fixed income Exchange Traded Fund, long-term treasury Exchange Traded Fund, and short-term Exchanged Traded Fund. Each of these categorical securities is also selected randomly for the addition. Once added to the base portfolio, compositional changes to the portfolio in terms of expected return, volatility, and Sharpe ratio are compared.

These categorical securities consisting of equities and fixed-income Exchange Traded Funds are selected from various investment indexes and benchmarks. They are randomly picked using the automated data processing capability of Open Icon. The historical data for each asset will range from December 1st 2007 to November 1st 2014. This historical return of each asset is obtained from Yahoo finance database.

Descriptive Statistics

Because the Great Recession occurred roughly from late 2007 to mid-2008, it is important to note and keep in mind how such volatile time-period affected the research data set. The following is the descriptive statistical summary for the base portfolio:

Table 1: Descriptive Statistics of Base Portfolio

MEAN	0.016
SD	0.135
MAX	0.702
MIN	-0.691
MEDIAN	0.017
KURTOSIS	3.371
SKEW	0.231

Skewness is a measure of symmetry. The following positive skewness of 0.231 suggests that the distribution is skewed moderately to the right, where the mean is higher than the median. The skewness to the right would indicate that right tail is longer relative to the left tail. This would also indicate that there is higher volume of asset returns that is lower than the mean return.

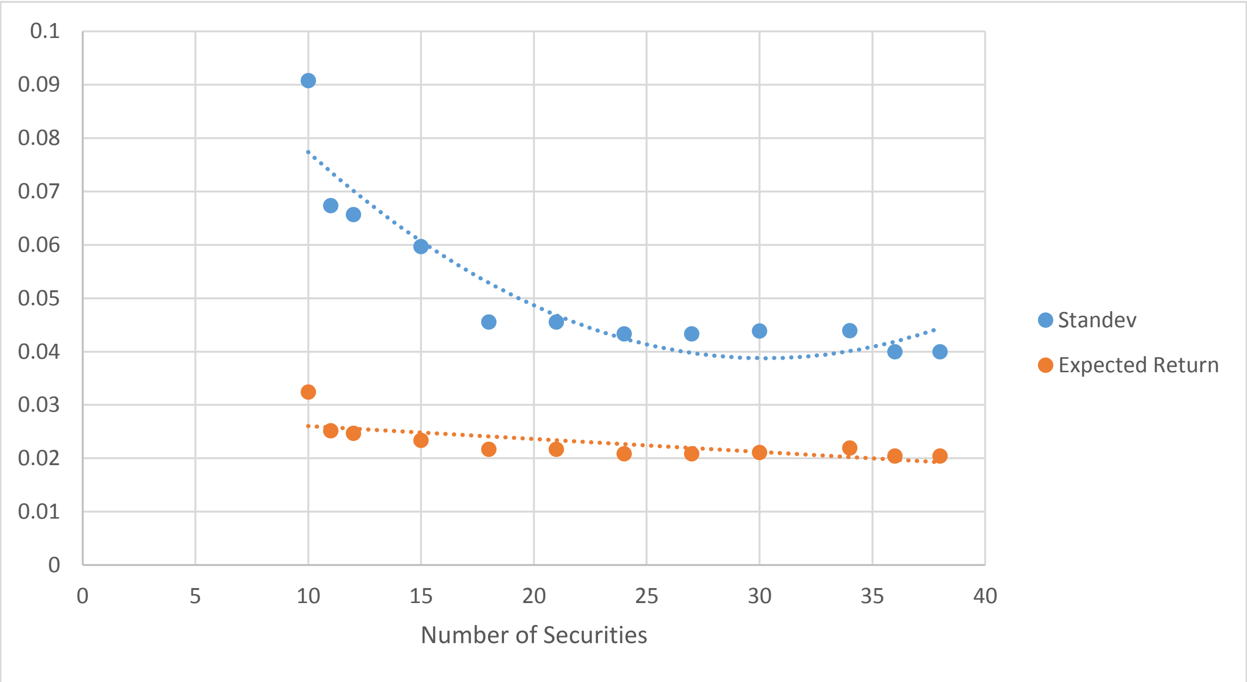
Kurtosis provides a measurement of extremities. In other words, it measures how extreme the data set is spread out in either tail end of the normal distribution curve. A kurtosis of 3.371 indicates the existence of high level of extreme outliers. Moreover, a low mean of 0.016 and relatively high standard deviation of 0.135 point out how large the data point spread is from the mean.

The descriptive statistics seem to lay out that bulk of asset return lie to the left of mean return. In addition, there seems to be much spread and extreme outliers. This can overall be explained by the volatility and price depression brought upon by the Great Recession. Thus, it is important to keep in mind such influence while interpreting research findings.

Risk Measures

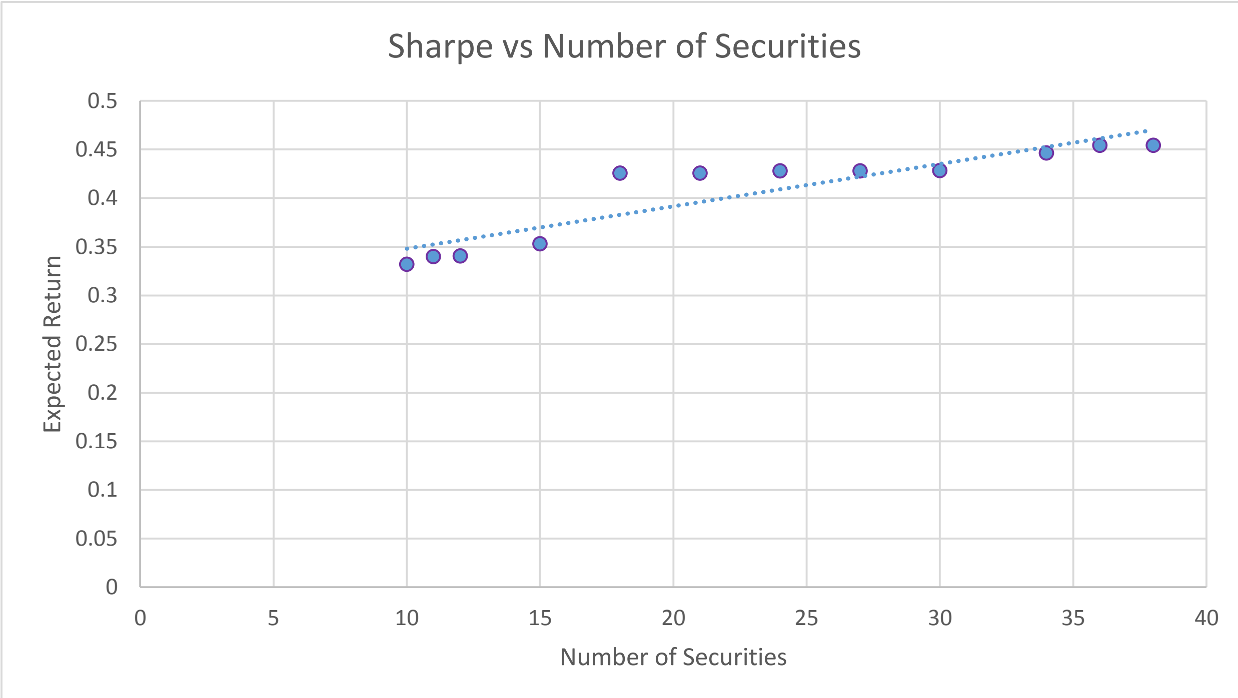
In order to assess the risk measures of the base portfolio, it is of great significance to identify the dichotomy point of idiosyncratic and systemic risks that are inherent in the base portfolio. More precisely, as more assets are included in the portfolio, the more diversified the portfolio will be, eventually eliminating systemic risk of the portfolio. However, the undiversifiable risk that is idiosyncratic risk will always remain to exist. Figuring out how many assets added to the portfolio will result in such diversion will help us gain an idea of how many assets would be needed to attain specific risk and reward profiles

Figure 1: Systemic and Idiosyncratic Risks of Base Portfolio



As random assets were added onto the base portfolio, standard deviation started plateauing off at roughly 0.04, representing the idiosyncratic risk. This is also when around 20 to 25 securities were in the portfolio. Expected return of portfolio somewhat stayed a constant level with a slope of -0.00002 at around 0.02. Systemic risk seems to range from 0.09 to 0.04. A well-diversified base portfolio would ideally have around 20 to 25 securities with a return of 0.02 and standard deviation of 0.04.

Figure 2: Sharpe Ratio Trend of Base Portfolio



Naturally, with standard deviation declining and expected return constant as more assets are included in the portfolio, the Sharpe ratio is expected to rise. The slope of trend-line is 0.004, exhibiting a steady incline. In addition, with 20-25 securities optimal for diversification, the Sharpe ratio stands at roughly 0.42.

ANALYSIS & FINDINGS

Assumptions

The modern portfolio theory takes into account several assumptions that may not always be true in the real world. One assumption often made in the derivation of the model is that asset returns are normally distributed random variables, indicating that the return and variance of asset classes converge into normally distributed random variables if the number of variables is sufficiently large. This idea centers on the central limit theorem that essentially imposes average expected return and standard deviation of mass to individual financial assets.

Another assumption includes that investors have access to the same information, which can also be far from the truth. These assumptions, along with several others, should be an indicator to investors that the minimum variance frontier is a theoretical and objective portfolio analytical metric that may not necessarily mold well into the real investment world.

The following is a complete list of its assumptions:

1) Asset returns are normally distributed random variables
2) Investors have access to the same information regarding investment decisions
3) Investors are rational and risk-averse
4) Taxes and trade commission fees are non-existent
5) Investors have access to borrow or lend unlimited amount money at the risk-free rate
6) No single or few investors have big enough influence on market for price movements

These assumptions are important to take into consideration as this research explores investment practicality and application of the minimum variance frontier in a real world setting. It is without question, however, that the modern portfolio theory serves to be an academically objective experimental analysis of portfolio analytics.

Methodology

Simply put, the minimum variance frontier can be calculated through computing the expected return, variance, and covariance matrix of the portfolio. By adjusting the weights of each asset in the portfolio, the minimum variance frontier is formulated through minimizing the standard deviation for every level of expected return. This methodology section will illustrate that a combined effort of understanding asset pricing theory as well as utilizing the solver excel functionality will be imperative in producing the minimum variance frontier based on the modern portfolio theory.

In order to demonstrate the asset-pricing model involved with calculating the expected return and standard deviation of a portfolio, the research will consider a portfolio composed of three assets, A, B, and C. The return on the portfolio will follow a normal distribution centered on the sample mean with spread of standard deviation.

Table 2: Three Asset Example

Stock	Expected Return	Standard Deviation	Pair	Covariance	Weight
A	0.05	0.1	(A,B)	0.0018	0.25
B	0.08	0.2	(A,C)	0.0011	0.25
C	0.03	0.05	(B,C)	0.0021	0.50

Table 2 provides data on the expected return, standard deviation, and covariance of stock A, B, and C. The example will go over methodology involving the computing of expected return and standard deviation of the portfolio.

Let X_A, X_B, and X_C denote the weight or share of capital for asset A, B, and C respectively. Similarly, let R_A, R_B, and R_C denote the expected return of asset A, B, and C respectively. This would naturally indicate that $X_A+X_B+X_C=1$. Additionally, the portfolio return is expressed as the product of weight and expected return for each financial asset:

$$E[R_P] = X_A R_A + X_B R_B + X_C R_C$$

$$E[R_P] = (0.25)(0.05) + (0.25)(0.08) + (0.50)(0.03)$$

$$E[R_P] = 0.0475$$

The above result of 0.0475 represents the expected return of the portfolio. This result captures the weighted average return of assets A, B, and C. Furthermore, the variance of the portfolio calculates as follows:

$$\mathrm{Var}[R_P] = X_A^2 * \mathrm{var}[R_A] + X_B^2 * \mathrm{var}[R_B] + X_C^2 * \mathrm{var}[R_C] + 2 * X_A * X_B * \mathrm{Cov}[R_A, R_B] + 2 * X_A * X_C * \mathrm{Cov}[R_A, R_C] + 2 * X_B * X_C * \mathrm{Cov}[R_B, R_C]$$

$$\mathrm{Var}[R_p] = (0.25)^2(0.1)^2 + (0.25)^2(0.2)^2 + (0.5)^2(0.05)^2 + 2(0.25)(0.25)(0.0018) + 2(0.25)(0.50)(0.0011) + 2(0.25)(0.50)(0.0021)$$

$$\mathrm{Var}[R_P] = 0.0047$$

The above result of 0.0047 represents the variance of the portfolio. This result captures the variance and covariance among assets A, B, and C. Because expected return of portfolio follows a random variable normal distribution, the average return of the portfolio would center on 0.0475 with variance of 0.0047.

The illustrated methodology of calculating expected return and variance of portfolio reaches a limitation in terms of how many assets can be included in the portfolio due to overly drawn out and convoluted calculations. To combat this issue and generate a much more efficient calculation procedure, this research will be utilizing matrix notations as its primary portfolio analytics methodology.

Given data provided in table 2, the matrix notation methodology will produce equivalent results as the prior methodology. These following three matrices represent expected return, weights, and covariance matrix of assets A, B, and C:

$$R = \begin{bmatrix} R_A \\ R_B \\ R_C \end{bmatrix} \qquad X = \begin{bmatrix} X_A \\ X_B \\ X_C \end{bmatrix} \qquad COV = \begin{bmatrix} Var[A] & Cov(R_A, R_B) & Cov(R_A, R_C) \\ Cov(R_B, R_A) & Var[B] & Cov(R_B, R_C) \\ Cov(R_C, R_A) & Cov(R_C, R_B) & Var[C] \end{bmatrix}$$

Following reflects information present in table 1.1:

$$R = \begin{bmatrix} 0.05 \\ 0.08 \\ 0.03 \end{bmatrix} \qquad X = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix} \qquad COV = \begin{bmatrix} 0.01 & 0.0018 & 0.0011 \\ 0.0018 & 0.04 & 0.0021 \\ 0.0011 & 0.0021 & 0.0025 \end{bmatrix}$$

The expected return of the portfolio can be described as multiplying X by R^T (R^T is the transpose of R). The corresponding Excel equation is =MMULT(TRANSPOSE(range of matrix R_P), range of matrix X). Following is the portfolio expected return:

$$E[R_P] = X * R^T = \begin{bmatrix} X_A \\ X_B \\ X_C \end{bmatrix} * \begin{bmatrix} R_A & R_B & R_C \end{bmatrix} = X_A R_A + X_B R_B + X_C R_C$$

$$E[R_P] = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix} * \begin{bmatrix} 0.05 & 0.08 & 0.03 \end{bmatrix} = 0.0475$$

Similarly, variance of the portfolio describes as multiplying W^T, COV, and W together. The corresponding Excel equation is =MMULT(MMULT(TRANSPOSE(range of matrix X), range of matrix COV), range of matrix X). Following is the portfolio variance:

$$\mathrm{Var}[\mathrm{R_P}] = \mathrm{W^T} * \mathrm{COV} * \mathrm{W} = [\mathrm{X_A} \quad \mathrm{X_B} \quad \mathrm{X_C}] * \begin{bmatrix} \mathrm{Var[A]} & \mathrm{Cov(R_A, R_B)} & \mathrm{Cov(R_A, R_C)} \\ \mathrm{Cov(R_B, R_A)} & \mathrm{Var[B]} & \mathrm{Cov(R_B, R_C)} \\ \mathrm{Cov(R_C, R_A)} & \mathrm{Cov(R_C, R_B)} & \mathrm{Var[C]} \end{bmatrix} * \begin{bmatrix} \mathrm{X_A} \\ \mathrm{X_B} \\ \mathrm{X_C} \end{bmatrix}$$

$$\mathrm{Var}[\mathrm{R_P}] = \mathrm{X_A^2} * \mathrm{var}[\mathrm{R_A}] + \mathrm{X_B^2} * \mathrm{var}[\mathrm{R_B}] + \mathrm{X_C^2} * \mathrm{var}[\mathrm{R_C}] + 2 * \mathrm{X_A} * \mathrm{X_B} * \mathrm{Cov}[\mathrm{R_A}, \mathrm{R_B}] + 2 * \mathrm{X_A} * \mathrm{X_C} * \mathrm{Cov}[\mathrm{R_A}, \mathrm{R_C}] + 2 * \mathrm{X_B} * \mathrm{X_C} * \mathrm{Cov}[\mathrm{R_B}, \mathrm{R_C}]$$

$$Var[R_P] = [0.25 \quad 0.25 \quad 0.50] * \begin{bmatrix} 0.01 & 0.0018 & 0.0011 \\ 0.0018 & 0.04 & 0.0021 \\ 0.0011 & 0.0021 & 0.0025 \end{bmatrix} * \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix} = 0.0047$$

The matrix notation methodology enables for analyses of countless assets in a single portfolio, bypassing the convoluted formulaic calculations with the help of Excel functions. This methodology will most assuredly be advantageous, as the research will focus on analyzing portfolios consisting of more than ten financial assets.

To demonstrate the mathematical as well as practical derivation of the minimum variance frontier, the next several parts of the research will hypothetically focus on assuming two assets in a portfolio as opposed to three illustrated in the previous examples. This will serve to make the calculation much simpler in terms of drafting the theoretical asset pricing model describing the variance minimization process and determining weight of each financial asset for the risky portfolio through the Lagrangian constraint. Once expected return and variance of the portfolio is calculated, a rational and risk-averse investor would look to minimize variance for any given level of return. This formulation would lead to creating the minimum variance frontier, which is established through the minimization of a quadratic program mathematically.

The objective function of the quadratic program would seek to minimize variance subjected to two constraints as follows (two assets):

$$\mathrm{MIN}\ \sigma_P^2 = X_A^2 * \sigma_A^2 + X_B^2 * \sigma_B^2 + 2 * X_A * X_A * \mathrm{Cov}(R_A, R_B) = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \rho_{ij} \sigma_i \sigma_j$$

Subject to

$$\sum_{i}^{n} X_i R_i = E[R_P]$$

$$\sum_{i}^{n} X_i = 1.0$$

The first constraint indicates that portfolio expected return is set to equal a target expected return. The second constraint indicates that summation of weight for each asset would equal to one. The minimum variance frontier is plotted through pairing each set point of expected return and variance that satisfies the above variance minimization function and constraints. More specifically and mathematically, the Lagrangian function will be derived to solve corresponding asset weights assigned for every investment opportunity set, represented by plotted set points on the minimum variance frontier. Taking the partial derivative of the following Lagrangian function with respect to weights (X_1, X_2, X_3,…,X_n) will yield in weights that minimize variance of portfolio subjected to its constraint.

$$\mathit{Lagrangian\ Function} = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \mathit{Cov}(R_i R_j) + \lambda_1 \left[R_p - \sum_{i=1}^{n} X_i E[R_i]\right] + \lambda_2 \left[1 - \sum_{i=1}^{n} X_i\right]$$

Deriving the minimum variance frontier theoretically and conceptually may pose to be quite time consuming and difficult, but computing it graphically can be easily done through programs like Excel and MathLab. This research will utilize Excel solver functionality to plot out each portfolio set point for the minimum variance frontier. Having established the minimum variance frontier for given risky assets, determining the optimal risky portfolio for an investor will depend on that particular investor's degree of risk aversion. This is where the integration of risk-free investment instruments like U.S. government bond come in as risk-averse option to counterbalance the amount of risky assets an investor takes on according to the investment opportunity sets available on the minimum variance frontier. By and large, the minimum variance frontier provides risky investment opportunities for an investor, but depending on the risk-averseness of that particular investor, more weight can be shifted to investing in risk-free assets. The following utility function captures an investor's degree of risk-aversion:

$$U = E[R_P] - 0.005A\sigma_P^2$$

The U stands for utility value derived and calculated from $E[R_P]$ (expected return of the portfolio), A (index of investor's risk aversion), σ_P^2 (variance of the portfolio), and 0.005 scaling factor that allows for percentage rather than decimal representation of expected return and variance. The mechanism behind the utility function is very clear where higher expected return generates higher utility, while higher variance or risk aversion generates lower utility for the investor. The utility function is depicted as an indifference curve. Given the investor's risk tolerance, the optimal portfolio for an investor lies on the tangential point of the utility curve and the minimum variance frontier. When it comes to selecting an optimally risky portfolio that neutralizes investor risk preference, but objectively assesses which portfolio generates highest amount of return for a given level of risk, the capital allocation line is used. The capital allocation line highlights every possible combination of risky and risk-free investment opportunities for the investor. Instead of adjusting weights of just risky assets to satisfy investor's risk preference by moving along the minimum variance frontier, the capital allocation line introduces the addition of risk-free assets that optimize allocation between risky and risk-free asset in achieving most favorable return-risk tradeoff. This tradeoff of reward-to-variability ratio is the slope of the capital allocation line and is known as the Sharpe ratio.

$$\text{Sharpe Ratio} = \frac{E[R_P] - R_f}{\sigma_p}$$

In order to find the optimal risky portfolio on the minimum variance frontier or the tangential point of the minimum variance frontier and the capital allocation line, the following Sharpe ratio needs to be maximized with the constraint that summation of weights equal one. Still assuming that the portfolio consists of two risky assets, the following remains to be true:

$$E[R_P] = X_A R_A + X_B R_B$$

$$\sigma_P^2 = X_A^2 * \sigma_A^2 + X_B^2 * \sigma_B^2 + 2 * X_A * X_A * \mathrm{Cov}(R_A, R_B)$$

$$X_A = 1 - X_B$$

After substituting the following equations in for the Sharpe Ratio, differentiating the resulting Sharpe Ratio equation with respect to X_A and solving for X_B would yield the weights for the optimal risky portfolio, which reflects the tangential point between capital allocation line and minimum variance frontier. The weights of optimal risky portfolio can be expressed as follows:

$$X_A = \frac{(E[R_A] - R_f)\sigma_B^2 - (E[R_B] - R_f)\rho_{AB}\sigma_A\sigma_B}{(E[R_A] - R_f)\sigma_B^2 + (E[R_B] - R_f)\sigma_A^2 - (E[R_A] - R_f + E[R_B] - R_f)\rho_{AB}\sigma_A\sigma_B}$$

$$X_A = 1 - X_B$$

Given the optimal risky portfolio, capital allocation line, risk averse index, A, and the risk-free asset, the optimal complete portfolio can be derived. The optimal complete portfolio represents the combination of risky and risk-free assets a particular investor would be comfortable investing in depending on the investor's risk tolerance level. The following expresses the expected return and variance of the optimal complete portfolio respectively:

$$\mathrm{E}[\mathrm{R_C}] = \mathrm{R_f} + \mathrm{y}(\mathrm{E}[\mathrm{R_P}] - \mathrm{R_f})$$

$$\sigma_C^2 = y^2\sigma_p^2$$

The variable y represents the allocation level of risky assets. The complete portfolio optimization objective utility function for any particular investor can be obtained by substituting in the complete portfolio's expected return for $E[R_C]$:

$$\text{MAX U} = \mathrm{E}[\mathrm{R_C}] - 0.005\mathrm{A}\sigma_P^2$$

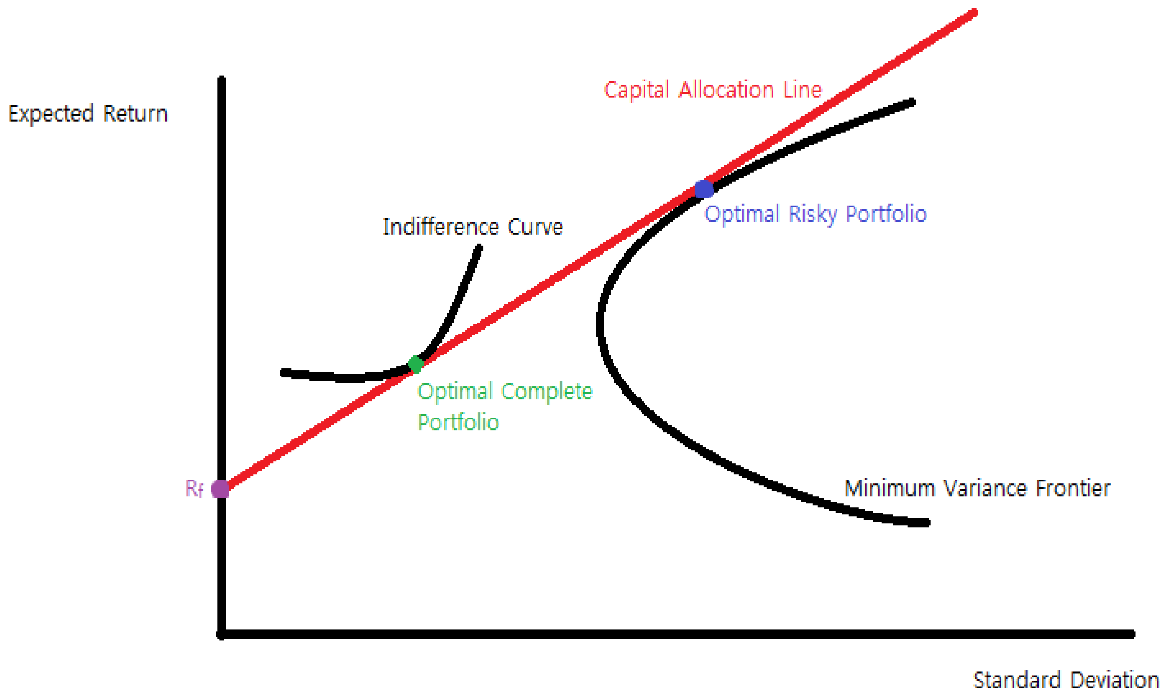
$$\text{MAX U} = \mathrm{R_f} + \mathrm{y}(\mathrm{E}[\mathrm{R_P}] - \mathrm{R_f}) - 0.005\mathrm{A}\sigma_P^2$$

Setting the derivative of the following utility objective function and then solving for y will yield the optimal weight of risky assets, y*, within the optimal complete portfolio. The following equation for y* reflects the appropriate weight any particular risk-inverse investor would place in risky assets versus risk-free assets depending on the level of A:

$$y^* = \frac{\mathrm{E}[\mathrm{R_P}] - \mathrm{R_f}}{0.01\mathrm{A}\sigma_P^2}$$

The optimal y* formula makes much sense, because as A, risk averseness, increases, weight on risky assets will decrease. Similarly, as expected return on the risky asset portfolio increases, weight on risky assets will increase. Lastly, as variance on the risky asset portfolio increases, weight on the risky assets will decrease. Once y* is calculated, the remaining weight, 1-y*, will represent the weight place on risk-free assets.

Figure 3: Graphical Representation of Optimally Complete Portfolio



The following graph summarizes all the main methodological factors visually. The research will seek to plot out the minimum variance frontier, and calculate the optimal risky portfolio and optimal complete portfolio opportunity sets. In order to calculate the optimal complete portfolio, the research will assume the risk aversion index, A, to be 4. In addition, we will assume that short selling is restricted and risk-free rate is at 0.23%. Lastly, this paper will place at least a 9% allocation level to the additional asset of the portfolio in order to see the distinct compositional changes in the Sharpe ratio, volatility, and overall expected return brought about from adding on the asset. Placing a 9% constraint on additional asset may serve to weaken the full effectiveness of diversification, but it prevents the portfolio from being too concentrated on only several equities that may possess better return-risk profiles than other securities including the additional security. Having such situation will result in an overly concentrated portfolio that exposes itself to higher shock impacts from idiosyncratic and liquidity risks. With all these constraints in place, the model will still seek to maximize the Sharpe ratio by locating tangential point of the Capital Allocation Line on the minimum variance frontier.

Findings

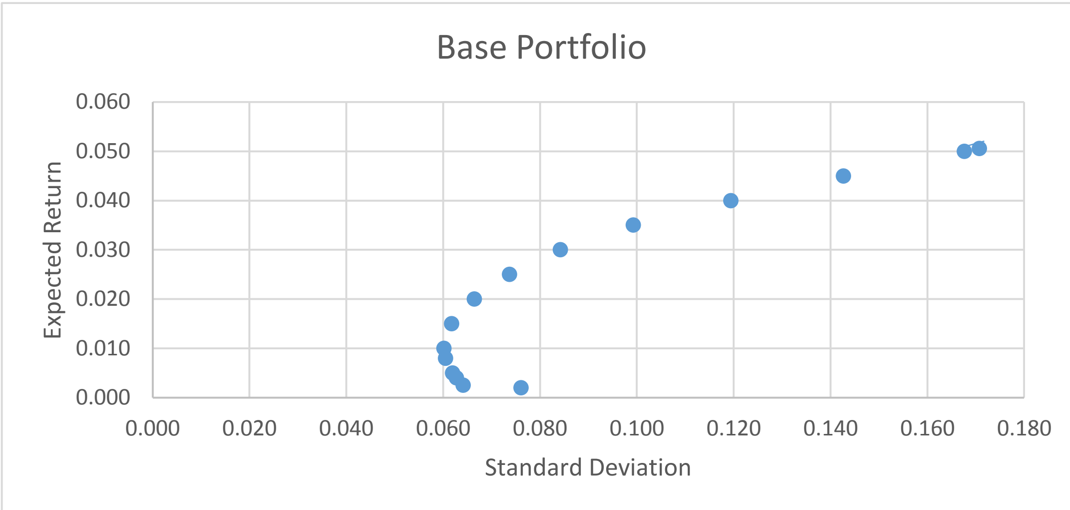
Per the methodology of research I have described prior to this section, the first step involves creating a base portfolio with ten random equities selected from the NASDAQ composite. Using the automated data processing system, ten random equities are chosen. The following table summarizes the individual assets in the base portfolio with respect to each expected return and standard deviation from December 1st 2006 to November 1st 2014.

Table 3: Expected Return & Standard Deviation of Each Asset in Base Portfolio

	Exp. Ret	ST. Dev.
EXPE	0.024	0.130
TWIN	0.016	0.175
CRME	-0.001	0.203
GMCR	0.051	0.172
PNC	0.005	0.091
CGI	0.008	0.119
BWA	0.019	0.106
SPLS	-0.005	0.083
RUSHB	0.020	0.120
DISCA	0.020	0.084

With the data from Table 3, the expected return and standard deviation of the portfolio are calculated using matrices framework illustrated in the methodology section. Minimizing the standard deviation for every level of return generates the minimum variance frontier shown on figure 4. The portfolio on the farther left peak of the curve represents the minimum variance portfolio, whose volatility is lowest. Opportunity sets above the minimum variance portfolio represent the efficient frontier portfolios with minimum volatility for higher level of expected return than the opportunity sets below the minimum variance portfolio. The cornerstone of minimum variance frontier is finding a portfolio that benefit optimally from diversification.

Figure 4: Minimum Variance Frontier of Base Portfolio



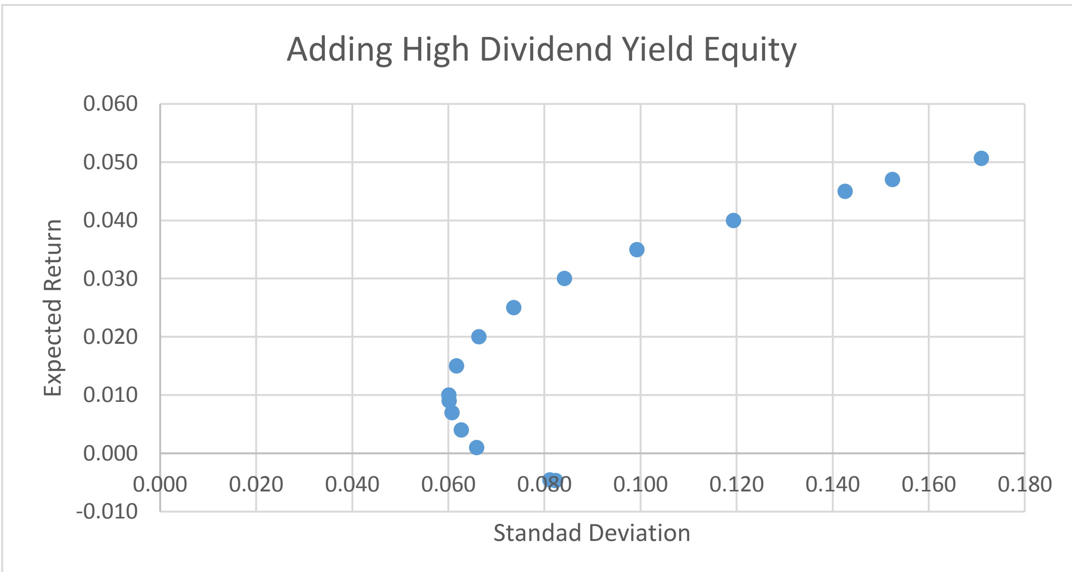
Then, maximizing the Sharpe ratio function given the expected return and standard deviation of the base portfolio will reflect the optimal risky portfolio. The optimal risky portfolio represents the investment opportunity set that is at the tangential point between the Capital Allocation line and the minimum variance frontier. Any opportunity set on the capital allocation line offers the highest level of risk-to-variability ratio that is the Sharpe ratio. Last step includes constructing the optimal complete portfolio that invests in both risky and risk-free assets given that risk-free rate is 0.23% and risk aversion index is 4. Solving for weights assigned to the risky portfolio versus risk-free assets is in the methodology section. Table 4 illustrates the expected return, standard deviation, weight, and Sharpe ratio for optimal risky portfolio and optimal complete portfolio.

	Risk-free	Optimal Risky Portfolio	Optimal Complete Portfolio
E(P)	0.230	3.242	2.986
SD(P)	0.000	9.073	8.300
Weight	0.085	0.915	1.000
Sharpe	0.332		

Table 4: Optimal Risky Portfolio & Optimal Complete Portfolio of Base Portfolio

The first asset to be included in the base portfolio is SDRL, an equity with high dividend yield. When SDRL was added, the minimum variance shifted downward, causing lower expected return and greater volatility for the optimal risky portfolio. Although the standard deviation of optimal complete portfolio declined, the expected return declined with it. The minimum variance frontier also flattened, resulting in a lower Sharpe ratio of 0.314, a drop from 0.332.

Figure 5: Minimum Variance Frontier after SDRL is added

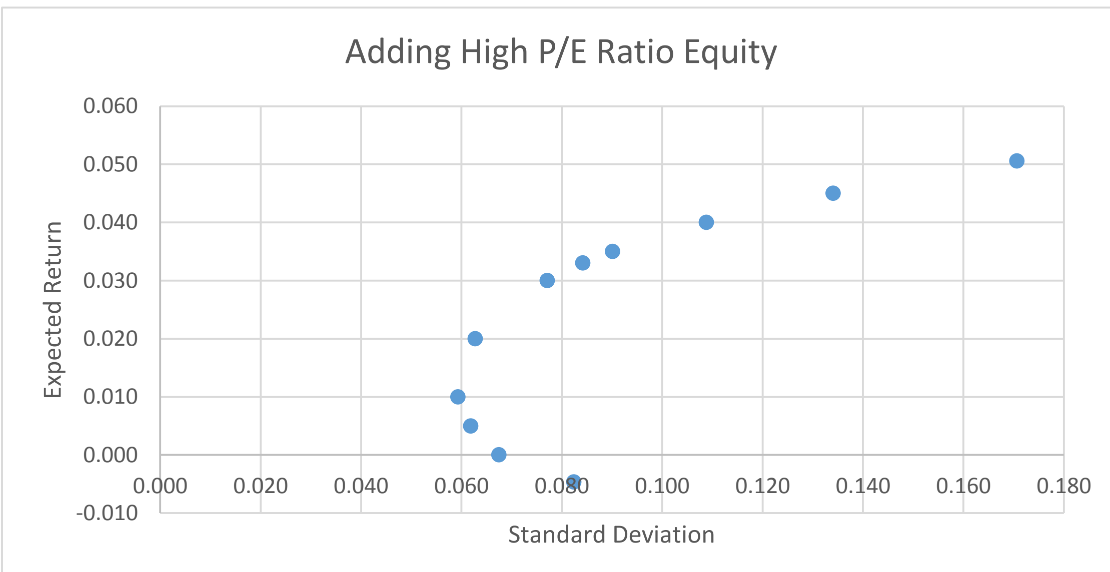


	Risk-free	Optimal Risky Portfolio	Optimal Complete Portfolio
E(P)	0.230	3.156	2.700
SD(P)	0.000	9.310	7.858
Weight	0.156	0.844	1.000
Sharpe	0.314		

Table 5: Optimal Complete Portfolio after SDRL is added

The second asset to be included in the base portfolio is AMZN, an equity with high price/earnings ratio. When AMZN was added, the minimum variance frontier shifted up, resulting in a lower level of volatility and higher expected return. The standard deviation of the optimal complete portfolio increased from 8.3 to 9.12, but the minimum variance frontier also steepened. The Sharpe ratio increased from 0.332 to 0.365, a 10% increase. It is also interesting to notice that it is borrowing from the risk-free asset and investing 108.5% in the optimal risky portfolio, which might have contributed to the increase in volatility in optimal complete portfolio.

Figure 6: Minimum Variance Frontier after AMZN is added

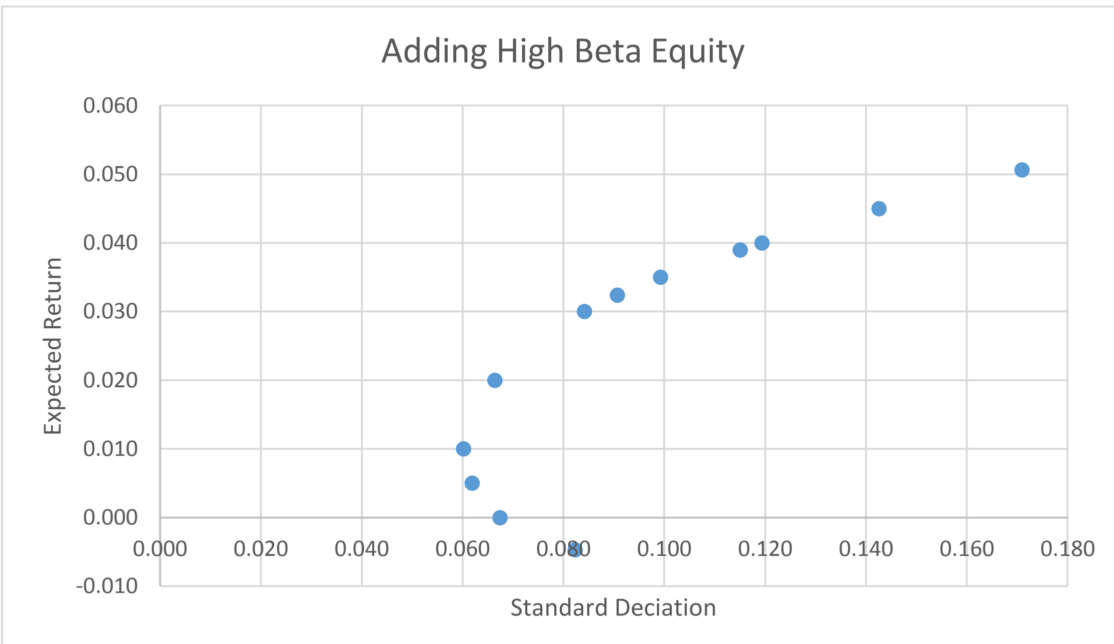


	Risk-free	Optimal Risky Portfolio	Optimal Complete Portfolio
E(P)	0.230	3.302	3.562
SD(P)	0.000	8.415	9.127
Weight	-0.085	1.085	1.000
Sharpe	0.365		

Table 6: Optimal Complete Portfolio after AMZN is added

The third asset to be included in base portfolio is STX, an equity with high beta. Although it is subtle change, expected return of the optimal risky portfolio decreased while standard deviation increased. Similarly, for optimal complete portfolio, expected return dropped, but standard deviation also dropped. The movement in the minimum variance frontier was a very subdued downward shift. Moreover, the curve flattened a bit with the Sharpe ratio dropping from 0.332 to 0.325. Lastly, weight for risk-free asset increased from 0.085 to 0.121, which explains the drop in standard deviation for optimal complete portfolio from 8.30 to 8.11.

Figure 7: Minimum Variance Frontier after STX is added

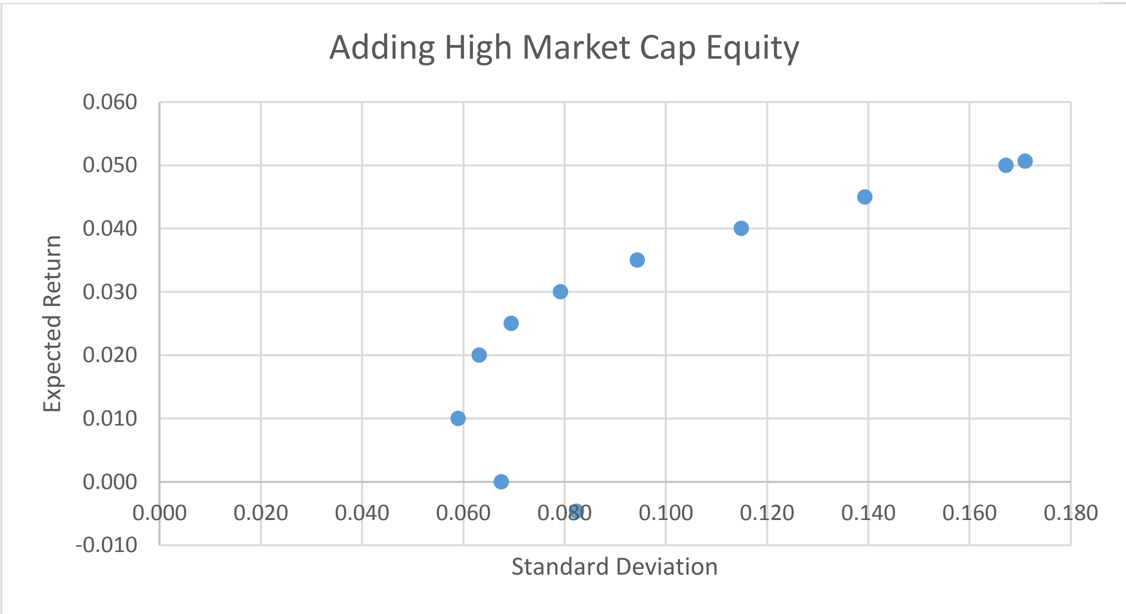


	Risk-free	Optimal Risky Portfolio	Optimal Complete Portfolio
E(P)	0.230	3.230	2.866
SD(P)	0.000	9.238	8.118
Weight	0.121	0.879	1.000
Sharpe	0.325		

Table 7: Optimal Complete Portfolio after STX is added

The fourth asset to be included in the base portfolio is AAPL, an equity with high market capitalization. Interestingly, the expected return and standard deviation of the optimal risky portfolio dropped. At the same time, weight for optimal risky portfolio increased to 1.053, meaning weight for risk-free is -0.053. Despite downward shift in the minimum variance curve, the curve steepened, resulting in the Sharpe ratio to increase from 0.332 to 0.352, a 6% increase. Overall, the expected return and standard deviation of the optimal complete portfolio increased.

Figure 8: Minimum Variance Frontier after AAPL is added

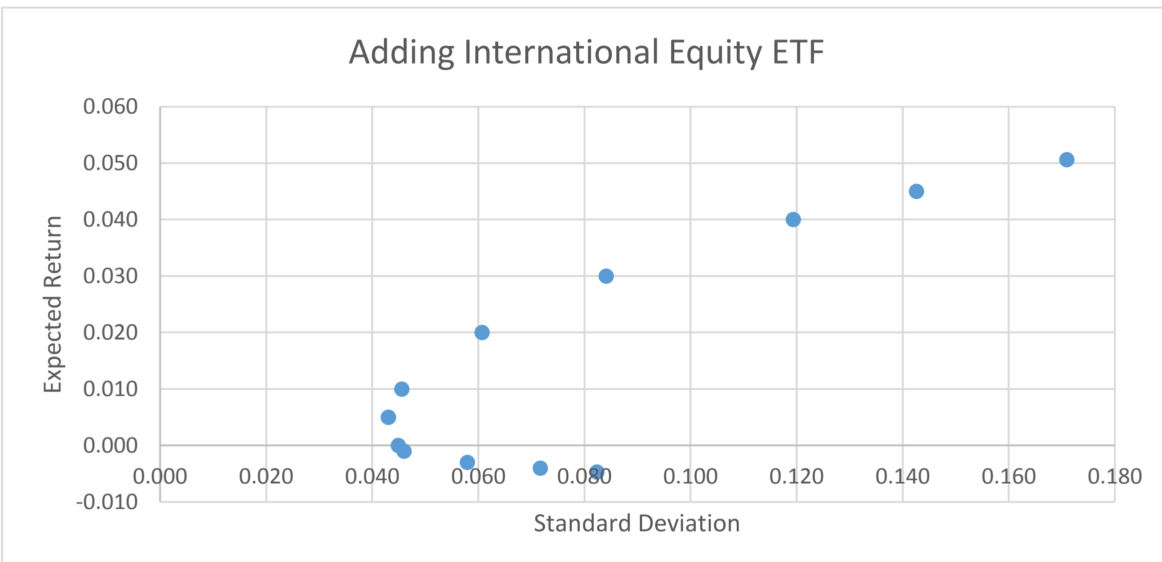


	Risk-free	Optimal Risky Portfolio	Optimal Complete Portfolio
E(P)	0.230	3.173	3.328
SD(P)	0.000	8.359	8.801
Weight	-0.053	1.053	1.000
Sharpe	0.352		

Table 8: Optimal Complete Portfolio after AAPL is added

The fifth asset to be included in the base portfolio is NOINX, an international Exchange Traded Fund. When the ETF was added, the expected return and standard deviation for both optimal risky portfolio and optimal complete portfolio dropped. In addition to the downward shift on the minimum variance frontier, the Sharpe ratio also dropped from 0.332 to 0.327 causing a flattening of the curve. The only metric that seemed to increase was the weight allocation to optimal risky portfolio.

Figure 9: Minimum Variance Frontier after NOINX is added

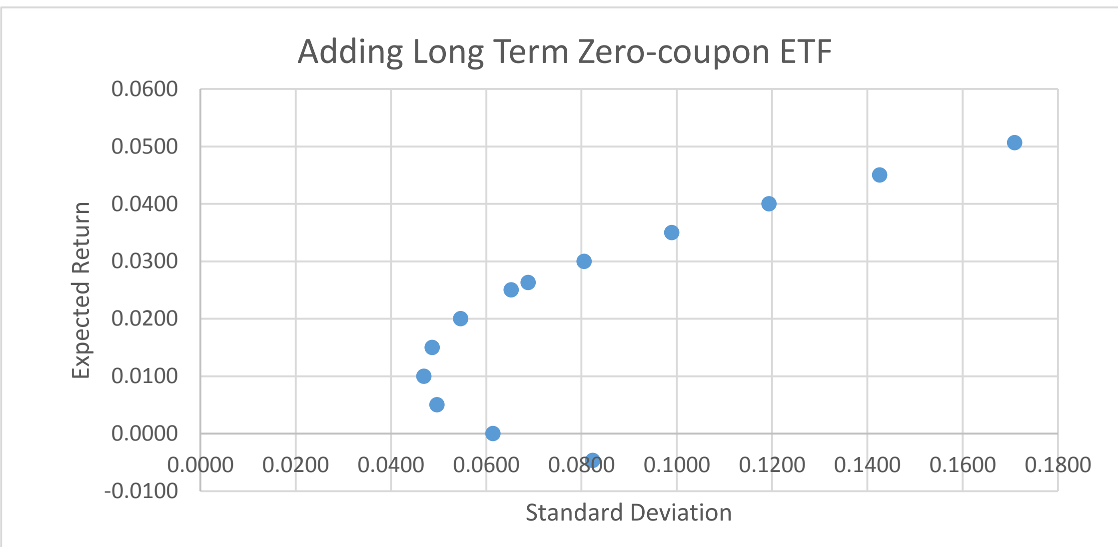


	Risk-free	Optimal Risky Portfolio	Optimal Complete Portfolio
E(P)	0.230	2.982	2.905
SD(P)	0.000	8.414	8.177
Weight	0.028	0.972	1.000
Sharpe	0.327		

Table 9: Optimal Complete Portfolio after NOINX is added

The sixth asset to be included in the base portfolio is ZROZ, a long-term zero coupon fixed income Exchange Traded Fund. After the ETF was added, expected return and standard deviation of the optimal risky portfolio dropped. Conversely, expected return and standard deviation of optimal complete portfolio increased. Overall the Sharpe ratio increased from 0.332 to 0.349, a 5% increase. Weight allocation for optimal risky portfolio increased to 1.267, because of the fact that standard deviation dropped much more than expected return had, resulting in higher Sharpe ratio. Thus, despite a downward shift in the minimum variance frontier, the curve had steepened much more.

Figure 10: Minimum Variance Frontier after ZROZ is added

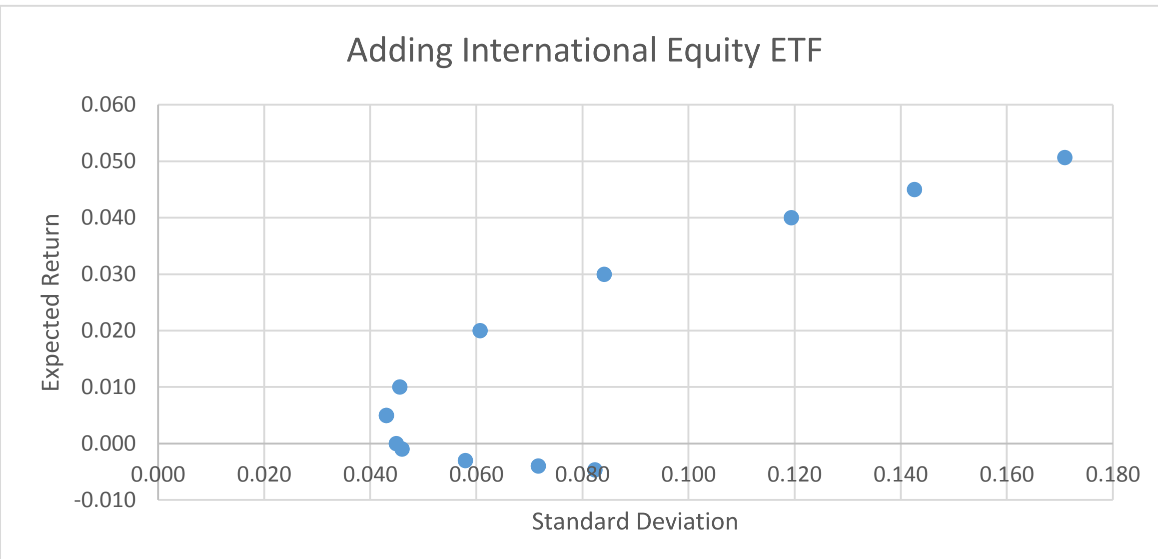


	Risk-free	Optimal Risky Portfolio	Optimal Complete Portfolio
E(P)	0.230	2.629	3.270
SD(P)	0.000	6.879	8.718
Weight	-0.267	1.267	1.000
Sharpe	0.349		

Table 10: Optimal Complete Portfolio after ZROZ is added

The seventh asset to be included in the base portfolio is TLO, a long-term treasury Exchanged Traded Fund. When TLO was added, expected return and standard deviation of the optimal risky portfolio dropped. However, weight allocation to optimal risky portfolio increased to 1.635, because standard deviation declined relatively more than expected return had, resulting in an increased Sharpe ratio. Sharp ratio increased from 0.332 to 0.348, a 4.8% increase. Naturally, the expected return and standard deviation increased for the optimal complete portfolio.

Figure 11: Minimum Variance Frontier after TLO is added

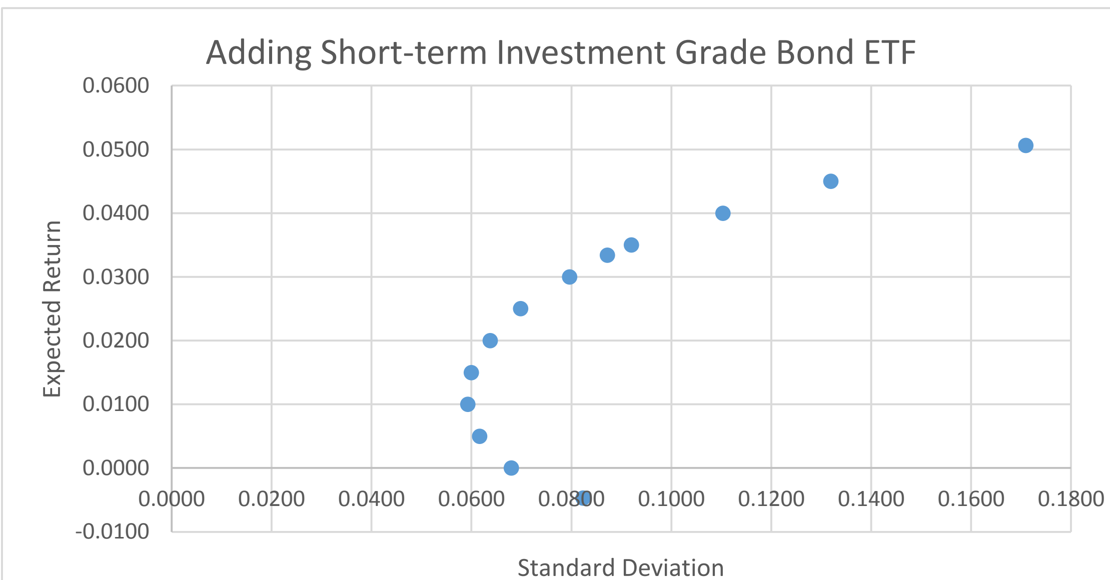


	Risk-free	Optimal Risky Portfolio	Optimal Complete Portfolio
E(P)	0.230	2.982	2.905
SD(P)	0.000	8.414	8.177
Weight	0.028	0.972	1.000
Sharpe	0.327		

Table 11: Optimal Complete Portfolio after TLO is added

The eighth asset to be included in the base portfolio is SRHQX, a short-term investment grade fixed income Exchange Traded Fund. Interestingly, when the security was added to the portfolio, the expected return of optimal risky portfolio increased while its standard deviation dropped. Despite such attractive return-risk profile, the weight allocation on the portfolio only increased to 1.007. Meanwhile, the expected return on the optimal complete portfolio increased whereas its standard deviation dropped. Overall, the minimum variance frontier exhibited San upward shift and steepened as Sharpe ratio increased from 0.332 to 0.356, a 7.2% increase.

Figure 12: Minimum Variance Frontier after SRHQX is added



	Risk-free	Optimal Risky Portfolio	Optimal Complete Portfolio
E(P)	0.230	3.378	3.400
SD(P)	0.000	8.840	8.903
Weight	-0.007	1.007	1.000
Sharpe	0.356		

Table 12: Optimal Complete Portfolio after SRHQX is added

Implications

After the addition of AMZN, AAPL, ZROZ, TLO, and SRHQX, the resulting portfolio experienced 10%, 6%, 5%, 4.8%, and 7.2% increase in the Sharpe ratio respectively. In addition, after AMZM, and SRHQX were added, minimum variance frontier of the optimal risky portfolio shifted upward, resulting in a higher expected return and lower standard deviation than the base portfolio. Subsequently, the addition of AMZN, AAPL, ZROZ, and SRHQX resulted in borrowing at the risk-free rate, leading to over 100% asset allocation in the optimal risky portfolio. The addition of TLO produced minimum variance portfolio with the lowest standard deviation. Finally, adding SDRL resulted in the portfolio with the lowest Sharpe ratio that dropped 5.4%.

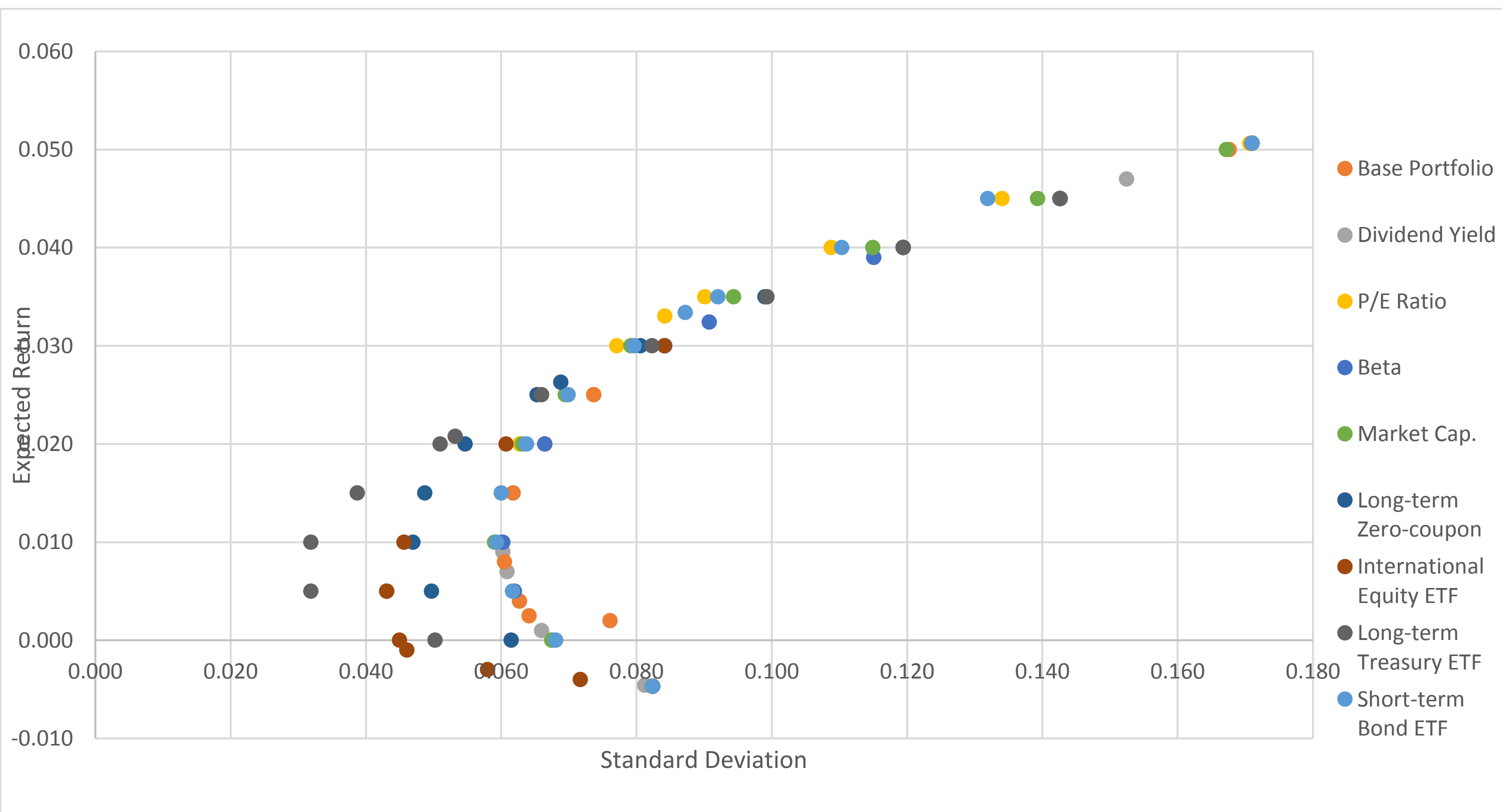


Figure 13: Minimum Variance Frontier Implications

CONCLUSION

The minimum variance frontier of the modern portfolio theory serves to be an investment analytical toolset that generates an optimally diversified portfolio. This research explores what effect compositional changes of such portfolio has on the minimum variance frontier as well as the optimally risky and complete portfolio. Looking at eight different financial categories, the analysis of output hinges on reviewing the post-addition expected return, standard deviation, weight allocation, and Sharpe ratio. It has found that some asset categories generated significantly better performance metrics than others. The high price-to-earnings ratio equity added to the portfolio generated the highest Sharpe ratio, whereas the high dividend yielding equity generated the lowest Sharpe ratio for the portfolio.

Although there are many reasons high price-to-earnings ratio equity performed better than the high dividend yielding equity, this result should be adapted with a grain of salt. This is because some of the assumptions that this experiment was subject to could be unrealistic in the actual investment world. For example, subjecting a 9% allocation level for each additional asset can be detrimental to attaining an optimally diversified portfolio. In addition, having randomly picked each categorical asset, hoping that it is best representative of each category may lack the statistical significance and validity. Lastly, some financial categories perform based on the direct consequence of the stock's performance. A high price-to-earnings ratio already indicates a high historical stock performance, whereas metrics like beta or dividend yield do not have direct relations to its historical performance in regards to past expected return. With these assumptions in mind, this research provides brief insight into portfolio compositional changes the minimum variance frontier is or is not vulnerable to in terms of achieving higher return and better diversification based on historical data.

Appendix A

Minimum Variance Frontier Weight Allocations

	Exp. Ret	ST. Dev.	Weight
EXPE	0.0239205	0.1302197	0.105252
TWIN	0.0160702	0.1749876	0
CRME	0.0012093	0.2033808	0
GMCR	0.0506448	0.1718629	0.3899912
PNC	0.0052037	0.0905659	0
CGI	0.0076678	0.1186928	0
BWA	0.018606	0.1056943	0
SPLS	0.0046769	0.0827916	0
RUSHB	0.0195943	0.1204572	0.0147158
DISCA	0.0201351	0.0844448	0.4900419

	Exp. Ret	ST. Dev.	Weight
EXPE	0.02392	0.13021974	0.074396
TWIN	0.01607	0.17498761	0
CRME	-0.001209	0.20338082	0
GMCR	0.050645	0.1718629	0.406092
PNC	0.005204	0.09056593	0
CGI	0.007668	0.11869277	0
BWA	0.018606	0.10569427	0
SPLS	-0.004677	0.08279165	0
RUSHB	0.019594	0.1204572	0
DISCA	0.020135	0.0844448	0.429512
SDRL	0.006298	0.12742347	0.09

	Exp. Ret	ST. Dev.	Weight
EXPE	0.02392	0.13022	0.022381
TWIN	0.01607	0.174988	0
CRME	-0.00121	0.203381	0
GMCR	0.050645	0.171863	0.325174
PNC	0.005204	0.090566	0
CGI	0.007668	0.118693	0
BWA	0.018606	0.105694	0
SPLS	-0.00468	0.082792	0
RUSHB	0.019594	0.120457	0
DISCA	0.020135	0.084445	0.296252
AMZN	0.028219	0.127423	0.356193

	Exp. Ret	ST. Dev.	Weight
EXPE	0.02392	0.130219738	0.0651229
TWIN	0.01607	0.174987605	0
CRME	-0.00121	0.203380823	0
GMCR	0.050645	0.171862905	0.3955117
PNC	0.005204	0.090565932	0
CGI	0.007668	0.118692768	0
BWA	0.018606	0.105694273	0
SPLS	-0.00468	0.082791648	0
RUSHB	0.019594	0.120457201	0
DISCA	0.020135	0.084444804	0.4493654
STX	0.018464	0.157601945	0.09

	Exp. Ret	ST. Dev.	Weight
EXPE	0.02392	0.130219738	0.031682
TWIN	0.01607	0.174987605	0
CRME	-0.00121	0.203380823	0
GMCR	0.050645	0.171862905	0.327535
PNC	0.005204	0.090565932	0
CGI	0.007668	0.118692768	0
BWA	0.018606	0.105694273	0
SPLS	-0.00468	0.082791648	0
RUSHB	0.019594	0.120457201	0
DISCA	0.020135	0.084444804	0.30598
AAPL	0.024551	0.096507842	0.334803

	Exp. Ret	ST. Dev.	Weight
EXPE	0.0239205	0.13021974	0.0849645
TWIN	0.01607024	0.17498761	0
CRME	-0.0012093	0.20338082	0
GMCR	0.05064485	0.1718629	0.3650741
PNC	0.00520371	0.09056593	0
CGI	0.00766775	0.11869277	0
BWA	0.01860602	0.10569427	0
SPLS	-0.0046769	0.08279165	0
RUSHB	0.01959435	0.1204572	0.005039
DISCA	0.0201351	0.0844448	0.4549224
NOINX	0.00047629	0.05844367	0.09

	Exp. Ret	ST. Dev.	Weight
EXPE	0.02392	0.13022	0.061921
TWIN	0.01607	0.174988	0
CRME	-0.00121	0.203381	0
GMCR	0.050645	0.171863	0.288571
PNC	0.005204	0.090566	0
CGI	0.007668	0.118693	0
BWA	0.018606	0.105694	0
SPLS	-0.00468	0.082792	0
RUSHB	0.019594	0.120457	0
DISCA	0.020135	0.084445	0.334763
ZROZ	0.01097	0.068328	0.314746

	Exp. Ret	ST. Dev.	Weight
EXPE	0.02392	0.13022	0.031465
TWIN	0.01607	0.174988	0
CRME	-0.00121	0.203381	0
GMCR	0.050645	0.171863	0.222254
PNC	0.005204	0.090566	0
CGI	0.007668	0.118693	0
BWA	0.018606	0.105694	0
SPLS	-0.00468	0.082792	0
RUSHB	0.019594	0.120457	0
DISCA	0.020135	0.084445	0.288029
TLO	0.006478	0.035438	0.458252

	Exp. Ret	ST. Dev.	Weight
EXPE	0.02392	0.13022	0.088491
TWIN	0.01607	0.174988	0
CRME	-0.00121	0.203381	0
GMCR	0.050645	0.171863	0.369969
PNC	0.005204	0.090566	0
CGI	0.007668	0.118693	0
BWA	0.018606	0.105694	0
SPLS	-0.00468	0.082792	0
RUSHB	0.019594	0.120457	0
DISCA	0.020135	0.084445	0.451539
SRHQX	0.042581	0.409482	0.09

Appendix B

Minimum Variance Frontier Excel Outputs

Base Portfolio

SD(p)	E(p)
0.060545713	0.005
0.060390211	0.0053
0.060292475	0.0055
0.060199578	0.0057
0.060069269	0.006
0.059876431	0.0065
0.059714249	0.007
0.059582983	0.0075
0.059482833	0.008
0.05937646	0.009
0.059370412	0.0095
0.059372976	0.0096
0.059376807	0.0097
0.059388212	0.0099
0.059395807	0.010

Dividend Yield

SD(p)	E(p)
0.065927	0.001
0.063585	0.003
0.062699	0.004
0.061944	0.005
0.061315	0.006
0.060816	0.007
0.060617	0.0075
0.060451	0.008
0.06032	0.0085
0.060223	0.009
0.06016	0.0095
0.060132	0.01
0.06018	0.011
0.060689	0.013
0.061723	0.015
0.066397	0.02
0.084179	0.03

P/E

SD(p)	E(p)
0.065978	0.001
0.061828	0.005
0.059989	0.008
0.058428	0.009
0.058364	0.0093
0.058327	0.0095
0.058294	0.0097
0.05828	0.0098
0.058266	0.0099
0.059309	0.01
0.062418	0.02
0.075387	0.03
0.108765	0.04
0.160239	0.07
0.208279	0.09

Beta

SD(p)	E(p)
0.0659784	0.001
0.0635924	0.003
0.0619436	0.005
0.060223	0.009
0.0601323	0.01
0.0575443	0.010001
0.0577939	0.011
0.058164	0.012
0.0586596	0.013
0.0592749	0.014
0.06001	0.015
0.0608485	0.016
0.061798	0.017
0.0652392	0.02
0.0841789	0.03
0.1709654	0.07
0.2276858	0.09

Market Cap.

SD(p)	E(p)
0.062771	0.001
0.060777	0.003
0.059196	0.005
0.058062	0.007
0.057401	0.009
0.057254	0.01
0.05723	0.011
0.05733	0.012
0.058363	0.015
0.062361	0.02
0.076925	0.03
0.119277	0.05
0.167954	0.07

International ETF

SD(p)	E(p)
0.170966	0.050645
0.082358	-0.004677
0.060704	0.020000
0.084094	0.030000
0.119402	0.040000
0.045596	0.010000
0.043044	0.005000
0.142615	0.045000
0.044927	0.000000
0.043044	0.005000
0.046037	-0.001000
0.071674	-0.004000
0.057943	-0.003000
0.099251	0.035000
0.090733	0.032424

Long-Term Zero

SD(p)	E(p)
0.1710	0.0506
0.0824	-0.0047
0.0614	0.0000
0.0469	0.0100
0.0546	0.0200
0.0806	0.0300
0.1194	0.0400
0.1426	0.0450
0.0653	0.0250
0.0990	0.0350
0.0497	0.0050
0.0487	0.0150
0.0688	0.0263

Long-Term Treasury

SD(p)	E(p)
0.1710	0.0506
0.0824	-0.0047
0.0502	0.0000
0.0318	0.0050
0.0318	0.0100
0.0387	0.0150
0.0509	0.0200
0.0660	0.0250
0.0823	0.0300
0.0993	0.0350
0.1194	0.0400
0.1426	0.0450
0.0532	0.0208

Short-term Bonds

SD(p)	**E(P)**
0.1710	0.0506
0.0824	-0.0047
0.0680	0.0000
0.0593	0.0100
0.0600	0.0150
0.0637	0.0200
0.0796	0.0300
0.0920	0.0350
0.1103	0.0400
0.1319	0.0450
0.0699	0.0250
0.061662	0.005
0.0872	0.0334

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ACADEMIC VITA

TONY PARK

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EDUCATION:

The Pennsylvania State University — ***University Park, PA***
The Schreyer Honors College — Graduation: Spring 2016
Smeal College of Business: Bachelor of Science, Finance
College of the Liberal Arts: Bachelor of Arts, Economics Minor

Recipient of the President's Freshman Award, the President's Sparks Award, the Don & Carol Mielke Scholarship, the Pierce Family Trustee Scholarship in the Smeal College of Business to Honor Chuck Snow and Shirley Kovach, Betty J. Lockington Memorial Scholarship, Schreyer Honors College Scholarship, and Ralph H. Wherry Student Service Award.

RELEVANT EXPERIENCE:

Bloomberg L.P. — ***New York, NY***
Equity Research Summer Analyst — *May 29th 2015-August 7th 2015*

- Conduct fundamental research and analysis of healthcare industries and equities through applying core valuation methodologies
- Create refined market and industry oriented healthcare investment framework to drive value-added research for Bloomberg clients
- Perform quantitative and qualitative analyses as part of due diligence to create in-depth research reports that are pitch to associates

Penn State Asset Management Group — ***University Park, PA***
Co-Founder, Chief Investment Officer — *December 2014-Present*

- Oversee fixed income investment techniques and overall decisions through developing in-depth valuation models and analyses
- Generate macro-thematic investment ideas through top down valuation methodology on economic, industry, and company level
- Update over 200 active members on ongoing macroeconomic and industry trends that reflect the fund's current strategic positions
- Manage and lead 7 fixed income research analysts on wide array of educational topics from valuations to professional networking

Korean Investment Business Club — ***University Park, PA***
Co-Founder, Lead Research Analyst — *August 2012-Present*

- Publish monthly journal reports and research articles that analyze potential investment ideas and mispricing in the Asian market
- Issue research reports to investment bankers, economists, and journalists in Asia for insightful feedback and networking opportunity
- Support over 30 research analysts or 5 research groups on the progress and development of their research efforts

DGB Financial Group — ***Daegu, South Korea***
Commercial Banking Summer Analyst — *July 1st 2014-August 1st 2014*

- Assist in the expansionary efforts of opening up another branch in Vietnam through interpretation of legal leasing documents
- Create transparent facilitation between clients and the bank in regards to financial and international regulations such as the FATCA
- Structure informative interaction for clients and firms in currency exchanges, loans, accounts, transfers, and financial products

Nittany Consulting Group — ***University Park, PA***
Consultant — *December 1st 2013-May 2015*

- Analyze business plans, management teams, industry trends, existing markets, objectives, and models to address various business specific problems and seek opportunities of improvement for clients
- Meet with management regularly to ensure that progress of short-term objectives are aligned properly to meet long-term objectives
- Promote entrepreneurship through constructing a comprehensive marketing plan for NCG, and conduct thorough client research

Penn State Investment Association — ***University Park, PA***
Healthcare Sector Analyst — *December 1st 2013-May 2014*

- Participate in a 10-week intensive session on the basics of investing techniques coupled with leading macroeconomic indicators and data, valuation techniques, current and future market conditions, and healthcare research to evaluate healthcare equities

LEADERSHIP EXPERIENCE:

Guided Study Group — ***University Park, PA***
Guided Study Group Leader — *August 2013-Present*

- Lead and facilitate three 1-hour business calculus sessions every week for underclassmen ranging from 30 to 100 students
- Create learning environments where students collaborate, discuss, interact, and generate ideas on solving business calculus problems
- Host extensive exam review sessions for over 300 students, and have increased overall exam average for the course

Sapphire Leadership Organization — ***University Park, PA***
Member — *August 2012-Present*

- Represent top 8% of Smeal and actively engage in social, professional, and personal activities to foster leadership development
- Required to satisfy corporate relation, community service, fundraising, and leadership development credits every semester

SKILLS & INTERESTS

- Familiar with Bloomberg, Minitab, Morningstar, and Microsoft Office (Excel, Word, PowerPoint)
- Interests include traveling, basketball, soccer, golf, skiing, fitness, guitar, personal investing & tutoring