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PERSISTENCE OF DYNAMIC FAIRNESS AND ALTRUISM IN THE ULTIMATUM GAME

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Abstract

In this paper, we explore a dynamic version of the Ultimatum Game and study its results with regards to population dynamics, stability, and preference set construction. The Ultimatum Game is a famous economic experiment that studies human belief structures and behaviors, such as equality, fairness, and cooperation. Previous studies of the game have shown that theoretical results for the game are not often mirrored experimentally. While many models were produced to take account of such differences, emphasis was put on explaining decision-making through individual optimization of strategies alone. Our treatment addresses how populations of players with different preferences form offers, accept or reject proposals, and rationalize interactions over time. We propose that belief systems for fairness and unfairness can be understood by establishing dependence between how players assign utility to offers received and offers made to others. We then analyze the dynamics of a population of players who adapt their trading behaviors based upon population size. We find that a model where groups in a population interact with population dynamics in mind will exhibit tendencies to approach proportional equilibria and that the growth and stability of the overall population of players depends upon a delicate balance between the preferences of the group and the individual.

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Chapter 1

Introduction

The Ultimatum Game has a rich history in experimental economics due to its simplicity in theoretical design and complexity in experimental results. The game itself is designed as follows. Consider two players that want to divide a certain amount of money M. Each player is assigned a role, either to make an offer for dividing the money or responding to the offer. Player I begins by making an offer x between 0 and M to player II, who can either accept or reject the offer. If player II accepts, player II gets x and player I keeps $M - x$. But if player II rejects the offer, both players receive nothing. Assuming rationality amongst both players, the equilibrium (often called the subgame perfect equilibrium) for this game is that player II should accept whatever player I offers, as player II, regardless of what player I offers, will be no worse off than before.

Historical treatments of the game have shown, however, that such an equilibrium in the Ultimatum Game is not always reached. This experimental evidence, as will be discussed in **Chapter 2**, indicates that there are subjective factors not being taken into account by researchers that manifest themselves in interaction games, Ultimatum Game included.

In our treatment of the Ultimatum Game, we seek to understand player strategies in a different context than usual - through population dynamics and ecological stability. Populations of players playing similar strategies, interacting in a controlled environment, can offer interesting analysis of decision-making processes. We aim to create a population of players who are exchanging resources via the Ultimatum Game, highlighting the contrast between players with different belief sets and, therefore, different utility functions. Our end goal is to provide a model that assesses the rate at which groups of players and their strategies are changing over time, an indicator in our model of the stability and likelihood a population could sustain potentially extreme preferences.

In **Chapter 3**, we outline a model for utility and expectation of return given a predefined Ultimatum Game environment. Next, we extend our game over a population of players. These players, however, do not uniformly play with the same strategy or have completely distinct belief sets. We assign two subgroups of the population to two strategies, one in which altruistic offerers will try to give more than they take, and the other in which greedy offerers will try to take more than they give. We create an intimate relationship between these two subgroups through defining ecologically-based growth equations to each subgroup. These equations measure the rate at which subgroup sizes change with respect to what each subgroup seeks to exchange during interactions. Our result is that stable proportional equilibrium emerges in populations where competition on an individual level is tied to growth equations of the subgroups, and that the magnitude of offers is the most important factor affecting the pace at which population equilibrium is attained.

Chapter 4 introduces dynamic variables for each subgroups' preference sets. These dynamic variables affect each subgroups' ability to make theoretically rational decisions during games, an attempt to mimic experimental results. Our dynamic variables take into account the very population dynamics we highlight in **Chapter 3** by placing emphasis on strategies that are formed relating population proportions and payoffs. We also define four possible strategies that could be played by the subgroups, based upon previous treatments, and provide graphical analysis of the change over time of player offers and subgroup size. Our results are that environments in which extreme greed or extreme altruism exist amongst symbiotic populations tend to rapidly shrink in size as they approach proportional equilibrium. We conclude that the experimental trading behaviors historically observed in the Ultimatum Game can be explained theoretically through our model.

We end the paper with a brief conclusion of results and future discussions in **Chapter 5**.

Chapter 2

Literature Review

2.1 Ultimatum Game: Theoretics and Experiments

In *Learning to be Imperfect* (1995) [1] from Gale, Binmore, and Samuelson, the authors highlight previous treatments of the Ultimatum Game in laboratory settings where experimental results did not match theoretical expectations. The authors reference the work of Guth *et al.* (1982) [2] demonstrating that the modal, or most common, offer was $\frac{1}{2}$ of the maximum offer, and offers below $\frac{1}{3}$ of the maximum offer were rejected around 50% of the time, drawing comparisons to Binmore (1989) [3] with similar results. They go on to list several other experiments conducted around that time to show the breadth of experimental evidence supporting the discrepancy between theoretical expectations and experimental results (Bolton and Zwick (1993) [3], Giath and Tietz (1990) [4], and Thaler (1988) [5]). They argue that the reasoning used by players in experiments is not formed from rational, deductive interpretations of strategies but by evolutionary learning, and thus dependent on the number of times the game is played [1].

They propose that expectations of player behavior be partitioned into broadly defined categories prior to experimentation, based upon the length of time players have to "learn" their strategies, e.g. short-run, medium-run, long-run, and ultra long-run. By creating these categories, researchers can isolate player behavior and study equilibria that emerge at each time stage during games based upon the games overall length [1]. For example, they provided evidence for medium-run games in which a modal offer of 20% is reached, and for long-run games in which the subgame perfect equilibrium is reached. With no constraints put on how to make offers other than knowledge of the other player's previous offers, a result was that as time increases the likelihood increases that player payoffs will approach theoretical equilibrium unique to model or environmental constraints [1]. They offer, however, a disclaimer that experiments are seldom run in long-run and ultra long-run cases. To the authors, it seems unlikely that most experimental results will exhibit the subgame perfect equilibrium symmetry as proposed in their model.

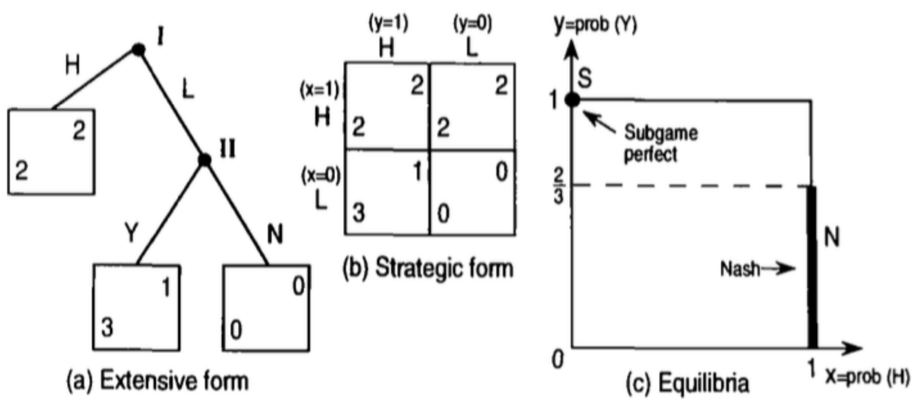


Figure 2.1: Ultimatum Game Minigame from Gale, Binmore, and Samuelson

Finally, they elaborate on the idea of fairness, replicator dynamics, and the biological motifs running through time-dependent versions of the Ultimatum Game. They show that the probability that players make high or low offers depends directly on the likelihood of blind acceptance by the responder in any given interaction. The more likely a responder is to accept, the lower an offer becomes [1]. This kind of behavior, they explained, could be illustrated through strategy mappings similar to that in Figure 2.1. Additionally, it is proposed that the faster a group of players learns a strategy, the quicker an approach towards modal offers or a specific level of equilibrium emerges in the game environment [1]. They conclude that results are entirely dependent on initial conditions and the ability of players to learn over time.

2.2 Fairness and Reciprocity

The behavior of players inside the Ultimatum Game that differ from theoretical predictions has stimulated substantial research and speculation. One underlying concept in the behavioral approach to game theory is fairness and reciprocity. A theory put forth by Falk and Fischbaker in *A Theory of Reciprocity* (2005) [6] states that agents in games make actions that are not grounded in completely rational or objective analyses but rather in behavioral responses to actions perceived as kind or unkind. They claim that a model for a reciprocal action is based upon how other people evaluate the subjective kindness or unkindliness of an action. An example of how acceptance thresholds could change between knowing and not knowing information about player behaviors can be seen in Figure 2.2. At the core of their model rests two ideas: the understanding of someone's intentions, and the evaluation of their action's consequences [6].

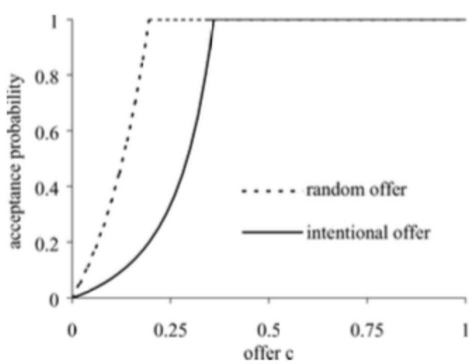


Figure 2.2: Difference in Acceptance Probability p Based Upon Information About Player Behavior from Falk and Fischbaker

Experimental evidence has shown, argue Falk and Fischbaker, that these two concepts can be used to accurately and efficiently explain seemingly irrational behavior in the Ultimatum Game and similar bargaining games [Brandts and Sola (2001) [7], Falk et al. (2003) [8]; McCabe et al. (2003) [9]; Offerman (2002) [10]; Greenberg and Frisch (1972) [11]; Goranson and Berkowitz (1966) [12]. The conclusions drawn by players based upon expected returns, choice sets, and intentions of each player significantly impact decision distributions during a game. Falk and Fischbaker conducted experiments and provided a model for reciprocity that sought to quantify how people reward kind actions and punish unkind ones, with particular emphasis put on players determining the fairness of offers based upon the expected returns all other players experience.

They apply their theoretical model of reciprocity to the Ultimatum Game in an attempt to explain the historic theoretical and experimental discrepancies. Arguing that individuals will make offers based upon the perceived reciprocity beliefs of other players, Falk and Fischbaker conclude that players will try to make offers that maximize their own utility based upon the lowest possible offer that will be accepted with probability $p = 1$ [6]. In other words, if a greedy player plays against someone who is likely to reciprocate current behavior in the future, then the greedy player will offer more in the current interaction; likewise, if a greedy player plays against someone unlikely to reciprocate current behavior in the future, then the greedy player will offer little in the current interaction [6]. This idea of reciprocity will be used in our model to construct trading behaviors for players that play against unfair or greedy players.

2.3 Altruism and Motivation

In *Altruism in Anonymous Dictator Games* (1995) [13] from Catherine Eckel and Philip Grossman, altruism as a motivator in games similar to the Ultimatum Game was studied. Altruism, which is present in games in which strategic decisions can be made over repeated games, is alleged to be the result of ”other-regarding behavior” [13]. Eckel and Grossman go on to argue that trading behavior, as present in situations like the Ultimatum Game, can be understood through a measure of ”deservingness” of the receiving party. In other words, if one is donating to a charity, one is more likely to give more than to an anonymous individual, which implies that information about the characteristics and qualities of the individuals playing the game have a role in the game's outcome.

Their experiment design focuses on the Ultimatum Game played between three players: one giver and two receivers, an anonymous individual and a charity. In the first treatment, between the giver and the anonymous individual, Eckel and Grossman finds that the offers tend to be close to the subgame perfect equilibrium; 62.5% of givers kept the entire amount. In the second treatment, between the giver and the charity, Eckel and Grossman finds that only 27.1% kept the entire amount.

These results indicate that, with information about parties engaged in a game, strategies of givers change depending on the qualities (or lackthereof) of the receivers [13]. In systems in which information about all parties is known to a degree, altruism tends to emerge when a player believes external benefits, whether outside of the game for themselves or other players and parties, are present. While what the external benefits are can be trivialized, the knowledge of such benefits impacts strategies over the course of dynamic games.

Chapter 3

Basic Model

3.1 Introduction to The Basic Model

We begin our analysis into the population dynamics of Ultimatum Game players by defining the variables inside of our model. There is a population of players N divided into two subgroups. The maximum amount of money or goods available to be offered or kept by a player during each interaction will be known as U. We define every player to have a preference set $\{r_m, \theta_m\}$ where $m \in \{1, 2\}$ designates to which subgroup the player belongs, $0 \leq r_m \leq 1$ is the proportion of the universal maximum amount U that one subgroup's player offers, and θ_m is a threshold for acceptance of the offer by the responding player in accordance with the responding player's subgroup preferences.

For the scope of our basic model, the preference set for each subgroup will be time independent. A return to preference set construction will occur in **Chapter 4**.

3.2 Generalized Interaction Model

Given the interaction characteristics of the classical Ultimatum Game, we will now define the structure for interactions for the entire population for our model.

In our model, we assume that players are paired together randomly from the pool of players in the population. There is no preference given to matching players inside or outside the same subgroup. Once matched, each player is randomly assigned role in the game, D or R. These normalizing assumptions will hold throughout the entire length of time the population is interacting.

For any given interaction in our population at time step t, we define the player who makes the offer as D and the person responding to the offer R. Let D offer $Ur_m(t)$. R will make a decision to accept or reject the offer by assigning utility to the offer. The utility received from an offer is not simply the magnitude of the offer, but an analysis of the magnitude of the offer with respect to R's subgroup preferences. The preferences of the subgroup, and therefore the individual, determine a real-valued utility based upon the distance from a 50/50 split of U, the offer that is recognized experimentally as "fair" [1]. Thus, utility at time step $t \geq 0$ is defined as

$$s[(Ur_m(t)) - \theta_n(Ur_m(t) - \frac{U}{2})^2] \tag{3.1}$$

where n is the subgroup identity of R, and s is an indicator function defined as

$$\begin{cases} s = 1 & \theta_n \leq \frac{Ur_m(t)}{(Ur_m(t) - \frac{U}{2})^2 + 1} \\ s = 0 & otherwise \end{cases}$$

The condition to satisfy acceptance of the offer from D is that the utility as determined by (3.1) is above the acceptance threshold of R. Such a threshold has been seen and measured experimentally by researchers, as the threshold acts as an irrationality barrier in theoretical treatments of the game. As discussed in **Chapter 2**, theoretical predictions for the game assert that, by the definition of rationality and assumptions about personal utility, rational responding agents in the Ultimatum Game will accept any offer $0 \leq c \leq U$, as the responding agent will be no worse off than before should they be offered nothing.

The utility equation (3.1) uses the convex function $(Ur_m(t) - \frac{U}{2})^2$ to establish a measurable deviation from the classically asserted fair offer of $\frac{U}{2}$ for the Ultimatum Game [1]. We employ this function to establish a population-wide, implicit decision-making structure. This function, moreover, guarantees smoothness across all possible offers. With the magnitude of offers playing a significant but not the entire role in determining utility, we leave room for subgroup preferences to shrink or enlarge the range of possible acceptable offers at every time step upon introduction of dynamic variables in **Chapter 4**.

3.3 Expectations and Fitness of Subgroups

For the purpose of analyzing population dynamics, it is necessary to define the fitness equations for the subgroups. A *fitness equation* is an equation modeling the ability of a population to reproduce, or grow, over time. Each subgroup in our population has its own set of preferences, and these preferences will manifest themselves in determining the size and growth of the subgroups over time.

The two subgroups treated in our basic model, $A(t)$ and $F(t)$, are defined in the following ways:

Definition 3.3.1. *Altruists (A(t)) have a belief system of fairness towards other players, independent of subgroup. Altruists in our model will exhibit a tendency to try to give more than they will keep. The amount offered by Altruists will be dependent on their threshold of acceptance.*

Altruists in our model can be seen as an extension of those in [13], with a belief system that applies value to fair offers. However, for generalization purposes, we leave to what Altruists apply value, outside of money, open for applications in different environments.

Definition 3.3.2. *Freeloaders (F(t)) have a belief system of selfishness towards other players, but dependent on the proportions of players in each subgroup. Freeloaders in our model will exhibit a tendency to try to take more than they will give. The amount offered by Freeloaders will be independent of their threshold of acceptance.*

We now need to define two new variables that represent the proportion of U each subgroup would like to keep given any interaction. From our variables defined in the General Interaction Model **Section 3.2**, we define these amounts to be

$$v_A = U(1 - r_A(t)) \tag{3.2}$$

$$v_F = U(1 - r_F(t)) \tag{3.3}$$

In our basic model, we begin with the three assumptions regarding interactions:

Basic Model Assumption 1. *For all interactions,* $v_A \leq \frac{U}{2} \leq v_F$.

Basic Model Assumption 2. *All players are acting as rational agents. Any offer proposed is accepted, and no player makes offers based upon future expectations of cooperation* ($\theta_m = 0$)*.*

Basic Model Assumption 3. *All players interact with another player (in other words, having an odd population size is trivial given the overall large size of the population)*

3.4 Expected Returns

These three assumptions are made for the Basic Model to provide an initial structure for interactions. It follows from **Assumption 1** that Altruists will never offer more than Freeloaders, and that the two subgroups experience varying levels of disparity between offers made to other players depending on the choice of r_m at the onset of the game. Additionally, due to the bounds on v_A, v_F, we have the following result:

$$r_F \leq \frac{1}{2} \leq r_A \tag{3.4}$$

It is clear from **Assumption 2** that the Basic Model is ignoring nonzero thresholds of acceptance. This assumption is true only for the Basic Model and will not be true for the Dynamic Model. **Assumption 3** eliminates the possibility of non-interacting players and simplifies the interaction model.

Given our assumptions, we can define the expectation equations for single game payoffs. To get an equation for the expected amount of money a player will receive during a game in which the receiver always accepts the offer, we need to combine the proportion of players in the population employing each strategy with the amount of money each subgroup collectively wants to keep (v_A, v_F) given an exchange.

Modeling the expected amount of money received during any interaction, we get:

$$E_A = \frac{1}{2}((U - v_A)(\frac{A}{F+A})) + (U - v_F)(\frac{F}{F+A})) + \frac{1}{2}v_A \tag{3.5}$$

$$E_F = \frac{1}{2}((U - v_A)(\frac{A}{F+A})) + (U - v_F)(\frac{F}{F+A})) + \frac{1}{2}v_F \tag{3.6}$$

We see that the expected return of any agent given any interaction relies upon the probability of the agent's role, the role of the other interacting agent, and the subgroup preferences. The expected return of any agent, more importantly, indicates the knowledge of other players' preferences from the onset of the game. Players can use the expectation of returns during each interaction as a measure of success or change at each time step.

3.5 Logistic Growth Equations

In our model, we will use logistic growth equations to capture competition and success, in relative terms, of each subgroup over the course of a time period. A *logistic growth equation* is a model for population growth over time and often take the form:

$$\frac{dx}{dt} = rx(1 - \frac{x}{k}) \tag{3.7}$$

where x is a population size; r is a parameter for growth or competition; and k is the carrying capacity, or maximum sustainable size, of the population in question.

Our model will utilize the features of logistic growth equations and expectation functions to capture information about the magnitude and direction of subgroup change over the course of time. With the given preferences of each subgroup and previous research on the Ultimatum Game and fairness, we know that individuals will seek to maximize payoff but not at the cost of intangible characteristics like fairness, honor, and respect. At the same time, an implicit level of competition arises between players and, given our subgroup preferences, by extension, our subgroups.

The competition between subgroups should be addressed before proceeding further. Given a starting population of players divided between our two subgroups, a competitive model for growth and success is most natural. From a biological perspective, carrying capacities are a product of measurable constraints to habitat, food sources, procreation, and various other factors. From an economic perspective, competing populations seek to maximize individual payoffs given preference sets in the population. Our model attempts to combine these two perspectives to create a dynamic model for preference set growth and decay, modeling the rise and fall of subgroups within a population competing for finite resources or goods in which individual tendencies during interactions are a product of general tendencies across the population.

With this in mind, let us propose our model for the fitness of each subgroup and population. The fitness equations for each subgroup and the population are as follows:

$$G(A,F) = \dot{A} = \frac{dA}{dt} = AE_A(1 - \frac{A}{F}) \tag{3.8}$$

$$H(A,F) = \dot{F} = \frac{dF}{dt} = FE_F(1 - \frac{F}{A}) \tag{3.9}$$

$$\dot{N} = \frac{dN}{dt} = \dot{A} + \dot{F} = \frac{(A-F)[-U(A^2-F^2) + AF(v_F - v_A)]}{2AF} \tag{3.10}$$

Notice several characteristics about these equations. Firstly, the parameter for subgroup growth is simply the expectation of return given any interaction at a time step. The greater the expectation of return, the larger in magnitude the change in the subgroup.

Note that, given our Basic Model Assumptions, the expected return for Freeloaders is greater than or equal to the expected return for Altruists.

Secondly, the magnitude of the parameter for population growth can be quite large should U be large relative to the overall size of the subgroup. Thirdly, and most importantly, note that the carrying capacity for each subgroup is the size of the other subgroup.

Given our belief about competition inside the game environment, it is a logical extension of biological competition to relate the sizes of the two subgroups and their growth equations. Because the number of Altruists and Freeloaders may change over time depending on initial allotments into each subgroup, the carrying capacities are, therefore, time varying. Unlike classical logistic growth equations, where the carrying capacity is constant due to exogenous factors, our logistic growth equations seek to capture the time-varying behavior of the two subgroups and make the role of subgroup growth or decay fatefully tied to the competing subgroup.

3.6 Systems of Differential Equations

We begin the analysis of the population dynamics of our Basic Model with several definitions and theorems.

Theorem 3.6.1. *(Existence and Uniqueness Theorem for First-Order Differential Equations) Any first order differential equation of the form* $\frac{dx}{dt} = f(x,t)$ *with initial conditions* $x(t_0) = x_0$*, where* $f(x,t)$ *is well-behaved, has a unique solution x.*

For our Basic Model, the growth equations for each population satisfy **Theorem 1**. Together, these two equations form a *system of differential equations* of the form

$$\frac{dA}{dt} = G(A, F) \tag{3.11}$$

$$\frac{dF}{dt} = H(A, F) \tag{3.12}$$

Often in studying systems of differential equations, it is important to identify the *nullclines* of our equations.

Definition 3.6.1. *A nullcline is locus of points such that the differential equation satisfies* $\frac{dx}{dt} = 0$

Using this system of differential equations and nullclines, we can establish relationships between A and F through an $A - F$ *plane*. This type of analysis lends itself to symbiotic or competitive environments like ours.

Let us now look at particular characteristics of each of these growth equations. Firstly, we can identify specific values for the subgroup size such that the growth rate for the subgroups and/or the population is zero, i.e. the nullclines of each differential equation.

Proposition 3.6.1. *For any values of* v_A, v_F*, the nullclines for Equations 3.8/3.11 and 3.9/3.12 are when A=F, including A=F=0.*

Proposition 3.6.2. *For any values of* v_A, v_F*, the nullclines for Equation 3.10 is when*

$$A = \frac{\frac{F(v_F - v_A)}{U} + F\sqrt{\frac{(v_A - v_F)^2}{U^2} + 4}}{2} \tag{3.13}$$

Proposition 1 holds by noticing that the carrying capacity is set to the size of the other subgroup, thus we have a nullcline whenever the population is split evenly between subgroups. It is also noteworthy that when either A or F are 0, the growth equation for that particular subgroup is 0 and the growth equation for the other subgroup is undefined. For the purposes of our model, a population in which the agents all have the same preference set is trivial to consider, so the following assumption holds:

Basic Model Assumption 4. *For all values of t,* $A(t) > 0$ *and* $F(t) > 0$.

Proposition 2 holds by setting the numerator of the growth equation for the population equal to zero. In other words, assuming $A \neq F$, by setting

$$-U(A^2 - F^2) + AF(v_F - v_A) = 0,$$

we can rearrange the terms so that

$$AF(v_F - v_A) = U(A^2 - F^2).$$

This equation, along with Assumption 4, allows us to solve the equality in terms of A.

With the nullclines determined, we can define the *fixed points* or *equilibrium points* of our system.

Definition 3.6.2. *Equilibrium in a system of differential equations is a point or set of points that satisfy* $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

Because the nullclines for our growth equations are equal across all values of A and F, our model exhibits a set of fixed points that span the entirety of both nullclines. More precisely, the nullclines are the same for each subgroup. This unique factor to our model captures a kind of symbiosis when the subgroups are of equal size. Despite preference sets that may be wildly different, when the subgroups are the same size, their growth equations are both 0.

Equilibrium in the population, however, is not as easy to summarize. It is obvious that when neither subgroup is growing or shrinking the population is constant. From our analysis of the nullcline for the population growth equation, we see that the number of A must be at a certain ratio to the number of F given by the maximum amount tradable in a given interaction U and the desired amount to keep kept during an interaction by each subgroup v_A, v_F. Using the known bounds of v_A, v_F, we can determine the range of values for A and F such that population growth 0.

When $v_F - v_A = 0$, we see that

$$A = F,$$

is a ratio of A and F such that population growth is 0. When $v_F - v_A = U$, we see that

$$A = \frac{F+F\sqrt{5}}{2},$$

implies the ratio of A and F such that population growth is 0.

The following proposition is immediate.

Proposition 3.6.3. *The nullcline for* $\frac{dN}{dt}$ *lies in between the straight lines* $A = F$ *and* $A = \frac{F+F\sqrt{5}}{2}$ *for any given values of* v_A, v_F.

Remark. *This proposition will hold for time-dependent or independent versions of* v_A, v_F. *Note, however, that the equilibria for the system is solely along the line* $A = F$.

3.7 Linearization and Population Dynamics

Given our equilibrium for the system of subgroup growth equations, let us define how we can further analyze the dynamics of the equilibrium over time.

Our system of equations $G(A, F), H(A, F)$ is *nonlinear*.

Definition 3.7.1. *Nonlinear equations are of the form* $\frac{dx}{dt} + xP(t) = Q(t)$, *where* $P(t)$ *and* $Q(t)$ *are some functions of only t.*

Because our growth equations contain other variables (namely, A and F are present in both growth equations), we must try to approximate our nonlinear system with linear equations. This is done to study how our system reacts when we apply small shifts, or *perturbations*, to the size of the subgroups. By doing this, we can get a better picture of how the subgroups react around equilibrium.

Given $F^* = A^*$ and $\frac{dN}{dt} = 0$, let us study small perturbations along the line of fixed points to see what the tendencies are for the growth functions for the two subgroups. Define $u = A - A^*$

and $v = F - F^*$ to represent small deviations from the line of fixed points. We now will look at $\dot{u}, \dot{v}$.

By setting $\dot{u} = G$ and $\dot{v} = H$, we can represent the derivative of the values u, v as:

$$\dot{u} = \frac{dA}{dt}(A^* + u, F^* + v) = \frac{dA}{dt}(A^*, F^*) + u\frac{\partial G}{\partial A} + v\frac{\partial G}{\partial F} \tag{3.14}$$

$$\dot{v} = \frac{dF}{dt}(A^* + u, F^* + v) = \frac{dF}{dt}(A^*, F^*) + u\frac{\partial H}{\partial A} + v\frac{\partial H}{\partial F} \tag{3.15}$$

By construction, the derivative of each subgroup with respect to (A^*, F^*) is equal to 0. This leaves us with the partial derivatives of G and H with respect to A and F multiplied by the magnitude of the perturbation per subgroup, u and v.

We find that:

$$\frac{\partial G}{\partial A} = \frac{-2U - v_A + v_F}{4} \tag{3.16}$$

$$\frac{\partial G}{\partial F} = \frac{2U + v_A - v_F}{4} \tag{3.17}$$

$$\frac{\partial H}{\partial A} = \frac{2U - v_A + v_F}{4} \tag{3.18}$$

$$\frac{\partial H}{\partial F} = \frac{-2U + v_A - v_F}{4} \tag{3.19}$$

From these equations, we notice that the partial derivatives with respect to each subgroup are in terms of the maximum amount tradable U and the preferred amount to be kept by each subgroup given any interaction v_A, v_F. The partial derivatives are independent of subgroup size. In other words, the rate of change of our linearized system from small perturbation from equilibrium is independent of the magnitude of some A^*, F^*. Relating this back to our game theoretic perspective, we can conclude that a deviation from equilibrium results in the rate of change in the subgroups being directly linked with the magnitude of the offers being made during any interaction.

Using our partial derivatives, we can write out our system in matrix form.

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2U - v_A + v_F & 2U + v_A - v_F \\ 2U + v_A - v_F & -2U + v_A - v_F \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \tag{3.20}$$

Having constructed a matrix of partial first-order derivatives, we can apply the definition of *Jacobian matrices* to find the eigenvalues of our system.

Definition 3.7.2. *The Jacobian Matrix is a matrix of all partial first-order derivatives of a vector-valued function. For a given function* $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ *that takes in a vector* $\boldsymbol{x} \in \mathbb{R}^n$ *and produces an output vector* $\boldsymbol{f(x)} \in \mathbb{R}^m$*, the Jacobian is an* $m \times n$ *matrix of the form*

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{3.21}$$

Because our Jacobian is 2×2, we can take the determinant of the matrix and solve for the eigenvalues of the matrix. The eigenvalues will provide us with information about the behavior of the system near the equilibrium points.

Taking the determinant of the matrix with eigenvalues along the diagonal, we solve for values of the two eigenvalues λ_1, λ_2:

$$\lambda_1 = 0 \tag{3.22}$$

$$\lambda_2 = -U \tag{3.23}$$

From our assumptions about v_A, v_F, we can determine the sign of both eigenvalues. For λ_1, regardless of the magnitude of v_A, v_F, we have a nonpositive eigenvalue. For λ_2, regardless of the magnitude of v_A, v_F, we have an eigenvalue of 0. Let us now classify the equilibrium points.

Proposition 3.7.1. *For all values of v_A, v_F, we have that $\lambda_1, \lambda_2 \leq 0$. Eigenvalues from our Jacobian that are both nonpositive indicate that small perturbations from our line of equilibrium $A = F$ will be attracted back towards the equilibrium points over time. We call this behavior **stable**.*

In summary, if we have initial values for the system such that $A(t_0) = F(t_0)$, then our system stays in equilibrium perpetually. Small deviations from equilibrium, however, will result in the system moving back towards equilibrium over time.

We can end this section with a theorem about our nullclines.

Theorem 3.7.1. *For a system in which $\frac{dx}{dt} = \frac{dy}{dt} = 0$ is a line of infinite fixed points with stable behavior, for $t \to \infty$, the x-nullcline and y-nullcline will never be crossed.*

Proof. Suppose the line of equilibrium could be crossed. Then a point on the line $X = Y$ would have either $\frac{dx}{dt} \neq 0$, $\frac{dy}{dt} \neq 0$, or both. This is a contradiction given the fact that the x-nullcine and y-nullcline overlap, implying that when $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ as well. $\square$

We can also study our model by inserting values for each subgroup and studying the growth characteristics and looking at the computation characteristics of our subgroups. In the first case, we can consider when the two subgroups are equal in size. This construction will provide us with a trivial solution to the growth equation, a state of non-growth amongst each subgroup and the population. This can be seen by noticing that the logistic function for each subgroup becomes zero when the size of the population meets the carrying capacity.

Secondly, when there are more Freeloaders than Altruists ($F > A$), the growth function for the entire population is less than zero. This implies that when there are more Freeloaders than Altruists in a given system, the rate at which the Freeloaders are shrinking in size is larger in magnitude than the rate at which Altruists are increasing in size. In other words, Freeloading behavior doesn't beget Freeloading behavior in our system. There isn't a kind of competitive exclusion occurring, and the entire population is actually suffering from too much freeloading.

To elaborate on this point, we can seek to understand the significance of the fact that too much freeloading can cause a decline in the entire size of the population. Given the fact that the each subgroup's size is directly tied to the size of the other, it is interesting that there are no trading patterns that emerge to keep the population at a constant level given a system in which there are more Freeloaders than Altruists. Looking at the growth equation for the entire population, we notice that the sign is strictly negative. Were it the case that each subgroup could swap trading parameters, i.e. Altruists begin to seek to keep more wealth per interaction than Freeloaders, we could find an equilibrium state of some kind where the subgroup growth rates combine to be zero. But because the preference set constraints are fixed, a decreasing population with a larger Freeloading subgroup results in more interspecies harm than good.

Thirdly, when there are more Altruists than Freeloaders ($A > F$), the growth function for the population is nonzero for most choices of v_A and v_F. Unlike in the case where there are more Freeloaders than Altruists, we see that the term inside the parentheses in the population growth equation could potentially change sign, implying that there might be an equilibrium state for the population when there are more Altruists than Freeloaders. This does not mean that the subgroups are also at equilibrium, as we've shown previously that the solutions to the differential equations are always when one subgroup is extinct or when the subgroups are of equal size.

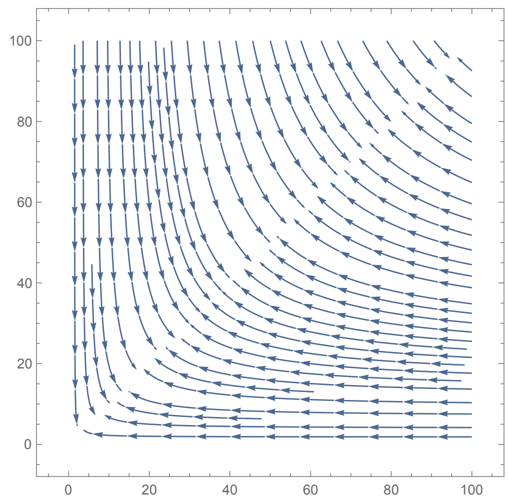


Figure 3.1: A phase diagram for $v_A = v_F$. Note the equilibrium at $A = F$ and the asymptotic attraction.

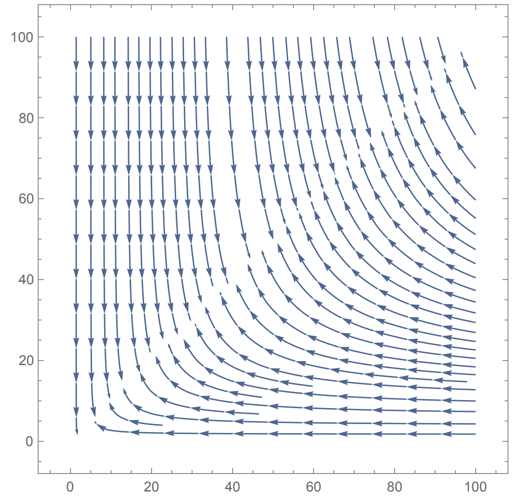
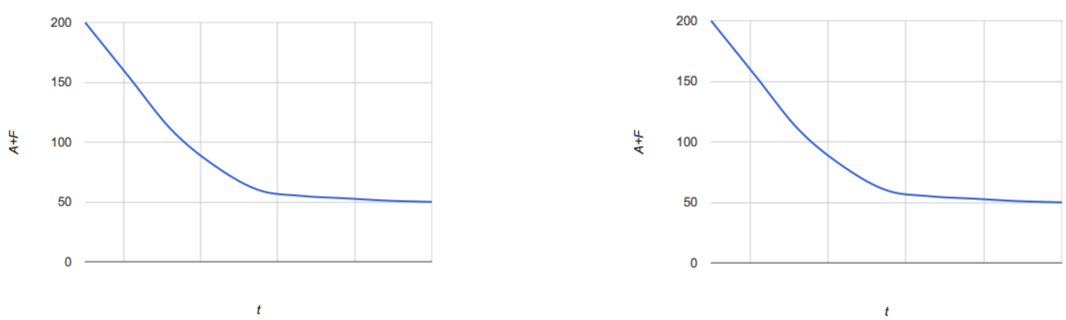


Figure 3.2: A phase diagram for $v_A = 0, v_F = U$. Note the steepness of the trajectories above equilibrium, due to the large expected value for Freeloaders and the low expected value of return for Altruists.

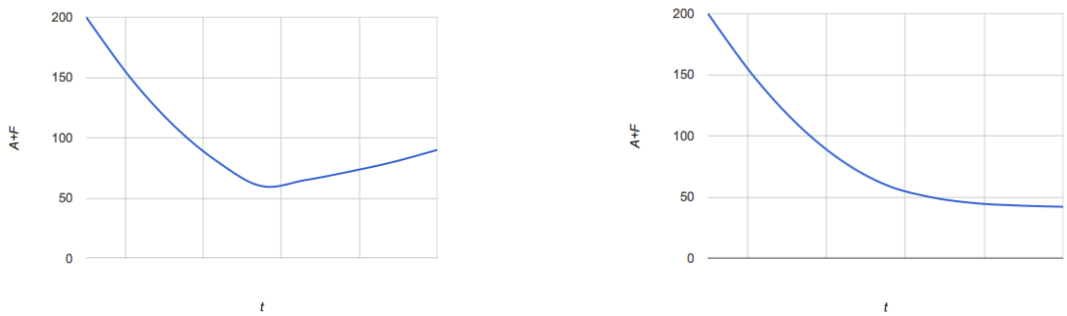
Figures 3.3 and 3.4 will be used to illustrate the population size changes over time given different initial conditions (here, $U = 100$):



(a) Initial Conditions: $A(0) = 150, F(0) = 50$

(b) Initial Conditions: $A(0) = 50, F(0) = 150$

Figure 3.3: Population Size Change at Minimum Disparity Trading Behavior.



(a) Initial Conditions: $A(0) = 150, F(0) = 50$ (b) Initial Conditions: $A(0) = 50, F(0) = 150$

Figure 3.4: Population Size Change at Maximum Disparity Trading Behavior.

It is clear to see that, given different initial conditions and trading preferences, the stability of the population size changes over time. Given trading behavior in which subgroups offer the same amount to each other, i.e. at minimum trading disparity, we see gradual population shrinking towards an equilibrium value over time in general. At maximum disparity, however, we see sharper decreases in the Freeloading population when Freeloaders initially outnumber Altruists and a relative population resurgence when Altruists initially outnumber Freeloaders. This is indicative of the population passing the moving into positive growth as the rate of increasing Freeloading is larger than the rate of decreasing Altruism.

3.8 Subgroup Proportions and Zero Population Growth

Given our construction of the population, we can also apply analysis to population proportions over time. Our system, comprised of only two subgroups, exhibits characteristically ecological structures with regards to population proportion changes over time.

We could write the differential proportion of each subgroup in the population as follows:

$$\frac{d}{dt}\left(\frac{A}{A+F}\right) = \frac{U}{2}\frac{(F-A)}{(A+F)} \tag{3.24}$$

$$\frac{d}{dt}\left(\frac{F}{A+F}\right) = \frac{U}{2}\frac{(A-F)}{(A+F)} \tag{3.25}$$

Note that the differential proportions do not depend on v_A, v_F. Despite the fact that we have a growth factor in our logistic growth equations that use v_A, v_F as parameters, the terms will cancel out and produce a proportion for the subgroup dependent only on the magnitude of the universal maximum amount U and the size of the other subgroup.

This type of relationship is characteristic of competitive systems in biology and ecology. When we have subgroups interacting with dependency in their individual growth equations, a subgroup proportion in terms only of subgroup size indicates that there is an underlying structure to population dynamics independent of preference sets. In other words, our subgroups are offering amounts of money or goods at each interaction, which on an individual basis affects subgroup size, but proportion trends are dependent only on the holistic view of subgroup trends.

Suppose we rewrote our equations in terms of A, N only. We would get:

$$\frac{d}{dt}\frac{A}{N} = \frac{da}{dt} = \frac{U}{2}(f - a) \tag{3.26}$$

where $f = 1 - a$. This gives us

$$\frac{da}{dt} = a' = \frac{U}{2}(1 - 2a) \tag{3.27}$$

Let $\alpha = 2a - 1$, giving $-\alpha = 1 - 2a$. Solving for a', we now have the rate of change in the proportion of Altruists as

$$a' = \frac{U}{2}(-\alpha) \tag{3.28}$$

Remark. *If $\alpha < 0$, we see that a' is positive, and if $\alpha > 0$, a' is negative.*

Chapter 4

Dynamic Offers and Acceptances

4.1 Dynamic Interaction Model and Assumptions

As defined in Section 3.2, our Generalized Interaction Model provides a time-step framework by which players in each subgroup are assigned a role and a decision-making function for each role. Given our assumption in Chapter 3 that the preference set values were constant, let us now propose a model for dynamic preference sets and behavior.

Let us revise our assumptions for the Dynamic Model from the Basic Model. Given

$$v_A = U(1 - r_A(t)) \tag{4.1}$$

$$v_F = U(1 - r_F(t)), \tag{4.2}$$

we will now claim:

Dynamic Model Assumption 1. *For all interactions,* $v_A \leq \frac{U}{2} \leq v_F$.

Dynamic Model Assumption 2. *Altruists and Freeloaders will exhibit time dependent decision-making across all interactions.*

Dynamic Model Assumption 3. *All players interact with another player (in other words, having an odd population size is trivial given the overall large size of the population)*

Assumption 2 is the main revision from the Basic Model Assumptions.

Consider again our Generalized Interaction Model:

$$s[(Ur_m(t)) - \theta_n(Ur_m(t) - \frac{U}{2})^2] = \begin{cases} s = 1, & \theta_n \leq \frac{Ur_m(t)}{(Ur_m(t) - \frac{U}{2})^2} \\ s = 0, & otherwise. \end{cases}$$

Each player, after being assigned their role in the interaction, will choose an amount to offer and a threshold for acceptance based upon their subgroup preference set $\{r_m, \theta_m\}$ where m designates to which subgroup the player is aligned. Thus, $0 \leq r_m \leq 1$ is the proportion of the universal maximum amount U that one subgroup's player offers, and θ_m is a threshold for acceptance of the offer by the responding player in accordance with the responding player's subgroup preferences.

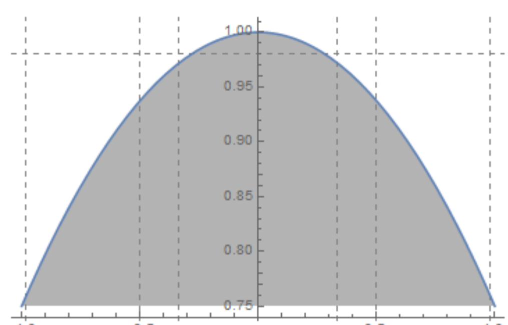


Figure 4.1: The gray area represents the amount of utility gained (proportional to the offer made) from every received offer. The dashed lines represent different boundary points for threshold values of acceptance.

Throughout this section, we propose several models for these interactions and demonstrate how different strategies played by the subgroups are dynamically related, based upon ideas from [6] and [13]. Let us define several terms that will be used to describe strategies played by agents in the game.

Definition 4.1.1. *(Reciprocated Fairness) A subgroup of a population exhibits reciprocated fairness if the subgroup, while having a relatively high threshold for acceptance, makes offers at or above $\frac{1}{2}U$. The preference set for the subgroup would be $\{r_m \geq \frac{1}{2}, \theta_m >> 0\}$.*

Definition 4.1.2. *(Unreciprocated Fairness) A subgroup of a population exhibits unreciprocated fairness if the subgroup, while having a low threshold for acceptance, makes offers at or above $\frac{1}{2}U$. The preference set for the subgroup would be $\{r_m \geq \frac{1}{2}, \theta_m \geq 0\}$*

Definition 4.1.3. *(Conditional Greed) A subgroup of a population exhibits conditional greed if the subgroup, while having no threshold for acceptance, makes offers at or below $\frac{1}{2}U$ but seeks to optimize offers based upon subgroup proportion sizes. The preference set for the subgroup would be $\{r_m \leq \frac{1}{2}, \theta_m = 0\}$.*

Definition 4.1.4. *(Unconditional Greed) A subgroup of a population exhibits unconditional greed if the subgroup, while having no threshold for acceptance, makes offers at or below $\frac{1}{2}U$ but doesn't seek to optimize offers based upon endogenous parameters. The preference set for the subgroup would be $\{r_m \leq \frac{1}{2}, \theta_m = 0\}$.*

4.2 Dynamic Preference Set Analysis

The dynamics of the Basic Game have shown the generalized dynamics of our game environment. We will now propose four example models, one for each Fairness/Greed pair, and demonstrate how our Dynamic Model assumptions model historical experimental data.

4.2.1 Reciprocated Fairness and Conditional/Unconditional Greed

Suppose we wanted a time dependent model for interactions in which Altruists have a Reciprocated Fairness strategy and Freeloaders have a Conditional Greed strategy. We can construct the preference sets for each subgroup as follows:

$$r_A(t) = 1 - 2af, \quad r_F(t) = 2af \tag{4.3}$$

where θ_A, θ_F are defined as:

$$\theta_A = \frac{1}{2Uaf}, \quad \theta_F = 0 \tag{4.4}$$

This game environment is characterized by high offers with high thresholds for Altruists when their subgroup proportion is low, and lower but fair offers and lower thresholds when their subgroup proportion is close to $\frac{1}{2}$. Additionally, Freeloaders will try to offer less when the proportion of Altruists is low, trying to take advantage of the fact that a majority of agents in the game won't turn down low offers. These strategies are adapted from similar strategies in [6] and [13].

We can also take partial derivatives of v_A, v_F to study the effects population size have on offers. For Reciprocated Fairness and Conditional Greed,

$$\frac{\partial Ur_A}{\partial A} = U\left(\frac{4AF}{(A+F)^3} - \frac{2F}{(A+F)^2}\right) \tag{4.5}$$

$$\frac{\partial Ur_A}{\partial F} = U\left(\frac{4AF}{(A+F)^3} - \frac{2A}{(A+F)^2}\right) \tag{4.6}$$

$$\frac{\partial Ur_F}{\partial A} = -\frac{4UAF}{(A+F)^3} + \frac{2UF}{(A+F)^2} \tag{4.7}$$

$$\frac{\partial Ur_F}{\partial F} = -\frac{4UAF}{(A+F)^3} + \frac{2UA}{(A+F)^2} \tag{4.8}$$

For a game in which the Altruists have a Reciprocated Fairness strategy and the Freeloaders have an Unconditional Greed strategy, we can model interactions in the following way:

$$r_A(t) = 1 - 2af, \quad r_F(t) = .1 \tag{4.9}$$

where θ_A, θ_F are defined as:

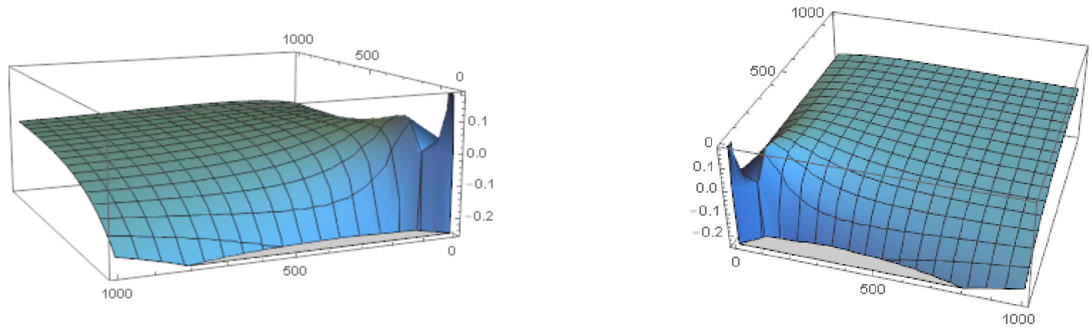
$$\theta_A = \frac{1}{2Uaf}, \quad \theta_F = 0 \tag{4.10}$$

We can also solve for the rate of change of the threshold value given the subgroup proportion of Altruists:

$$\frac{\partial \theta_A}{\partial A} = \frac{A+F}{UAF} - \frac{(A+F)^2}{2UA^2F} \tag{4.11}$$

$$\frac{\partial \theta_A}{\partial F} = \frac{A+F}{UAF} - \frac{(A+F)^2}{2UAF^2} \tag{4.12}$$

This game environment is characterized by high offers and high thresholds for Altruists when their subgroup proportion is low, and lower but still fair offers and lower thresholds when their subgroup proportion is close to $\frac{1}{2}$. Additionally, Freeloaders will make low offers (here characterized by a constant offer magnitude $Ur_F = \frac{U}{10}$) and no threshold for acceptance. This interaction dynamic mimics closely long-run dynamics mentioned in [1].

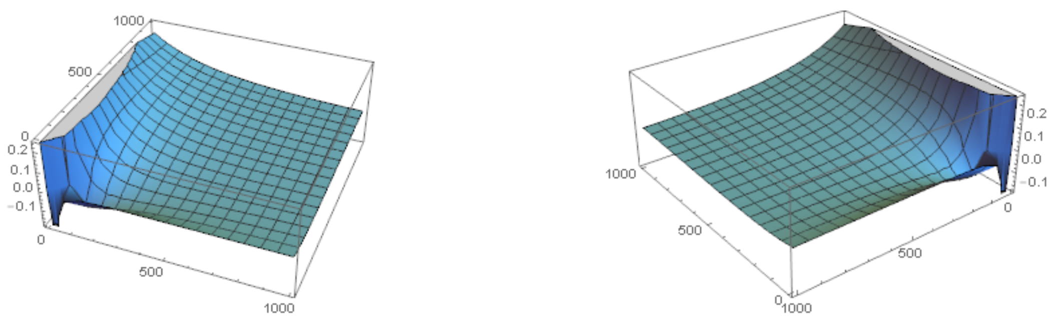


(a) Altruistic Offer Rate of Change with respect to A (U=100; $A, F > 0$)

(b) Altruistic Offer Rate of Change with respect to F (U=100; $A, F > 0$)

Figure 4.2: Altruistic Offer Rate of Change.

Figure 4.2 show the rate of change in offers from Altruists playing Unreciprocated Fairness with respect to changes in subgroup size of Altruists and Freeloaders. When the proportion of Altruists is large in the population, the rate of change of offers from Altruistic players is near 0. Conversely, when the proportion of Altruists is small in the population, the rate of change of offers from Altruistic players is negative. This indicates that Altruists will be more likely to give a greater amount during an interaction if they know that other players are more likely to be Altruists, and that Altruists reciprocate Freeloader offers with lowering their own offers.



(a) Freeloading Offer Rate of Change with respect to A (U=100; $A, F > 0$)

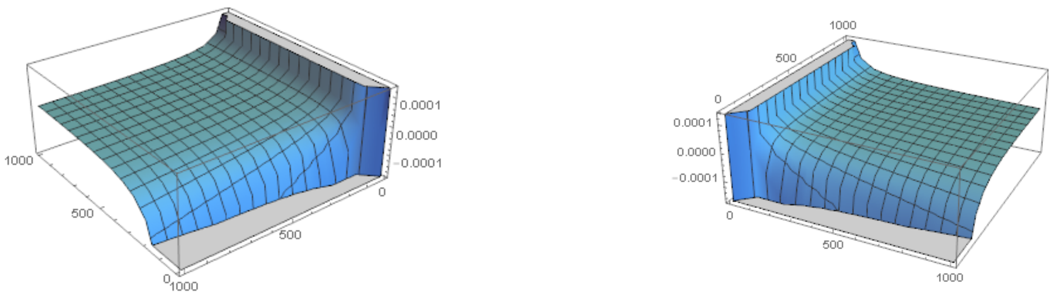
(b) Freeloading Offer Rate of Change with respect to F (U=100; $A, F > 0$)

Figure 4.3: Freeloading Offer Rate of Change.

Figure 4.3 show the rate of change in offers from Freeloaders playing Conditional Greed with respect to changes in subgroup size of Altruists and Freeloaders. When the proportion of Freeloaders is large in the population, the rate of change of offers from Freeloading players is positive. Similarly, when the proportion of Freeloaders is small in the population, the rate of change of offers from Freeloading players is positive.

Only once the proportions are close to equal do Freeloader offers begin to approach a constant value. This indicates that Freeloaders will try to take advantage of low numbers of Altruists or Freeloaders in the population. Using experimental evidence and perspectives from [1], we argue that, given the low acceptance rate of Altruists, Freeloaders will try unfair offers initially but, since Altruists will have a high threshold for acceptance, they can't repeatedly make unfair offers, so

Freeloaders make more fair offers as time goes on. When there are a large number of Freeloaders relative to Altruists, Freeloaders may try to compensate for the lack of fair play and initially become greedy themselves. The return to a 50/50 split could be seen as a domino effect from the first Freeloader being rejected by a reciprocating Altruist, as more and more Freeloaders would rather play with spite than try to stay greedy.

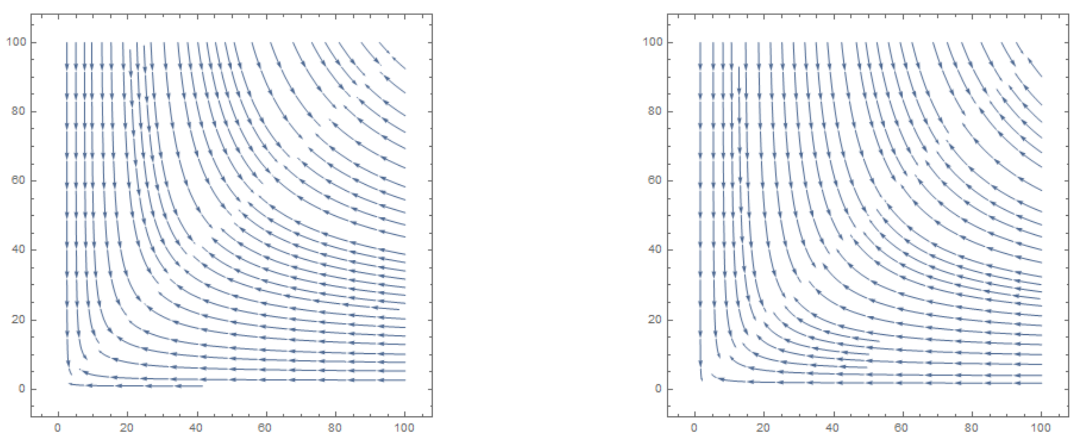


(a) θ_A with respect to A (U=100; $A, F > 0$)

(b) θ_A with respect to F (U=100; $A, F > 0$)

Figure 4.4: Rate of Change in θ_A given A, F.

Figure 4.4 shows the rate of change in the threshold value of acceptance for Altruists with respect to changes in subgroup size of Altruists and Freeloaders. At extreme levels of inequality between the number of Altruists and Freeloaders, we see exceptionally high thresholds of acceptance. As equilibrium between subgroups is approached, this threshold will fall, as fairness will be implicit across the population.



(a) Reciprocated Fairness and Conditional Greed with $U = 100$

(b) Reciprocated Fairness and Unconditional Greed with $r_F = .1$; $U = 100$

Figure 4.5: Phase Fields for Reciprocated Fairness.

Figure 4.5 illustrates the rates of change in the subgroups given our two preference set pairs. For the paired strategies Reciprocated Fairness and Conditional Greed in 4.5a, the expected return per interaction for each subgroup approach equality, as both subgroups will make offers close to a 50/50 split of U. This kind of equilibrium, in which subgroups are approaching not only the same size but also the same offers, is representative of a long-term preference set development in which players put emphasis on making trades that the other group will most likely accept.

For the paired strategies Reciprocated Fairness and Unconditional Greed in 4.5b, the expected return for Freeloaders is larger in comparison to that of Altruists, contributing to a steeper angle of the vectors in the phase field. This is indicative of more rapid decreases in Freeloaders than in Altruists, a result of Altruists lowering acceptance thresholds as the subgroup proportions approach equality. The inability for Altruists to change the behavior of Freeloaders, therefore, results in slight but significant differences in near-equilibrium behavior.

4.2.2 Unreciprocated Fairness and Conditional/Unconditional Greed

Altruists who exhibit Unreciprocated Fairness can be seen as giving away virtually everything in a trade, without a strong desire to turn down offers that don't give much in return.

For a game in which the Altruists have an Unreciprocated Fairness strategy and the Freeloaders have a Conditional Greed strategy, we can model interactions in the following way:

$$r_A(t) = \frac{3}{4} + \theta_A, \quad r_F(t) = 2af \tag{4.13}$$

where θ_A, θ_F are defined as:

$$\theta_A = af, \quad \theta_F = 0 \tag{4.14}$$

A game environment characterized by high offers with no threshold for Altruists regardless of subgroup size can be viewed as Unreciprocated Fairness. Additionally, Freeloaders will try to offer less when the proportion of Altruists is low, trying to take advantage of the fact that a majority of agents in the game won't turn down low offers.

For Unreciprocated Fairness and Conditional Greed, we find the partial derivatives of the offers are:

$$\frac{\partial U r_A}{\partial A} = U\left(\frac{-2AF}{(A+F)^3} - \frac{F}{(A+F)^2}\right) \tag{4.15}$$

$$\frac{\partial U r_A}{\partial F} = U\left(\frac{-2AF}{(A+F)^3} - \frac{A}{(A+F)^2}\right) \tag{4.16}$$

$$\frac{\partial U r_F}{\partial A} = -\frac{4UAF}{(A+F)^3} + \frac{2UF}{(A+F)^2} \tag{4.17}$$

$$\frac{\partial U r_F}{\partial F} = -\frac{4UAF}{(A+F)^3} + \frac{2UA}{(A+F)^2} \tag{4.18}$$

We can also solve for the rate of change of the threshold value given the subgroup proportion of Altruists:

$$\frac{\partial \theta_A}{\partial A} = \left(\frac{-2AF}{(A+F)^3} - \frac{F}{(A+F)^2}\right) \tag{4.19}$$

$$\frac{\partial \theta_A}{\partial F} = \left(\frac{-2AF}{(A+F)^3} - \frac{A}{(A+F)^2}\right) \tag{4.20}$$

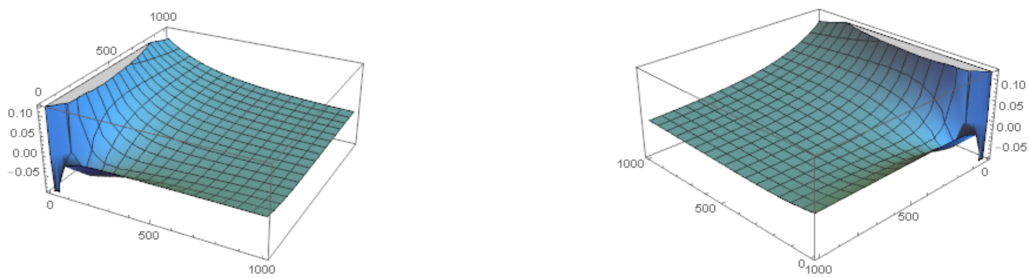
For an environment in which Altruists have an Unreciprocated Fairness strategy and Freeloaders have an Unconditional Greed strategy, we can model interactions in this way:

$$r_A(t) = \frac{3}{4} + \theta_A, \quad r_F(t) = .1 \tag{4.21}$$

where θ_A, θ_F are defined as:

$$\theta_A = af, \quad \theta_F = 0 \tag{4.22}$$

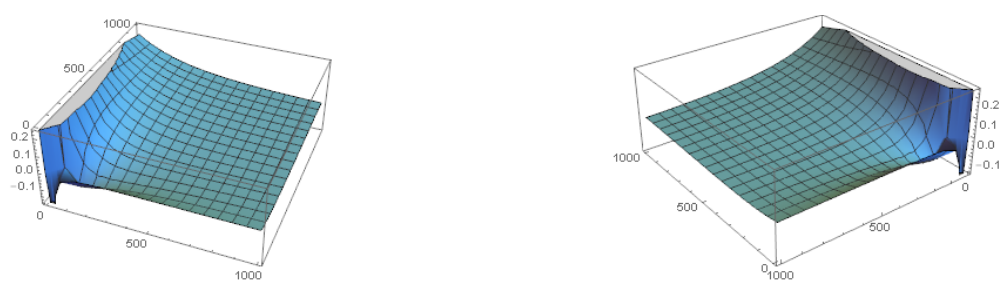
This game is characterized by high offers with low thresholds for Altruists when their subgroup proportion is low, and lower but still fair offers and lower thresholds when their subgroup proportion is close to $\frac{1}{2}$. Additionally, Freeloaders will make low offers (here characterized by a constant offer magnitude $Ur_F = \frac{U}{10}$ and no threshold for acceptance.



(a) $U - v_A$ (U=100; $A, F > 0$) (b) $U - v_A$ (U=100; $A, F > 0$)

Figure 4.6: Rate of Change in $U - v_A$ under Unreciprocated Fairness and Conditional Greed with respect to A, F.

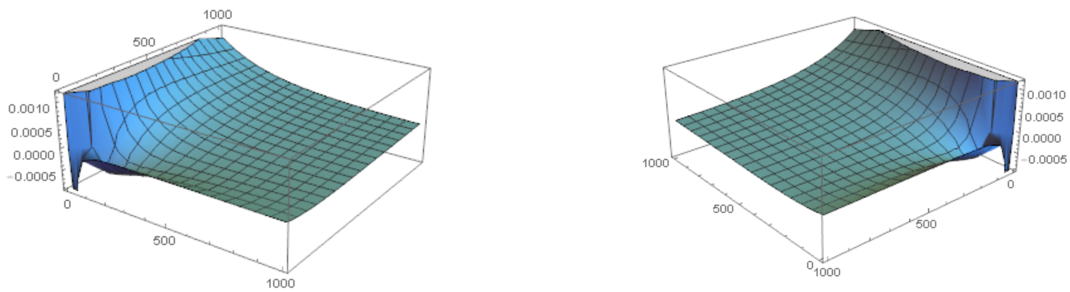
Figure 4.6 shows the rate of change in offers from Altruists with respect to changes in subgroup size of Altruists and Freeloaders. Because Altruists are overwhelmingly generous in their offers, and because the threshold for acceptance is close to zero for all time, there is very little change over time in offers that begin near subgroup equality. If, however, initial conditions lead to one subgroup largely outnumbering the other, we could see relatively large increases in the offer. There is a slight dip near the origin, indicating that for smaller subgroup sizes the rate of change of the offer could be temporarily negative.



(a) $U - v_F$ (U=100; $A, F > 0$) (b) $U - v_F$ (U=100; $A, F > 0$)

Figure 4.7: Rate of Change in $U - v_F$ under Unreciprocated Fairness and Conditional Greed with respect to A, F.

Figure 4.7 shows the rate of change in offers from Freeloaders with respect to changes in subgroup size of Altruists and Freeloaders. The rate of change of Freeloader offers will be generally be near 0 for most subgroup sizes. Like for Altruists, there is a small area near the origin in which offers may decrease.

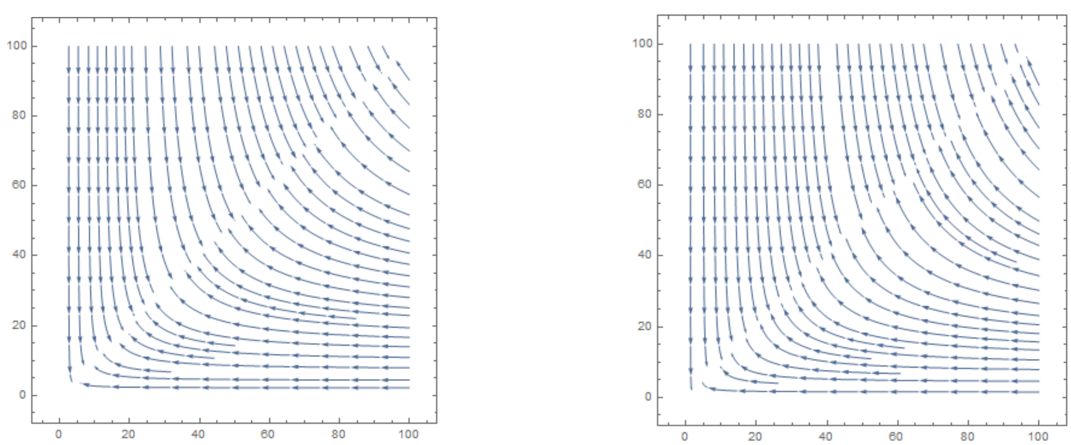


(a) θ_A with respect to A (U=100; $A, F > 0$)

(b) θ_A with respect to F (U=100; $A, F > 0$)

Figure 4.8: Rate of Change in θ_A given A, F.

Figure 4.8 shows the rate of change in the threshold value of acceptance for Altruists with respect to changes in subgroup size of Altruists and Freeloaders. The threshold for acceptance is so small relative to the magnitude of offers that, given virtually any change in subgroup sizes, the threshold for acceptance will stay near 0.



(a) Unreciprocated Fairness and Conditional Greed

(b) Unreciprocated Fairness and Unconditional Greed

Figure 4.9: Phase Fields for Unreciprocated Fairness.

Figure 4.9 illustrates the differences in rate of change in population sizes given different preference sets. For paired strategies Unreciprocated Fairness and Conditional Greed in 4.9a, the number of Freeloaders decreases at a significantly faster rate than Altruists. Since Altruists give a significant portion during each trade, with low acceptance thresholds for both subgroups, we approach a subgroup equilibrium with Altruists giving almost all of their wealth and Freeloaders making 50/50 splits. This kind of behavior could be seen as an extension of the theories in [6], where Freeloaders want to make trades that guarantee acceptance from Altruists. It is important to note, however, that in our example Freeloaders are not optimizing their offers based upon the exact value necessary to pass the acceptance threshold for Altruists. In fact, this would be a case where Freeloaders are causally making fair offers and reciprocating for the Altruists high offers.

If Freeloaders play with Unconditional Greed, as represented in 4.9b, we have a case similar to the maximum disparity example from **Chapter 3**. Freeloading behavior is constant through the interactions, unaffected by subgroup proportion changes. When this type of behavior is paired with Unreciprocated Fairness, we have a exploitative environment in which one subgroup is giving more than they are receiving but not attempting to address the inequity. In situations where Freeloaders greatly outnumber Altruists, the Freeloader subgroup decreases in size rapidly. When there are significantly more Altruists than Freeloaders, the Altruists also decrease in size, but the population begins to increase in size near equilibrium.

Chapter 5

Conclusion and Future Discussions

Our model seeks to study the affects of fairness, altruism, and greed in the Ultimatum Game through the lens of population dynamics. As such, it serves to capture important dynamical characteristics of two interacting subgroups in a population. Our **Basic Model** shows that, pitting Altruists and Freeloaders against each other, that if offers are constant across all time and all offers are accepted, the population will approach proportional equality. We also showed that populations in which strictly 50/50 offers are made will approach equilibrium at different rates than if Freeloaders keep the maximum amount and Altruists give everything away. This behavior of the system indicates that a population of players will change in size differently as it approaches equilibrium dependent on trading behaviors.

Importantly, the 50/50 split is assumed to be the most fair trading behavior in our game environment. Concerns about the validity of this assumption can be raised and investigated, but such concerns were not the focus of this thesis.

A changing size in the population, moreover, speaks to the sustainability of a game environment in which the growth of each subgroup is intimately tied to that of the other. As noted in [6], reciprocity in the system is inherent through the growth equations, and we see that with 50/50 offers the subgroup trajectories are almost mirrored across the line of equilibria. However, in the maximum disparity situation, we do not see such mirroring across the equilibria. Instead, the phase plane shows two different situations arising given different initial conditions. Our conclusion is that Freeloading behavior is more heavily punished by the population dynamics when there are more Freeloaders than Altruists.

Our model features logistic growth equations that have dynamic carrying capacities. By setting the size of one subgroup as the carrying capacity of the other, we establish a powerful relationship between the growth and decay of the subgroups over time. Further analysis into the biological or ecological implications of such a relationship could yield potentially interesting findings.

In our **Dynamic Model**, the four groups of strategies, based upon dynamic preference sets with beliefs in altruism or greed, found that in populations where the offer and acceptance threshold a player exhibits is related to subgroup sizes and proportions, that fairness in trading behavior can emerge in systems where players place value on equality and balance in a system. Reciprocation, modeled from the ideas in [1] and [6], could be due to players placing value on equal subgroup proportions, or even directly on the offers and perceived preferences of the other subgroup. Conditional greed, like that implicit unfairness towards anonymous players in [13], could result from players not placing value on such balance in an environment.

There is room for growth inside this dynamic model. First and foremost, there is the potential to study the affects of playing the Ultimatum Game with larger or smaller values for U. Additionally, further study could be applied to subgroup proportion analysis and stability given our current assumptions on trading behavior or without. Finally, there is room for more strategies across the population and more care given to the importance, ecologically, of sustainable game systems and environments.

Chapter 6

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