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EXTENDING FIRST ORDER LOGIC TO QUANTIFY OVER FORMULAS

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Abstract

In standard first order logic there is no limit on the cardinality of a set of axioms. All proofs, however, must be finite. This means that for infinite sets of axioms it isn't possible to use them all in a given proof. Given a set Γ of first order formulas is there a way to modify our logic so that we can have a set Γ' in our new logic that has finitely many axioms yet can prove the same statements as Γ?

The natural idea here is that of compactness. If Γ is "nice enough," can we find a finite set of similar axioms that proves the same things that Γ proves, e.g. a "cover" of Γ? Here we consider the special case where all but finitely many axioms in Γ do not fall into one of finitely many structures. The obvious example would be the induction scheme on the natural numbers. In first order logic we would need infinitely many axioms of the form $(\phi(0) \wedge (\phi(n) \rightarrow \phi(n+1))) \rightarrow \forall n \phi(n)$ where ϕ is a well formed formula. We will construct a new type of logic with the means of collapsing these infinitely many formulas into one formula. We will check if the analogous structures in our new language have the same nice properties of first order logic, namely soundness and completeness. We will then investigate which first order statements we can prove with our new logic.

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Acknowledgements

I would like to thank Dr. Jason Rute Penn State, my thesis adviser, for providing ideas to guide this thesis when I became stuck.

I would also like to thank Dr. Herbert Enderton of UCLA (1936-2010), whom I will never meet. Nevertheless, Chapter 2 of his *A Mathematical Introduction to Logic* [1] served as my first glimpse into mathematical logic and his discussion on first order logic in it was used as a guide for my own logic in this thesis.

Chapter 1

Introduction

First order logic is one of the primary logics used when considering the foundations of mathematics. Given specific set of symbols $\mathcal{L}$, which we call a *language*, first order logic has mechanisms for talking about one type of object and what statements are true of these objects. For example, our language may talk about numbers and contain the relations Ex and Ox, where Ex says "x is even" and Ox says "x is odd." Then when thinking about the natural numbers, "$\forall x \; Ex$" would be a false statement and "$\forall x \; Ex \vee Ox$" would be a true statement.

Our goal in this thesis is to slightly extend the idea of a formal first order language $\mathcal{L}$ to a new language $\mathcal{L}^*$ by adding a new type of variable that is interpreted as a formula in $\mathcal{L}$. This new form of logic will have mechanisms to talk about objects *and* formulas of our original language and evaluate truth or falsity. We will also add the ability to quantify over all $\mathcal{L}$-formulas and perform substitutions like we do for regular first order variables. We do this hoping to achieve three things. One, we want our new language to be able to represent a first order axiom scheme, that is an infinite set of first order axioms which consists of a single format applied to all formulas, in one sentence by quantifying over the set of the $\mathcal{L}$-formulas. Second, we desire $\mathcal{L}^*$ to have analogues of soundness, compactness, and completeness, which are all important and useful properties of first order logic. Finally, we want the "first order part" of our $\mathcal{L}^*$ structures to be exactly the corresponding $\mathcal{L}$ structure in a natural way, that is to say we can prove the same first order statements in both logics. This is done to eliminate the need to assume infinitely many axioms in certain "nice" axiom systems despite the fact that only finitely many axioms can be used in a given proof. If the extended language turns out to be no more powerful than the standard language then we will have shown that certain sets of infinite axioms can be reduced to finitely many axioms without using more powerful objects, suggesting that the need for infinitely many axioms can be a restraint imposed by first order logic in some instances.

In Chapter 2 we will review the literature and quickly visit propositional logic and first order logic. We will demonstrate one of the weaknesses of propositional logic and use first order logic as an attempt to eliminate this weakness. We will discuss what composes a first order language $\mathcal{L}$, the components of an $\mathcal{L}$-structure $\mathfrak{A}$, and the equivalence of provability and "truth" in first order logic. We will then discuss a perceived weakness of first order logic and attempt to fix this with our new logic in the next chapter.

In Chapter 3, the **Foundation** chapter, we shall take a first order language $\mathcal{L}$ and create from it a new extended language $\mathcal{L}^*$. $\mathcal{L}^*$ will have a new type of variable which is meant to be interpreted as well formed formulas of $\mathcal{L}$. We shall define what it means for a structure to satisfy a $\mathcal{L}^*$-formula and how we can go about proving things in the new logic. This is a new type of construction, although it uses the format of Enderton's [1] construction of first order logic as inspiration.

In Chapter 4 we will demonstrate an example of how this process works to make certain infinite sets of axioms into finite ones in a fairly natural way. We will use first order Peano Arithmetic as a guide and turn the induction scheme into a single $\mathcal{L}^*$ formula.

In Chapter 5, the **Results** chapter, we shall address the second issue of desirable results for our new class of languages. Specifically we will verify the soundness of our new logic, which is to say that everything that we can prove is "true," i.e. satisfied by our structures. We will not prove completeness, everything that is "true" is provable, but we will show that it is equivalent to a particularly technical statement. Assuming completeness will give us two other results almost immediately.

We shall finish with a conclusion that discusses the ramifications of the results and what further questions could be investigated.

Chapter 2

First Order Logic

The majority of this thesis takes heavy inspiration from Dr. Herbert Enderton's *A Mathematical Introduction to Logic* [1]. This chapter is a summary of Propositional and First Order Logic, and more detail can be found in Enderton [1] Chapters 1 and 2.

We shall use $\frown$ to denote the concatenation of strings/vectors throughout.

2.1 Propositional Logic

Definition 2.1.1. *We say the set $L = \{P_n : n \in \mathbb{N}\} \cup \{(,), \neg, \to\}$ is the language of propositional logic. The P_n's are called propositional variables, and We call the elements of L propositional symbols. (The informed reader may wonder why we only use $\neg, \to$ and not including symbols like $\wedge$. This is addressed in Enderton [1] Section 1.5)*

Definition 2.1.2. *We say that a Well Formed Formula (abbreviated as WFF) of propositional logic is a finite string ψ of propositional symbols such that ψ is in the set F defined inductively as follows: (For a more general look at induction and recursion, see Enderton [1] Section 1.4)*

- *$P_n \in F$ for all $n \in \mathbb{N}$.*
- *F is closed under the function $F_\neg : L^{<\mathbb{N}} \to L^{<\mathbb{N}}$, where $F_\neg(A) = \langle (, \neg, \rangle \frown A \frown \langle) \rangle$. We denote this string by $(\neg A)$, or $\neg A$ if there is no ambiguity, and we say "not A."*
- *F is closed under the function $F_\to : L^{<\mathbb{N}} \times L^{<\mathbb{N}} \to L^{<\mathbb{N}}$, where $F_\to(A, B) = \langle (\rangle \frown A \frown \langle \to \rangle \frown B \frown \langle) \rangle$. We denote this string by $(A \to B)$, or $A \to B$ if there is no ambiguity, and we say "A implies B" or "If A, then B."*

Informally we think of WFF's as statements that can either be true or false. Formally, we define a function on the WFF's whose value will denote "truth" or "falsity."

Definition 2.1.3. *A truth assignment is a function* $T\colon \{P_n : n \in \mathbb{N}\} \to \{0,1\}$*. The interpretation is meant to be that* 1 *is "true" and* 0 *is "false." Then we define the function* $\overline{T}\colon F \to \{0,1\}$ *as an extension of* T *which evaluates the "truth" or "falsity" of any propositional formula. This is defined recursively as follows:*

- $\overline{T}(P_i) = T(P_i)$ *for every* $i \in \mathbb{N}$*.*
- $\overline{T}(\neg A) = 1 - \overline{T}(A)$*. Notice that this is always the opposite value of* $\overline{T}(A)$*.*
- $\overline{T}(A \to B) = \overline{T}(B) + (1 - \overline{T}(B))(1 - \overline{T}(A))$*. This computation is exactly saying that if* $\overline{T}(B) = 1$*, then* $\overline{T}(A \to B) = 1$*. If it is* 0*, then* $\overline{T}(A \to B)$ *has the value opposite of* A*.*

Now we have statements and a way to discuss whether or not those statements are true. One important notion that we will use later can now be introduced: the notion of tautology. Informally, a tautology is a WFF that is always true.

Definition 2.1.4. *A tautology is a WFF* A *such that for any truth assignment* T*,* $\overline{T}(A) = 1$*.*

One example of a tautology is $P_1 \to P_1$: Let T be a truth assignment. Then if $T(P_1) = 0$, we have $\overline{T}(P_1 \to P_1) = T(P_1) + (1 - T(P_1))(1 - T(P_1)) = 0 + (1)(1) = 1$. If $T(P_1) = 1$, then $\overline{T}(P_1 \to P_1) = T(P_1) + (1 - T(P_1))(1 - T(P_1)) = 1 + (0)(0) = 1$. Thus $\overline{T}(P_1 \to P_1) = 1$ for any T. This, however, is not a terribly interesting example: this says that "If P_1, then P_1" is always true. A slightly more interesting example is $(\neg P_2 \to \neg P_1) \to (P_1 \to P_2)$. Let us check that this is a tautology.

- If $T(P_1) = T(P_2) = 1$: $\overline{T}(P_1 \to P_2) = 1$, $\overline{T}(\neg P_2 \to \neg P_1) = 1$, so $\overline{T}((\neg P_2 \to \neg P_1) \to (P_1 \to P_2)) = 1$.
- If $T(P_1) = 1$ and $T(P_2) = 0$: $\overline{T}(P_1 \to P_2) = 0$, $\overline{T}(\neg P_2 \to \neg P_1) = 0$, so $\overline{T}((\neg P_2 \to \neg P_1) \to (P_1 \to P_2)) = 1$.
- If $T(P_1) = 0$ and $T(P_2) = 1$: $\overline{T}(P_1 \to P_2) = 1$, $\overline{T}(\neg P_2 \to \neg P_1) = 1$, so $\overline{T}((\neg P_2 \to \neg P_1) \to (P_1 \to P_2)) = 1$.
- If $T(P_1) = T(P_2) = 0$: $\overline{T}(P_1 \to P_2) = 1$, $\overline{T}(\neg P_2 \to \neg P_1) = 1$, so $\overline{T}((\neg P_2 \to \neg P_1) \to (P_1 \to P_2)) = 1$.

(The astute reader will observe that this is an informal version of a *truth table*, which is discussed in Enderton [1] Section 1.2) This very useful, as it shows that the contrapositive of an implication implies the original implication. Notice that the argument does not rely at all on what P_1 and P_2 represent.

2.2 First Order Logic

There are some things propositional logic cannot discuss well. For example, suppose I want to say "If every mathematician is human and I am a mathematician, then I am human." In propositional logic I could let P_1 represent the statement "Every mathematician is human," let P_2 represent the statement "I am a mathematician," and let P_3 represent the statement "I am a human." Then the statement could be represented by $\neg(P_1 \to \neg P_2) \to P_3$. The problem is that this does not accurately represent the logical relationship between P_1, P_2, and P_3. We could define a truth assignment T such that $T(P_1) = T(P_2) = 1$ and $T(P_3) = 0$. Then

$$\overline{T}(P_1 \to \neg P_2) = (1 - T(P_2)) + (1 - (1 - T(P_2))(1 - T(P_1)) = 0 + 1(1 - T(P_1)) = 0$$

$$\overline{T}(\neg(P_1 \to \neg P_2)) = 1 - \overline{T}(P_1 \to \neg P_2) = 1$$

so

$$\overline{T}(\neg(P_1 \to \neg P_2) \to P_3) = T(P_3) + (1 - T(P_3))(1 - \overline{T}(\neg(P_1 \to \neg P_2))) = 0 + 1(0) = 0$$

This means that the statement is false under the truth assignment T.

First Order Logic, on the other hand, allows us to talk about objects in a manner that can avoid problems like these, which stem from our inability to "connect" the statements P_1, P_2 and P_3 sufficiently in propositional logic.

Definition 2.2.1. *We say a set $\mathcal{L}$, which is different from L, the language of propositional logic, is a first order language, if*

$$\mathcal{L} = \{\to, \neg, \forall, (,), =\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{F} \cup \{v_i : i \in \mathbb{N}\}$$

where $\{\to, \neg, \forall, (,), =\}$, $\{v_i : i \in \mathbb{N}\}$, $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{F}$ are all pairwise disjoint and no element of one is a finite string of elements from the others. The elements of $\mathcal{P}$ will be n-place predicate symbols, or relations, for any $n > 1$, the elements of $\mathcal{C}$ will be constant symbols, and the elements of $\mathcal{F}$ will be function symbols. Each of the v_i will be first order variable symbols.

Definition 2.2.2. *The set T of terms is the smallest subset of $\mathcal{L}^{<\mathbb{N}}$ such that $\mathcal{C} \subseteq T$, $v_i \in T$ for all $i \in \mathbb{N}$, and T is closed under the functions F_f for all $f \in \mathcal{F}$, where for a k-place function symbol f, $F_f(t_1, t_2, \ldots, t_k) = \langle f \rangle \frown t_1 \frown t_2 \frown \cdots \frown t_k$. We write this as $ft_1t_2 \ldots t_k$.*

Definition 2.2.3. *An atomic formula of $\mathcal{L}$ is any string of the form $= t_1t_2$ for $t_1, t_2 \in T$ (written as $t_1 = t_2$) or $Pt_1t_2 \ldots t_k$ for $t_1, t_2, \ldots, t_k \in T$ and $P \in \mathcal{P}$ a k-place predicate symbol.*

Definition 2.2.4. *Let $WFF_{\mathcal{L}}$ denote the set of $\mathcal{L}$-WFF's. Formally, this is the smallest subset of $(\mathcal{L})^{<\mathbb{N}}$ containing all atomic formulas and closed under $F_\neg$, $F_\to$, and $F_{\forall_n}$ for all $n \in \mathbb{N}$. Here each $F_\Box$ ($\Box = \neg, \to, \forall_n$) is a function $F_\Box : ((\mathcal{L})^{<\mathbb{N}})^k \to (\mathcal{L})^{<\mathbb{N}}$ for $k \in \mathbb{N}$ such that:*

- *$F_\neg(\psi) = \langle \neg \rangle \frown \psi$, which we write as $\neg\psi$.*
- *$F_\to(\psi, \lambda) = \psi \frown \langle \to \rangle \frown \lambda$, which we write as $\psi \to \lambda$.*
- *$F_{\forall_n}(\psi) = \langle \forall, v_n \rangle \frown \psi$, which we write as $\forall v_n\, \psi$.*

2.3 Satisfiability and Provability

Now that we have a language, we will define the notion of structure, where a structure is a "world" that our language can discuss.

Definition 2.3.1. *An $\mathcal{L}$-structure $\mathfrak{A}$ is a nonempty set $|\mathfrak{A}|$ along with interpretations for the symbols in $\mathcal{F}$, $\mathcal{P}$, and $\mathcal{C}$: specifically, for each constant symbol $c \in \mathcal{C}$ there is a value $c^{\mathfrak{A}} \in |\mathfrak{A}|$, for each k-place predicate symbol $P \in \mathcal{P}$ there is a relation $P^{\mathfrak{A}} \subseteq |\mathfrak{A}|^k$, and for each k-place function symbol $f \in \mathcal{F}$ there is a function $f^{\mathfrak{A}} : |\mathfrak{A}|^k \to |\mathfrak{A}|$.*

Definition 2.3.2. *Let $e\colon \{v_n \mid n \in \mathbb{N}\} \to |\mathfrak{A}|$ be a function which assigns a value to each v_i. This will serve as an interpretation of all variables. We define the extension $\overline{e} : T \to |\mathfrak{A}|$ by:*

- $\overline{e}(v_n) = e(v_n)$
- $\overline{e}(c_\alpha) = c_\alpha^{\mathfrak{A}}$
- $\overline{e}(f_\beta t_1 t_2 \dots t_k) = f_\beta^{\mathfrak{A}}(\overline{e}(t_1), \overline{e}(t_2), \dots \overline{e}(t_k))$

Definition 2.3.3. *We will now discuss what it means for something to be "true" or "false" with regards to our structure. We say $\mathfrak{A} \models_e \sigma$ for $\sigma \in WFF_{\mathcal{L}}$, or $\mathfrak{A}$ satisfies σ with e, as follows:*

- *$\mathfrak{A} \models_e =t_1 t_2$ if and only if $\overline{e}(t_1) = \overline{e}(t_2)$. (If the equality symbol is in $\mathcal{L}$.)*
- *$\mathfrak{A} \models_e Pt_1 t_2 \dots t_k$ for a k-place predicate symbol P if and only if $\langle \overline{e}(t_1), \overline{e}(t_2), \dots, \overline{e}(t_k) \rangle \in P^{\mathfrak{A}}$.*
- *$\mathfrak{A} \models_e \neg\sigma$ if and only if $\mathfrak{A} \not\models_e \sigma$.*
- *$\mathfrak{A} \models_e \sigma \to \psi$ if and only if $\mathfrak{A} \not\models_e \sigma$ or $\mathfrak{A} \models_e \psi$.*
- *$\mathfrak{A} \models_e \forall v_n\ \psi$ if and only if for all $x \in |\mathfrak{A}|$, $\mathfrak{A} \models_{e_{n,x}} \psi$, where $e_{n,x}$ sends v_n to x and agrees with e everywhere else.*

Definition 2.3.4. *For $\Gamma \subseteq WFF_{\mathcal{L}}$, we say $\Gamma \models \sigma$, or Γ logically implies σ, if and only if for all $\mathcal{L}$-structures $\mathfrak{A}$ and variable evaluation functions e, $\mathfrak{A} \models_e \gamma$ for all $\gamma \in \Gamma$ implies that $\mathfrak{A} \models_e \sigma$.*

Before we can discuss formal provability, we need to define free variables, substitution, and substitutability. This is done formally later for our extended language, so here we will point the reader to Enderton [1, Section 2.4] for a rigorous construction. Informally, free variables are variables in a $\mathcal{L}$-WFF which are not quantified over. In the formula $(v_1 = c) \to (\forall v_2\ Pv_2)$, for example, v_1 is free and v_2 is not. Substitution and substitutability then dictate when it is or is not safe to replace things in our formula. To use the example from Enderton [1], if we were to replace v_1 by v_2 in $\neg\forall v_2\ v_1 = v_2$, then we could run into issues because the substitution yields $\neg\forall v_2\ v_2 = v_2$, which is quite unbelievable. Effectively, something is substitutable if and only if it will not be captured by a quantifier.

Lastly we will discuss formal provability in $\mathcal{L}$. First we define the set of (first order) *logical axioms* Δ to contain exactly generalizations of the following:

- All tautologies of $\mathcal{L}$. (A tautology from propositional logic with propositional variables replaced by $\mathcal{L}^*$-WFF's)
- $\forall x\, \psi \to \psi^x_t$, where t is substitutable for x in ψ
- $\forall x\, (\psi \to \gamma) \to (\forall x\, \psi \to \forall x\, \gamma)$
- $\psi \to \forall x\, \psi$, where x does not occur free in ψ
- $x = x$, where x is a variable symbol. (If the equality symbol is in $\mathcal{L}$)
- $x = y \to (\psi \to \psi')$, where x and y are variable symbols and ψ' is obtained by switching some occurrences of x for y in the atomic $\mathcal{L}$-formula ψ (If the equality symbol is in $\mathcal{L}$)

(A *generalization* of the WFF ψ in $\mathcal{L}$ is $\forall x_1 \ldots \forall x_m\, \psi$, where the quantifiers $(\forall x_i)$ can appear in any order.)

Definition 2.3.5. *For $\Gamma \subseteq WFF_{\mathcal{L}}$, we say $\Gamma \vdash \sigma$, or Γ proves σ, if there is a finite sequence $\langle \chi_i \rangle_{i \le n}$ of $\mathcal{L}$-WFF's such that $\chi_n = \sigma$ and for each $i \le n$ one of the following holds:*

- $\chi_i \in \Delta$
- $\chi_i \in \Gamma$
- *There exists $j, k < i$ such that χ_j is of the form $\psi \to \gamma$ and χ_k is equal to ψ. (Modus Ponens)*

An important notion (that we will come back to when we approach completeness) is that of an *alphabetic variant* of a $\mathcal{L}$-WFF ψ. These allow us to overcome some technical minutiae of proving statements that arise from substitution.

Theorem 2.3.1. *Let ψ be a $\mathcal{L}$-WFF, t a term, and v_i a first order variable. Then there is a $\mathcal{L}$-WFF ψ' such that ψ' is the same as ψ except ψ' has different quantified first order variables, $\psi \vdash \psi'$, $\psi' \vdash \psi$, and t is substitutable for v_i in ψ'. Then ψ' is an alphabetic variant of ψ.*

Proof. Enderton [1, Section 2.4] □

The most important results in regards to first order logic are the following.

Theorem 2.3.2. *(Soundness) Let Γ be a set of $\mathcal{L}$-WFF's. If $\Gamma \vdash \psi$, then $\Gamma \models \psi$.*

Proof. Enderton [1, Section 2.5] □

Theorem 2.3.3. *(Gödel's Completeness Theorem) Let Γ be a consistent set of $\mathcal{L}$-WFF's. If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.*

Proof. Enderton [1, Section 2.5] □

Gödel's Completeness Theorem has an alternative formulation: Any consistent set of $\mathcal{L}$-WFF's is satisfiable. The equivalence is used by Enderton [1] to prove completeness. We shall also use it in Chapter 5.

The Soundness Theorem says, informally, that if we can prove ψ, then ψ is "true." Gödel's Completeness Theorem says that if ψ is "true," then we can prove it. While Soundness is significantly easier to prove than Completeness, it seems much more important. Without Soundness, we cannot have confidence in anything we prove, rendering our proof system relatively useless. Without Completeness, on the other hand, there would merely be statements that are true which we cannot prove.If we have Soundness but not Completeness then we can at least have confidence in whatever we are able to prove.

Chapter 3

Foundation

There are certain relatively simple collections of $\mathcal{L}$-WFF's that, despite their simplicity, have infinitely many axioms. (See Chapter 4 for an in-depth example.) As humans, however, we can never use infinitely many different ideas (not yet, anyway), so some of these axioms must be the same "idea," but first order logic does not have the strength to encode this idea into one statement. Take the induction scheme as an example: the induction scheme has infinitely many axioms of the form $(\phi(0) \wedge (\phi(n) \rightarrow \phi(n+1))) \rightarrow \forall n \phi(n)$, one for each well-formed formula, but this statement really only represents one idea. We are already quantifying over all formulas in some sense outside of our language, so can we encode a formal method to perform this quantification inside a formal system? We shall create a new logic that addresses this issue for certain "nice" sets, specifically those with axiom schema applying to all $\mathcal{L}$-WFF's.

In this chapter we will actually create a new logic. We will use Enderton [1] as a guide for inspiration, but this is a new and original construction. We will start by creating the new language that contains countably infinitely variables of a new type ("formula variables" ϕ_n), a new quantifier for the new variables which will quantify over all first order formulas, symbols for "substituting" in our variable symbols, and a countably infinite set of new first order variable symbols which do not occur in $\mathcal{L}$. (The new first order variables are added for technical reasons.) From there we will define the formulas in our new logic, create a notion of substitution and free variables, and define how the analogues of satisfiability and provability function.

3.1 Extended Languages

Definition 3.1.1. *We say a set $\mathcal{L}^*$ is an extended language, or e-language for short, if there is a first order language $\mathcal{L}$ such that*

$$\mathcal{L}^* = \mathcal{L} \cup \{\forall^*\} \cup \{\{\}_s, \boldsymbol{\phi}_n, v_{-n} \mid n \in \mathbb{N}, s \in (\{v_i : i \in \mathbb{Z} \setminus \{0\}\} \cup \mathcal{C})\}$$

where $\mathcal{L}$, $\{\forall^*\}$, and $\{\{\}_s, \boldsymbol{\phi}_n, v_{-n} \mid n \in \mathbb{N}, s \in (\{v_i : i \in \mathbb{Z} \setminus \{0\}\} \cup \mathcal{C})\}$ are all pairwise disjoint. We intend the $\forall^*$ symbol to act as a quantifier over the $\boldsymbol{\phi}_n$ variables, which will represent $\mathcal{L}$-WFF's. (Here $\{\}_s$ is treated as one symbol.)

Definition 3.1.2. *The set $T_{\mathcal{L}^*}$ of e-terms is the set containing all formula variables $\boldsymbol{\phi}_n$ and closed under the functions $F_{\{\}_t}$ for $t \in \{v_i : i \in \mathbb{Z} \setminus \{0\}\} \cup \mathcal{C}$, where $F_{\{\}_t}(\psi, s) = \{\}_t \frown \psi \frown s$, where ψ is an e-term and s is a term. (That is an $\mathcal{L}$-term.) We write this as $\psi\{s\}_t$.*

Definition 3.1.3. *Let $WFF_{\mathcal{L}^*}$ denote the set of e-WFF's. Formally, this is the smallest subset of $(\mathcal{L}^*)^{<\mathbb{N}}$ containing all $\mathcal{L}$-WFF's, ψ for all $\psi \in T_{\mathcal{L}^*}$, and closed under $F_\neg$, $F_\to$,$F_{\forall_n}$ for all $n \in \mathbb{N}$, and $F_{\forall^*_n}$ for all $n \in \mathbb{N}$. Here each $F_\Box$ ($\Box = \neg, \to, \forall_n, \forall^*_n$) is a function $F_\Box : ((\mathcal{L}^*)^{<\mathbb{N}})^k \to (\mathcal{L}^*)^{<\mathbb{N}}$ for $k \in \mathbb{N}$ such that:*

- *$F_\neg(\psi) = \langle\neg\rangle \frown \psi$, which we write as $\neg\psi$.*
- *$F_\to(\psi, \lambda) = \psi \frown \langle\to\rangle \frown \lambda$, which we write as $\psi \to \lambda$.*
- *$F_{\forall_n}(\psi) = \langle\forall, v_n\rangle \frown \psi$, which we write as $\forall v_n\, \psi$.*
- *$F_{\forall^*_n}(\psi) = \langle\forall^*, \boldsymbol{\phi}_n\rangle \frown \psi$, which we write as $\forall^*\boldsymbol{\phi}_n\, \psi$.*

Intuitively, these are just the $\mathcal{L}$-WFF's with the addition of $\boldsymbol{\phi}_n$ as an atomic formula for each n, all of the terms built up from the $\boldsymbol{\phi}_n$'s, and a quantifier over each atomic $\mathcal{L}^*$-term.

Definition 3.1.4. *An $\mathcal{L}^*$-structure $\mathfrak{A}^*$ is an $\mathcal{L}$-structure $\mathfrak{A}$. Here the intended interpretation is that $\forall^*$ is a quantifier over all of the $\mathcal{L}$-formulas.*

Definition 3.1.5. *We define what it means for v_n to occur free in a $\mathcal{L}^*$-WFF ψ recursively:*

- *If ψ is a $\mathcal{L}$-WFF, then v_n occurs free in ψ according to the definition in [1].*
- *v_n occurs free in $\psi = \boldsymbol{\phi}_i$ if and only if $n > 0$.*
- *v_n occurs free in $\psi\{t\}_s$ if and only if $v_n \neq s$ and v_n occurs free in ψ or v_n occurs in t and s occurs free in ψ.*
- *v_n occurs free in $\neg\psi$ if and only if v_n occurs free in ψ.*
- *v_n occurs free in $\sigma \to \psi$ if and only if v_n occurs free in σ or v_n occurs free in ψ.*
- *v_n occurs free in $\forall v_i\, \psi$ if and only if v_n occurs free in ψ and $i \neq n$.*
- *v_n occurs free in $\forall^*\boldsymbol{\phi}_i\psi$ if and only if v_n occurs free in ψ.*

Similarly we define what it means for ϕ_n to occur free in a $\mathcal{L}^*$-WFF ψ recursively:

- *If ψ is a $\mathcal{L}$-WFF, then ϕ_n does not occur free in ψ.*
- *ϕ_n occurs free in $\psi = \phi_i$ if and only if $i = n$.*
- *ϕ_n occurs free in $\psi\{t\}_s$ if and only if ϕ_n occurs free in ψ.*
- *ϕ_n occurs free in $\neg\psi$ if and only if ϕ_n occurs free in ψ.*
- *ϕ_n occurs free in $\sigma \to \psi$ if and only if ϕ_n occurs free in σ or ϕ_n occurs free in ψ.*
- *ϕ_n occurs free in $\forall v_i\ \psi$ if and only if ϕ_n occurs free in ψ.*
- *ϕ_n occurs free in $\forall^* \phi_i\ \psi$ if and only if ϕ_n occurs free in ψ and $i \neq n$.*

3.2 Extended Satisfiability and Extended Provability

Definition 3.2.1. *Let $e : \{v_n \mid n \in \mathbb{Z} \setminus \{0\}\} \to |\mathfrak{A}|$ and $r : \{\phi_n \mid n \in \mathbb{N}\} \to WFF_{\mathcal{L}}$ be functions. These will serve as an interpretation of all variables, where e interprets first order variables and r interprets formula variables. We define the extension $\overline{e} : T_{\mathcal{L}} \to |\mathfrak{A}|$ by:*

- $\overline{e}(v_n) = e(v_n)$
- $\overline{e}(c_\alpha) = c_\alpha^{\mathfrak{A}}$
- $\overline{e}(f_\beta t_1 t_2 \ldots t_k) = f_\beta^{\mathfrak{A}}(\overline{e}(t_1), \overline{e}(t_2), \ldots \overline{e}(t_k))$

The extension $\overline{r} : T_{\mathcal{L}^*} \to F_{\mathcal{L}}$ is defined similarly:

- $\overline{r}(\phi_n) = r(\phi_n)$
- $\overline{r}(\psi\{t\}_s) = \overline{r}(\psi)^s_t$. *(Use an alphabetic variant of $\overline{r}(\psi)$ is t is not substitutable for s in $\overline{r}(\psi)$.)*

Definition 3.2.2. *We say $\mathfrak{A}^* \models_{e,r} \sigma$ for $\sigma \in F_{\mathcal{L}^*}$, or $\mathfrak{A}^*$ satisfies σ with e and r, as follows:*

- *$\mathfrak{A}^* \models_{e,r} \psi$, $\psi \in F_{\mathcal{L}}$, if and only if $\mathfrak{A} \models_e \psi$.*
- *$\mathfrak{A}^* \models_{e,r} \psi$ if and only if $\mathfrak{A} \models_e \overline{r}(\psi)$, where $\psi \in T_{\mathcal{L}^*}$.*
- *$\mathfrak{A}^* \models_{e,r} \neg\sigma$ if and only if $\mathfrak{A}^* \not\models_{e,r} \sigma$.*
- *$\mathfrak{A}^* \models_{e,r} \sigma \to \psi$ if and only if $\mathfrak{A}^* \not\models_{e,r} \sigma$ or $\mathfrak{A}^* \models_{e,r} \psi$.*
- *$\mathfrak{A}^* \models_{e,r} \forall v_n\ \psi$ if and only if for all $x \in |\mathfrak{A}|$, $\mathfrak{A}^* \models_{e_{n,x},r} \psi$, where $e_{n,x}$ sends v_n to x and agrees with e everywhere else.*
- *$\mathfrak{A}^* \models_{e,r} \forall^* \phi_n\ \psi$ if and only if for all $\sigma \in WFF_{\mathcal{L}}$, $\mathfrak{A}^* \models_{e,r_{n,\sigma}} \psi$, where $r_{n,\sigma}$ sends ϕ_n to σ and agrees with r everywhere else.*

Logical implication is then defined similarly to the regular notion.

Definition 3.2.3. *For $\Gamma \subseteq WFF_{\mathcal{L}^*}$, we say $\Gamma \models \sigma$, or Γ logically implies σ, if and only if for all $\mathcal{L}^*$-structures $\mathfrak{A}^*$ and variable, sentence evaluation functions e, r, $\mathfrak{A}^* \models_{e,r} \gamma$ for all $\gamma \in \Gamma$ implies that $\mathfrak{A}^* \models_{e,r} \sigma$. A WFF ψ is valid if $\emptyset \models \psi$.*

Lastly we will discuss formal provability in $\mathcal{L}^*$. We need to define substitution and substitutability in $\mathcal{L}^*$ before we can have a clear picture of how proofs work.

Definition 3.2.4. *We define that t is substitutable for s in ψ, $s \in \{v_i : i \in \mathbb{Z} \setminus \{0\}\} \cup \mathcal{C}$ and what the substitution yields if it exists recursively:*

- *If ψ is a $\mathcal{L}$-WFF then we use the definition from Enderton [1].*
- *If $\psi \in T_{\mathcal{L}^*}$, then t is substitutable for s in ψ, and $\psi^s_t = \psi\{t\}_s$. If t is a first order variable symbol, $\psi = \boldsymbol{\phi}_n\{s\}_t$, and $s = v_{-i}$ for some $i \in \mathbb{N}$, then $\psi^s_t = \psi$.*
- *t is substitutable for s in $\neg\psi$ if and only if t is substitutable for s in ψ, and $(\neg\psi)^s_t = \neg\psi^s_t$.*
- *t is substitutable for s in $\sigma \to \psi$ if and only if t is substitutable for s in σ and ψ, and $(\sigma \to \psi)^s_t = \sigma^s_t \to \psi^s_t$.*
- *t is substitutable for s in $\forall v_i\ \psi$ if and only if s does not occur free in $\forall v_i\ \psi$ or v_i does not appear in t and t is substitutable for s in ψ. Then $(\forall v_i\ \psi)^s_t = \forall v_i\ \psi^s_t$ if $v_i \neq s$ and $\forall v_i\ \psi$ if $v_i = s$.*
- *t is substitutable for s in $\forall \boldsymbol{\phi}_i\ \psi$ if and only if t is substitutable for s in ψ and $(\forall^* \boldsymbol{\phi}_i\ \psi)^s_t = \forall^* \boldsymbol{\phi}_i\ \psi^s_t$.*

We say χ is substitutable for $\boldsymbol{\phi}_n$ in ψ, where χ is in $WFF_{\mathcal{L}}$, similarly:

- *If $\psi = \boldsymbol{\phi}_i$, then χ is substitutable for $\boldsymbol{\phi}_n$ in ψ and $\psi^{\boldsymbol{\phi}_n}_\chi = \chi$ if $i = n$ and ψ otherwise.*
- *If $\psi = \sigma\{t\}_s$ for $\sigma \in T_{\mathcal{L}^*}$, then χ is substitutable for $\boldsymbol{\phi}_n$ in ψ if and only if t is substitutable for s in χ and χ is substitutable for $\boldsymbol{\phi}_n$ in σ Then $\psi^{\boldsymbol{\phi}_n}_\chi = (\sigma^{\boldsymbol{\phi}_n}_\chi)^{v_i}_t$.*
- *χ is substitutable for $\boldsymbol{\phi}_n$ in $\neg\psi$ if and only if χ is substitutable for $\boldsymbol{\phi}_n$ in ψ, and $(\neg\psi)^{\boldsymbol{\phi}_n}_\chi = \neg\psi^{\boldsymbol{\phi}_n}_\chi$.*
- *χ is substitutable for $\boldsymbol{\phi}_n$ in $\sigma \to \psi$ if and only if χ is substitutable for $\boldsymbol{\phi}_n$ in σ and ψ, and $(\sigma \to \psi)^{\boldsymbol{\phi}_n}_\chi = \sigma^{\boldsymbol{\phi}_n}_\chi \to \psi^{\boldsymbol{\phi}_n}_\chi$.*
- *χ is substitutable for $\boldsymbol{\phi}_n$ in $\forall v_i\ \psi$ if and only if χ is substitutable for $\boldsymbol{\phi}_n$ in ψ. Then $(\forall v_i\ \psi)^{\boldsymbol{\phi}_n}_\chi = \forall v_i\ \psi^{\boldsymbol{\phi}_n}_\chi$.*
- *χ is substitutable for $\boldsymbol{\phi}_n$ in $\forall^* \boldsymbol{\phi}_i\ \psi$ if and only if $\boldsymbol{\phi}_n$ does not occur free in $\forall^* \boldsymbol{\phi}_i \psi$ or χ is substitutable for $\boldsymbol{\phi}_n$ in ψ. Then $(\forall^* \boldsymbol{\phi}_i\ \psi)^{\boldsymbol{\phi}_n}_\chi = \forall^* \boldsymbol{\phi}_i\ \psi^{\boldsymbol{\phi}_n}_\chi$.*

Now we define the set of "axioms" Δ to contain exactly generalizations of the following:

- All tautologies of $\mathcal{L}^*$. (A tautology from propositional logic with propositional variables replaced by $\mathcal{L}^*$-WFF's)

- $\forall x\,\psi \rightarrow \psi^x_t$, where t is substitutable for x in ψ
- $\forall^* \boldsymbol{\phi}_n\,\psi \rightarrow \psi^{\boldsymbol{\phi}_n}_\gamma$, where γ is an $\mathcal{L}$-formula and γ is substitutable for $\boldsymbol{\phi}_n$ in ψ.
- $\forall x\,(\psi \rightarrow \gamma) \rightarrow (\forall x\,\psi \rightarrow \forall x\,\gamma)$
- $\forall^* \boldsymbol{\phi}_n\,(\psi \rightarrow \gamma) \rightarrow (\forall^* \boldsymbol{\phi}_n\,\psi \rightarrow \forall^* \boldsymbol{\phi}_n\,\gamma)$
- $\psi \rightarrow \forall x\,\psi$, where x does not occur free in ψ
- $\psi \rightarrow \forall^* \boldsymbol{\phi}_n\,\psi$, where $\boldsymbol{\phi}_n$ does not occur free in ψ
- $x = x$, where x is a variable symbol. (If the equality symbol is in $\mathcal{L}$)
- $x = y \rightarrow (\psi \rightarrow \psi')$, where x and y are variable symbols and ψ' is obtained by switching some occurrences of x for y in the atomic $\mathcal{L}$-formula ψ (If the equality symbol is in $\mathcal{L}$)
- $(\chi \leftrightarrow \psi) \rightarrow (\chi^s_t \leftrightarrow \psi\{t\}_s)$, where $\psi \in T_{\mathcal{L}^*}$ and t is substitutable for s in χ. (Here $(A \leftrightarrow B)$ is shorthand for $(A \rightarrow B) \wedge (B \rightarrow A)$, where $(C \wedge D)$ is itself shorthand for $\neg(A \rightarrow \neg B)$. This is merely saying "If A, then B, and if B, then A," or "A if and only if B.")

(A *generalization* of the WFF ψ in $\mathcal{L}^*$ is $\forall^* \boldsymbol{\phi}_{n1} \ldots \forall^* \boldsymbol{\phi}_{nk} \forall v_1 \ldots \forall v_m\,\psi$, where the quantifiers can appear in any order.)

Definition 3.2.5. *For $\Gamma \subseteq WFF_{\mathcal{L}^*}$, we say $\Gamma \vdash \sigma$, or "Γ proves σ", if there is a finite sequence of $\langle x_i \rangle_{i \leq n}$ of $\mathcal{L}^*$-WFF's such that $x_n = \sigma$ and for each $i \leq n$ one of the following holds:*

- $x_i \in \Delta$
- $x_i \in \Gamma$
- *There exists $j, k < i$ such that x_j is of the form $\psi \rightarrow \gamma$ and x_k is equal to ψ. (Modus Ponens)*

Definition 3.2.6. *A set Γ of $\mathcal{L}^*$-WFF's is consistent if there does not exist a $\mathcal{L}^*$-WFF ψ such that $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$.*

Chapter 4

Example

4.1 Peano Arithmetic

First Order Peano Arithmetic serves as an excellent example of our motivation for adding a way to quantify over formulas. Peano Arithmetic (in its first order formulation) is a collection of axioms in the language of number theory.

Definition 4.1.1. *The language of arithmetic $\mathcal{L}_A$ is the language containing one constant symbol "0", two binary function symbols "$+$" and "$\cdot$", and one unary function symbol "S". (Our intended interpretation is that S is the "Successor" function, i.e. $S(x) = x + 1$.) For readability we write $+t_1t_2$ as $t_1 + t_2$ and $\cdot t_1t_2$ as $t_1 \cdot t_2$.*

Peano Arithmetic is then the following (infinite) set of axioms:

- $\forall v_1 \neg(Sv_1 = 0)$
- $\forall v_1 \forall v_2 (Sv_1 = Sv_2 \to v_1 = v_2)$
- $\forall v_1 (v_1 + 0 = v_1)$
- $\forall v_1 \forall v_2 (S(v_1 + v_2) = v_1 + Sv_2)$
- $\forall v_1 (v_1 \cdot 0 = 0)$
- $\forall v_1 \forall v_2 (v_1 \cdot Sv_2 = (v_1 \cdot v_2) + v_2)$
- For all $\psi \in WFF_{\mathcal{L}_A}$, $\neg(\psi_0^{v_1} \to \neg(\forall v_1 (\psi \to \psi_{Sv_1}^{v_1}))) \to \forall v_1 \psi$. Notice that this is in fact an *axiom scheme*, or infinitely many axioms each corresponding to a well-formed first order formula. Remember that $\neg(A \to \neg B)$ is a formal way of expressing "A and B," so this is

saying that if a statement (ψ) is true of 0 and if it is true of n then it is true of Sn, then it is true of all n.

Notice that while this set is infinite, there are only finitely many "ideas" at work here. Informally, these are:

- 0 is not the successor of anything
- If the successor of two things is the same, then those things are the same.
- Anything plus 0 is the same.
- The successor of a sum is the sum of the successor.
- Anything times 0 is 0.
- The successor preserves distributivity of multiplication.
- Given any statement, the principle of mathematical induction applies to the statement.

We are, in effect, already quantifying over the first order statements outside of our formal language. Our new logic is merely an attempt to code this into a formal system. In our new logic, let $\mathcal{L}_A^*$ be built from $\mathcal{L}_A$. Then replace the induction scheme of Peano Arithmetic with the $\mathcal{L}_A^*$-WFF $\forall^* \boldsymbol{\phi}_1 \, (\neg(\boldsymbol{\phi}_1\{0\}_{v_1} \to \neg(\forall v_1 \, (\boldsymbol{\phi}_1 \to \boldsymbol{\phi}_1\{Sv_1\}_{v_1}))) \to \forall v_1 \, \boldsymbol{\phi}_1)$.

Then we have a new finite set of axioms:

- $\forall v_1 \, \neg(Sv_1 = 0)$
- $\forall v_1 \, \forall v_2 \, (Sv_1 = Sv_2 \to v_1 = v_2)$
- $\forall v_1 \, (v_1 + 0 = v_1)$
- $\forall v_1 \, \forall v_2 \, (S(v_1 + v_2) = v_1 + Sv_2)$
- $\forall v_1 \, (v_1 \cdot 0 = 0)$
- $\forall v_1 \, \forall v_2 \, (v_1 \cdot Sv_2 = (v_1 \cdot v_2) + v_1)$
- $\forall^* \boldsymbol{\phi}_1 \, (\neg(\boldsymbol{\phi}_1\{0\}_{v_1} \to \neg(\forall v_1 \, (\boldsymbol{\phi}_1 \to \boldsymbol{\phi}_1\{Sv_1\}_{v_1}))) \to \forall v_1 \, \boldsymbol{\phi}_1)$

Now suppose we want to prove a first order statement ψ. If ψ is provable in standard first order logic, then let $\langle \chi_1, \ldots, \chi_n \rangle$ be a proof, where $\chi_n = \psi$. Then we can prove ψ in $\mathcal{L}_A^*$. To see this, first note that the logical axioms of first order logic are all logical axioms in our new logic. Thus if χ_i is a logical axiom, then we can still use it in the proof. If χ_i is obtained from Modus Ponens, then we can still use it so long as we can still use both of the components. Also notice that the first six axioms are not a problem because they are in the new set of axioms. Thus, the only thing we need consider is induction.

Suppose χ_i is a part of the induction scheme, where γ is the first order formula involved, e.g.

$$\chi_i = \neg(\gamma_0^{v_1} \to \neg(\forall v_1 \, (\gamma \to \gamma_{Sv_1}^{v_1}))) \to \forall v_1 \, \gamma$$

Then we can see that all γ will be substitutable for $\boldsymbol{\phi}_1$ in

$$\forall^* \boldsymbol{\phi}_1 \; (\neg(\boldsymbol{\phi}_1\{0\}_{v_1} \to \neg(\forall v_1 \; (\boldsymbol{\phi}_1 \to \boldsymbol{\phi}_1\{Sv_1\}_{v_1}))) \to \forall v_1 \; \boldsymbol{\phi}_1)$$

(This is because we will always be able to replace v_1 by 0 and Sv_1. The rest of the computation follows from our definitions.) Then we can always use the logical axiom

$$\forall^* \boldsymbol{\phi}_1 \; (\neg(\boldsymbol{\phi}_1\{0\}_{v_1} \to \neg(\forall v_1 \; (\boldsymbol{\phi}_1 \to \boldsymbol{\phi}_1\{Sv_1\}_{v_1}))) \to \forall v_1 \; \boldsymbol{\phi}_1) \to$$

$$(\neg(\boldsymbol{\phi}_1\{0\}_{v_1} \to \neg(\forall v_1 \; (\boldsymbol{\phi}_1 \to \boldsymbol{\phi}_1\{Sv_1\}_{v_1}))) \to \forall v_1 \; \boldsymbol{\phi}_1)^{\boldsymbol{\phi}_1}_{\gamma}$$

or in other words

$$\forall^* \boldsymbol{\phi}_1 \; (\neg(\boldsymbol{\phi}_1\{0\}_{v_1} \to \neg(\forall v_1 \; (\boldsymbol{\phi}_1 \to \boldsymbol{\phi}_1\{Sv_1\}_{v_1}))) \to \forall v_1 \; \boldsymbol{\phi}_1) \to$$

$$(\neg(\gamma^{v_1}_0 \to \neg(\forall v_1 \; (\gamma \to \gamma^{v_1}_{Sv_1}))) \to \forall v_1 \; \gamma)$$

Chapter 5

Results

5.1 Soundness

The following lemmas are proven because they will help with the proof of soundness.

Lemma 5.1.1. *Let $\mathfrak{A}^*$ be a $\mathcal{L}^*$-structure. Suppose that r is a sentence evaluation function and e_1 and e_2 are variable evaluation functions that agree on all free first order variables present in an $\mathcal{L}^*$-WFF ψ. Then $\mathfrak{A}^* \models_{e_1,r} \psi$ if and only if $\mathfrak{A}^* \models_{e_2,r} \psi$.*

Proof. Fix $\mathfrak{A}^*$. Let S be the set of all $\mathcal{L}^*$-WFF's ψ such that for all e_1 and e_2 that agree on all free first order variables in ψ and for all r, $\mathfrak{A}^* \models_{e_1,r} \psi$ if and only if $\mathfrak{A}^* \models_{e_2,r} \psi$.

If ψ is a $\mathcal{L}$-WFF, then the fact that $\psi \in S$ follows from [1] Theorem 22A. If $\psi \in T_{\mathcal{L}^*}$, then $\mathfrak{A}^* \models_{e_1,r} \psi$ if and only if $\mathfrak{A} \models_{e_1} \overline{r}(\psi)$ and $\mathfrak{A}^* \models_{e_2,r} \psi$ if and only if $\mathfrak{A} \models_{e_2} \overline{r}(\psi)$. $\overline{r}(\psi) \in WFF_{\mathcal{L}}$, so this follows from Enderton [1] Theorem 22A.

Let σ and ψ be elements of S. Then $\neg\sigma \in S$ and $\sigma \to \psi \in S$ follows immediately from the fact that the free variables in $\neg\sigma$ are the same free variables as those in σ and the free variables in $\sigma \to \psi$ are the same free variables as those in σ or ψ.

Let $\psi \in S$. Then consider $\forall v_n \psi$. Then the variables free in $\forall v_n \psi$ are exactly the variables free in ψ without v_n. Therefore we have that for any $a \in |\mathfrak{A}|$ e_1 agrees with e_2 on all free variables in ψ, so $e_{1_{n,a}}$ also agrees with $e_{2_{n,a}}$ on all free variables in ψ. Thus $\mathfrak{A}^* \models_{e_{1_{n,a}},r} \psi$ if and only if $\mathfrak{A}^* \models_{e_{2_{n,a}},r} \models \psi$. It then follows by definition that $\mathfrak{A}^* \models_{e_1,r} \forall v_n \psi$ if and only if $\mathfrak{A}^* \models_{e_2,r} \forall v_n \psi$.

Let $\psi \in S$. Then consider $\forall^* \boldsymbol{\phi}_n\, \psi$. $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\, \psi$ if and only if $\mathfrak{A}^*_{e,r_{n,\chi}} \psi$ for every $\chi \in WFF_{\mathcal{L}}$. Since $\psi \in S$, $\mathfrak{A}^* \models_{e_1,r} \psi$ if and only if $\mathfrak{A}^* \models_{e_2,r} \psi$ for every sentence evaluation function r, in particular for all $r_{n,\chi}$. It then follows that $\mathfrak{A}^* \models_{e_1,r} \forall^* \boldsymbol{\phi}_n\ \psi$ if and only if $\mathfrak{A}^* \models_{e_2,r} \forall^* \boldsymbol{\phi}_n\ \psi$, so $\forall^* \boldsymbol{\phi}_n\ \psi \in S$.

Thus we have shown that S contains $T_{\mathcal{L}^*}$ and $WFF_{\mathcal{L}}$ and is closed under $F_\neg$, $F_\rightarrow$, $F_{\forall_n}$, and $F_{\forall_n^*}$, so it is equal to $WFF_{\mathcal{L}^*}$. $\square$

Lemma 5.1.2. *Let $\mathfrak{A}^*$ be a $\mathcal{L}^*$-structure. Suppose that e is a variable evaluation function and r_1 and r_2 are sentence evaluation functions that agree on all free formula variables present in a $\mathcal{L}^*$-WFF ψ. Then $\mathfrak{A}^* \models_{e,r_1} \psi$ if and only if $\mathfrak{A}^* \models_{e,r_2} \psi$.*

Proof. Fix $\mathfrak{A}^*$. Let S be the set of all $\mathcal{L}^*$-WFF's ψ such that for all r_1 and r_2 that agree on all free sentence variables in ψ and all e, $\mathfrak{A}^* \models_{e,r_1} \psi$ if and only if $\mathfrak{A}^* \models_{e,r_2} \psi$.

If ψ is a $\mathcal{L}$-WFF, then $\mathfrak{A}^* \models_{e,r_1} \psi$ if and only if $\mathfrak{A} \models_e \psi$, and $\mathfrak{A}^* \models_{e,r_2} \psi$ if and only if $\mathfrak{A} \models_e \psi$. Thus it is clear that $\psi \in S$. If $\psi \in T_{\mathcal{L}^*}$, then we have $\overline{r_1}(\psi) = \overline{r_2}(\psi)$, since r_1 and r_2 agree on the only free sentence variable in ψ. $\mathfrak{A}^* \models_{e,r_1} \psi$ if and only if $\mathfrak{A} \models_e \overline{r_1}(\psi)$, which is the same as $\mathfrak{A} \models_e \overline{r_2}(\psi)$ since they are the same. But $\mathfrak{A} \models_e \overline{r_2}(\psi)$ if and only if $\mathfrak{A}^* \models_{e,r_2} \psi$, so $\psi \in S$.

Let σ and ψ be elements of S. Then $\neg\sigma \in S$ and $\sigma \rightarrow \psi \in S$ follows immediately from the fact that the free sentence variables in $\neg\sigma$ are the same free variables as those in σ and the free variables in $\sigma \rightarrow \psi$ are the same free variables as those in σ or ψ.

Let $\psi \in S$. Then consider $\forall v_n \psi$. $\mathfrak{A}^* \models_{e,r} \forall v_n \psi$ if and only if $\mathfrak{A}^*_{e_{n,a},r} \psi$ for every $a \in |\mathfrak{A}|$. Since $\psi \in S$, $\mathfrak{A}^* \models_{e,r_1} \psi$ if and only if $\mathfrak{A}^* \models_{e,r_2} \psi$ for every variable evaluation function e, in particular for all $e_{n,a}$. It then follows that $\mathfrak{A}^* \models_{e,r_1} \forall v_n \psi$ if and only if $\mathfrak{A}^* \models_{e,r_2} \forall v_n \psi$, so $\forall v_n \psi \in S$.

Let $\psi \in S$. Then consider $\forall^* \boldsymbol{\phi}_n\ \psi$. Then the variables free in $\forall^* \boldsymbol{\phi}_n\ \psi$ are exactly the variables free in ψ without $\boldsymbol{\phi}_n$. Therefore we have that for any $\chi \in WFF_{\mathcal{L}}$ r_1 agrees with r_2 on all free variables in ψ, so $r_{1n,\chi}$ also agrees with $r_{2n,\chi}$ on all free variables in ψ. Thus $\mathfrak{A}^* \models_{e,r_{n,\chi}} \psi$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\chi}} \psi$. It then follows by definition that $\mathfrak{A}^* \models_{e,r_1} \forall^* \boldsymbol{\phi}_n\ \psi$ if and only if $\mathfrak{A}^* \models_{e,r_2} \forall^* \boldsymbol{\phi}_n\ \psi$.

Thus we have shown that S contains $T_{\mathcal{L}^*}$ and $WFF_{\mathcal{L}}$ and is closed under $F_\neg$, $F_\rightarrow$, $F_{\forall_n}$, and $F_{\forall_n^*}$, so it is equal to $WFF_{\mathcal{L}^*}$. $\square$

Lemma 5.1.3. *If t is a term that is substitutable for v_n in the $\mathcal{L}^*$-WFF ψ, then $\mathfrak{A}^* \models_{e,r} \psi_t^{v_n}$ if and only if $\mathfrak{A}^* \models_{e_{n,\overline{e}(t)},r} \psi$.*

Proof. Fix $\mathfrak{A}^*$ and e. Let S be the set of all $\mathcal{L}^*$-WFF's ψ such that t is not substitutable for v_n in ψ or $\mathfrak{A}^* \models_{e,r} \psi_t^{v_n}$ if and only if $\mathfrak{A}^* \models_{e_{n,\overline{e}(t)},r} \psi$ for all r.

If ψ is a $\mathcal{L}$-WFF, then this is merely the substitution lemma from Enderton [1], so $\psi \in S$. If $\psi \in T_{\mathcal{L}^*}$, then if t is substitutable for v_n in ψ we have $\psi_t^{v_n} = \psi\{t\}_{v_n}$ by the definition of substitution and $\mathfrak{A}^* \models_{e,r} \psi\{t\}_{v_n}$ if and only if $\mathfrak{A} \models_e \overline{r}(\psi)_t^{v_n}$. As $\overline{r}(\psi)$ is a $\mathcal{L}$-WFF, apply the substitution lemma from Enderton [1] to get that $\mathfrak{A} \models_e \overline{r}(\psi)_t^{v_n}$ if and only if $\mathfrak{A} \models_{e_{n,\overline{e}(t)}} \overline{r}(\psi)$. By the definition of satisfiability, this is if and only if $\mathfrak{A}^* \models_{e_{n,\overline{e}(t)},r} \psi$. Thus $\psi \in S$.

Let $\sigma, \psi \in S$. Then $\neg\sigma \in S$ and $\sigma \to \psi \in S$ are immediate from the definitions of substitution, substitutability and the inductive hypothesis, since $(\neg\sigma)^{v_n}_t = \neg\sigma^{v_n}_t$ and $(\sigma \to \psi)^{v_n}_t = \sigma^{v_n}_t \to \psi^{v_n}_t$.

Let $\psi \in S$. Then consider $\forall v_i\ \psi$. If v_n does not occur free in $\forall v_i\ \psi$, then $(\forall v_i\ \psi)^{v_n}_t = \forall v_i\ \psi$. Then e and $e_{n,\overline{e}(t)}$ agree on all the free variables in $\forall v_i\ \psi$, so $\mathfrak{A}^* \models_{e,r} (\forall v_i\ \psi)^{v_n}_t$ if and only if $\mathfrak{A}^* \models_{e_{n,\overline{e}(t)},r} \forall v_i\ \psi$ by Lemma 5.1.1, so $\forall v_i\ \psi \in S$.
If v_n does occur free in $\forall v_i\ \psi$, then assume t is substitutable for v_n in $\forall v_i\ \psi$. (If not, then $\forall v_i\ \psi \in S$ by the definition of S.) Thus v_i does not occur in t by the definition of substitution. Therefore we have that $\overline{e}(t) = \overline{e_{i,a}}(t)$ for any $a \in |\mathfrak{A}|$ since the extension $\overline{e}$ of a term s relies solely on the value of the variables present in s. Since v_n occurs free in $\forall v_i\ \psi$, we know that $v_i \neq v_n$, so $(\forall v_i\ \psi)^{v_n}_t = \forall v_i\ \psi^{v_n}_t$ by the definition of substitution. Then $\mathfrak{A}^* \models_{e,r} (\forall v_i\ \psi)^{v_n}_t$ if and only if $\mathfrak{A}^* \models_{e,r} \forall v_i\ \psi^{v_n}_t$, which is if and only if $\mathfrak{A}^* \models_{e_{i,a},r} \psi^{v_n}_t$ for all $a \in |\mathfrak{A}|$. Then since $\overline{e}(t) = \overline{e_{i,a}}(t)$, we can apply the induction hypothesis to get that this is if and only if $\mathfrak{A}^* \models_{e_{i,a_{n,\overline{e}(t)}},r} \psi$ for all $a \in |\mathfrak{A}|$, which is exactly $\mathfrak{A}^* \models_{e_{n,\overline{e}(t)},r} \forall v_i\ \psi$, so $\forall v_i\ \psi \in S$.

Let $\psi \in S$. Then consider $(\forall^* \boldsymbol{\phi}_i\ \psi)^{v_n}_t$. Then if t is substitutable for v_n in ψ it is substitutable in $\forall^* \boldsymbol{\phi}_i\ \psi$ and the substitution yields $\forall^* \boldsymbol{\phi}_i\ \psi^{v_n}_t$. By definition $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_i\ \psi^{v_n}_t$ if and only if $\mathfrak{A}^* \models_{e,r_{i,\sigma}} \psi^{v_n}_t$ for all $\mathcal{L}$-WFF's σ. Then by the induction hypothesis this is if and only if $\mathfrak{A}^* \models_{e_{n,\overline{e}(t)},r_{i,\sigma}} \psi$. This is true for all σ, so we have $\mathfrak{A}^* \models_{e_{n,\overline{e}(t)},r} \forall \boldsymbol{\phi}_i\ \psi$. Thus $\forall \boldsymbol{\phi}_i\ \psi \in S$.

Thus we have shown that S contains $T_{\mathcal{L}^*}$ and $WFF_{\mathcal{L}}$ and is closed under $F_\neg$, $F_\to$, $F_{\forall_n}$, and $F_{\forall^*_n}$, so it is equal to $WFF_{\mathcal{L}^*}$ by induction. □

Lemma 5.1.4. *If χ is a $\mathcal{L}$-formula that is substitutable for $\boldsymbol{\phi}_n$ in the $\mathcal{L}^*$-WFF ψ, then $\mathfrak{A}^* \models_{e,r} \psi^{\boldsymbol{\phi}_n}_\chi$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\chi}} \psi$.*

Proof. Fix $\mathfrak{A}^*$ and r. Let S be the set of all $\mathcal{L}^*$-WFF's ψ such that χ is not substitutable for $\boldsymbol{\phi}_n$ in ψ or $\mathfrak{A}^* \models_{e,r} \psi^{\boldsymbol{\phi}_n}_\chi$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\chi}} \psi$ for all e.

If ψ is a $\mathcal{L}$-WFF, then $\psi^{\boldsymbol{\phi}_n}_\chi = \psi$, so it is clear that $\psi \in S$. If $\psi = \boldsymbol{\phi}_n$, then $\mathfrak{A}^* \models_{e,r} \psi^{\boldsymbol{\phi}_n}_\chi$ if and only if $\mathfrak{A} \models_e r(\boldsymbol{\phi}_n)$, which is if and only if $\mathfrak{A}^* \models_{e,r_{n,\chi}} \psi$ by definition. Let ψ be an e-term such that χ is substitutable for $\boldsymbol{\phi}_n$ in ψ and $\mathfrak{A}^* \models_{e,r} \psi^{\boldsymbol{\phi}_n}_\chi$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\chi}} \psi$. Then consider $\psi\{t\}_s$. If $\mathfrak{A}^* \models_{e,r} (\psi\{t\}_s)^{\boldsymbol{\phi}_n}_\chi$ if and only if $\mathfrak{A} \models_e \overline{r}((\psi\{t\}_s)^{\boldsymbol{\phi}_n}_\chi)$, which is if and only if $\mathfrak{A} \models_e \overline{r}(\psi^{\boldsymbol{\phi}_n}_\chi)^s_t$. By the induction hypothesis this is if and only if $\mathfrak{A}^* \models_{e,r_{n,\chi}} \psi\{t\}_s$.

Let $\sigma, \psi \in S$. Then $\neg\sigma \in S$ and $\sigma \to \psi \in S$ are immediate from the definitions of substitution, substitutability and the inductive hypothesis, since $(\neg\sigma)^{\boldsymbol{\phi}_n}_\chi = \neg\sigma^{\boldsymbol{\phi}_n}_\chi$ and $(\sigma \to \psi)^{\boldsymbol{\phi}_n}_\chi = \sigma^{\boldsymbol{\phi}_n}_\chi \to \psi^{\boldsymbol{\phi}_n}_\chi$.

Let $\psi \in S$. Then consider $(\forall v_i\ \psi)^{\boldsymbol{\phi}_n}_\chi$. Then if χ is substitutable for $\boldsymbol{\phi}_n$ then it is substitutable in $\forall v_i\ \psi$ and the substitution yields $\forall v_i\ \psi^{\boldsymbol{\phi}_n}_\chi$. By definition $\mathfrak{A}^* \models_{e,r} \forall v_i\ \psi^{\boldsymbol{\phi}_n}_\chi$ if and only if $\mathfrak{A}^* \models_{e_{i,a},r} \psi^{\boldsymbol{\phi}_n}_\chi$ for all $a \in |\mathfrak{A}|$. Then by the induction hypothesis this is if and only if $\mathfrak{A}^* \models_{e_{i,a},r_{n,\chi}} \psi$. This is true for all a, so we have $\mathfrak{A}^* \models_{e,r_{n,\chi}} \forall v_i\ \psi$. Thus $\forall v_i\ \psi \in S$.

Let $\psi \in S$. Then consider $\forall^* \boldsymbol{\phi}_i\ \psi$. If $\boldsymbol{\phi}_n$ does not occur free in $\forall^* \boldsymbol{\phi}_i\ \psi$, then $(\forall^* \boldsymbol{\phi}_i\ \psi)^{\boldsymbol{\phi}_n}_\chi = \forall^* \boldsymbol{\phi}_i\ \psi$.

Then r and $r_{n,\chi}$ agree on all the free variables in $\forall^* \boldsymbol{\phi}_i\ \psi$, so $\mathfrak{A}^* \models_{e,r} (\forall^* \boldsymbol{\phi}_i\ \psi)^{\boldsymbol{\phi}_n}_{\chi}$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\chi}} \forall^* \boldsymbol{\phi}_i\ \psi$ by Lemma 5.1.2, so $\forall^* \boldsymbol{\phi}_i\ \psi \in S$.
If $\boldsymbol{\phi}_n$ does occur free in $\forall^* \boldsymbol{\phi}_i\ \psi$, then assume χ is substitutable for $\boldsymbol{\phi}_n$ in $\forall^* \boldsymbol{\phi}_i\ \psi$. (If not, then $\forall^* \boldsymbol{\phi}_i\ \psi \in S$ by the definition of S.) Since $\boldsymbol{\phi}_n$ occurs free in $\forall^* \boldsymbol{\phi}_i\ \psi$, we know that $\boldsymbol{\phi}_i \neq \boldsymbol{\phi}_n$, so $(\forall^* \boldsymbol{\phi}_i\ \psi)^{\boldsymbol{\phi}_n}_{\chi} = \forall^* \boldsymbol{\phi}_i\ \psi^{\boldsymbol{\phi}_n}_{\chi}$ by the definition of substitution. Then $\mathfrak{A}^* \models_{e,r} (\forall^* \boldsymbol{\phi}_i\ \psi)^{\boldsymbol{\phi}_n}_{\chi}$ if and only if $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_i\ \psi^{\boldsymbol{\phi}_n}_{\chi}$, which is if and only if $\mathfrak{A}^* \models_{e,r_{i,\gamma}} \psi^{\boldsymbol{\phi}_n}_{\chi}$ for all $\mathcal{L}$-WFF's γ. Then we can apply the induction hypothesis to get that this is if and only if $\mathfrak{A}^* \models_{e,r_{i,\gamma_{n,\chi}}} \psi$ for all $\mathcal{L}$-WFF's γ, which is exactly $\mathfrak{A}^* \models_{e,r_{n,\chi}} \forall^* \boldsymbol{\phi}_i\ \psi$, so $\forall^* \boldsymbol{\phi}_i\ \psi \in S$.

Thus we have shown that S contains $T_{\mathcal{L}^*}$ and $WFF_{\mathcal{L}}$ and is closed under $F_{\neg}$, $F_{\rightarrow}$, $F_{\forall_n}$, and $F_{\forall^*_n}$, so it is equal to $WFF_{\mathcal{L}^*}$ by induction. □

Theorem 5.1.1. *Every $\mathcal{L}^*$-WFF in Δ is valid. (Remember Δ is the set of all logical axioms for provability in $\mathcal{L}^*$.)*

Proof. The proof is split up into multiple parts.

Part 1: We shall show that if $\emptyset \models \psi$, then $\emptyset \models \forall v_n \psi$ and $\emptyset \models \forall^* \boldsymbol{\phi}_n\ \psi$ for all $n \in \mathbb{N}$.

Suppose that $\emptyset \models \psi$. Then for any $\mathcal{L}^*$-structure $\mathfrak{A}^*$ and variable, sentence evaluation functions e, r we have that $\mathfrak{A}^* \models_{e,r} \psi$. In particular, for every e, r, n, and $\mathfrak{A}^*$ and every $x \in |\mathfrak{A}|$ we have $\mathfrak{A}^* \models_{e_{n,x},r} \psi$ because $e_{n,x}$ is also a variable evaluation function. Thus we have by definition $\emptyset \models \forall v_n \psi$. Similarly for every e, r, n, and $\mathfrak{A}^*$ and every $\sigma \in WFF_{\mathcal{L}}$ we have $\mathfrak{A}^* \models_{e,r_{n,\sigma}} \psi$ because $r_{n,\sigma}$ is also a sentence evaluation function. Thus we have by definition $\emptyset \models \forall^* \boldsymbol{\phi}_n\ \psi$.

Now we have that any generalization of a valid formula is valid, so it suffices to show that all of the non-generalizations in Δ are valid and we will have that everything in Δ is valid.

Part 2: All tautologies of $\mathcal{L}^*$ are valid.

Let $\mathfrak{A}^*$ be an $\mathcal{L}^*$-structure, e, r be variable,sentence evaluation functions, and let P be a propositional WFF obtained from a $\mathcal{L}^*$-WFF ψ by replacing $\mathcal{L}^*$-WFF's $\gamma_1, \ldots, \gamma_n$ with sentence variables $x_1, \ldots, x_n$. Then consider the truth assignment v of all sentence variables appearing in P as follows: $v(x_i) = T$ if and only if $\mathfrak{A}^* \models_{e,r} \gamma_i$. Then we claim that $v(P) = T$ if and only if $\mathfrak{A}^* \models_{e,r} \psi$, where P is obtained from ψ as described.

We argue this by induction on P. Let S be the set of sentential formulas A such that $v(A) = T$ if and only if $\mathfrak{A}^* \models_{e,r} \psi$, where A is obtained from ψ as described above. If $P = x_i$, then this follows immediately from our construction of v. Suppose $A \in S$. Then consider $\neg A$. Then $\neg A$ can be obtained from $\neg\psi$ with the same replacements. Then $v(\neg A) = T$ if and only if $\mathfrak{A}^* \models_{e,r} \neg\psi$, which is $\mathfrak{A}^* \not\models_{e,r} \psi$ by definition, so $\neg A \in S$. Now suppose $A, B \in S$. Then consider $A \rightarrow B$. Then this can be obtained from $\psi_A \rightarrow \psi_B$ by the same replacements. $\mathfrak{A}^* \models_{e,r} \psi_A \rightarrow \psi_B$ if and only if $\mathfrak{A}^* \not\models_{e,r} \psi_A$ or $\mathfrak{A}^* \models_{e,r} \psi_B$, which matches that $v(A \rightarrow B) = T$ if and only if $v(A) = F$ or $v(B) = T$, so $A \rightarrow B \in S$. Thus S contains all sentence variables and is closed under $\neg, \rightarrow$, so it contains all sentential formulas.

Now the desired result follows immediately, as tautologies are true for any truth assignment.

Part 3: $\forall v_n \psi \to \psi_t^{v_n}$, where t is substitutable for v_n in ψ is valid.

Let $\mathfrak{A}^*$ be an $\mathcal{L}^*$-structure and e, r variable, sentence evaluation functions. If $\mathfrak{A}^* \not\models_{e,r} \forall v_n \psi$, then $\mathfrak{A}^* \models_{e,r} \forall v_n \psi \to \psi_t^{v_n}$ by definition. Now suppose $\mathfrak{A}^* \models_{e,r} \forall v_n \psi$. Let t be substitutable for v_n in ψ. We know that $\mathfrak{A}^* \models_{e_{n,\overline{e}(t)},r} \psi$ by definition. Thus we have by Lemma 5.1.3 that $\mathfrak{A}^* \models_{e,r} \psi_t^{v_n}$, so $\mathfrak{A}^* \models_{e,r} \forall v_n \psi \to \psi_t^{v_n}$.

Part 4: $\forall^* \boldsymbol{\phi}_n\ \psi \to \psi_\gamma^{\boldsymbol{\phi}_n}$, where γ is an $\mathcal{L}$-formula substitutable for $\boldsymbol{\phi}_n$ in ψ.

Let $\mathfrak{A}^*$ be an $\mathcal{L}^*$-structure and e, r be variable, sentence evaluation functions. If $\mathfrak{A}^* \not\models \forall^* \boldsymbol{\phi}_n\ \psi$, then $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\ \psi \to \psi_\gamma^{\boldsymbol{\phi}_n}$ by definition. Now suppose $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\ \psi$. Let γ be substitutable for $\boldsymbol{\phi}_n$ in ψ. We know that $\mathfrak{A}^* \models_{e,r_{n,\gamma}} \psi$ by definition. Thus we have by Lemma 5.1.4 that $\mathfrak{A}^* \models_{e,r} \psi_\gamma^{\boldsymbol{\phi}_n}$, so $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\ \psi \to \psi_\gamma^{\boldsymbol{\phi}_n}$.

Part 5: $\forall v_n(\psi \to \chi) \to (\forall v_n \psi \to \forall v_n \chi)$ is valid for all $\psi, \chi \in WFF_{\mathcal{L}^*}$.

Let $\mathfrak{A}^*$ be a $\mathcal{L}^*$-structure and e, r be variable, sentence evaluation functions. If $\mathfrak{A}^* \not\models_{e,r} \forall v_n(\psi \to \chi)$, then $\mathfrak{A}^* \models_{e,r} \forall v_n(\psi \to \chi) \to (\forall v_n \psi \to \forall v_n \chi)$ by definition. If $\mathfrak{A}^* \models_{e,r} \forall v_n(\psi \to \chi)$, then consider $\forall v_n \chi$. If $\mathfrak{A}^* \models_{e,r} \forall v_n \chi$, then $\mathfrak{A}^* \models_{e,r} \forall v_n(\psi \to \chi) \to (\forall v_n \psi \to \forall v_n \chi)$. If not, then there is $a \in |\mathfrak{A}|$ such that $\mathfrak{A}^* \not\models_{e_{n,a},r} \chi$. But $\mathfrak{A}^* \models_{e_{n,a},r} \psi \to \chi$ by definition and our assumption, so we must have $\mathfrak{A}^* \not\models_{e_{n,a},r} \psi$. Therefore $\mathfrak{A}^* \not\models_{e,r} \forall v_n \psi$, so $\mathfrak{A}^* \models_{e,r} \forall v_n(\psi \to \chi) \to (\forall v_n \psi \to \forall v_n \chi)$.

Part 6: $\forall^* \boldsymbol{\phi}_n\ (\psi \to \chi) \to (\forall^* \boldsymbol{\phi}_n\ \psi \to \forall^* \boldsymbol{\phi}_n\ \chi)$ is valid for all $\psi, \chi \in WFF_{\mathcal{L}^*}$.

Let $\mathfrak{A}^*$ be a $\mathcal{L}^*$-structure and e, r be variable, sentence evaluation functions. If $\mathfrak{A}^* \not\models_{e,r} \forall^* \boldsymbol{\phi}_n\ (\psi \to \chi)$, then $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\ (\psi \to \chi) \to (\forall^* \boldsymbol{\phi}_n\ \psi \to \forall^* \boldsymbol{\phi}_n\ \chi)$ by definition. If $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\ (\psi \to \chi)$, then consider $\forall^* \boldsymbol{\phi}_n\ \chi$. If $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\ \chi$, then $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\ (\psi \to \chi) \to (\forall^* \boldsymbol{\phi}_n\ \psi \to \forall^* \boldsymbol{\phi}_n\ \chi)$. If not, then there is $\sigma \in WFF_{\mathcal{L}}$ such that $\mathfrak{A}^* \not\models_{e,r_{n,\sigma}} \chi$. But $\mathfrak{A}^* \models_{e,r_{n,\sigma}} \psi \to \chi$ by definition and our assumption, so we must have $\mathfrak{A}^* \not\models_{e,r_{n,\sigma}} \psi$. Therefore $\mathfrak{A}^* \not\models_{e,r} \forall^* \boldsymbol{\phi}_n\ \psi$, so $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\ (\psi \to \chi) \to (\forall^* \boldsymbol{\phi}_n\ \psi \to \forall^* \boldsymbol{\phi}_n\ \chi)$.

Part 7: $\psi \to \forall v_n \psi$ where v_n does not occur free in ψ is valid.

Let $\mathfrak{A}^*$ be a $\mathcal{L}^*$-structure and e, r be variable, sentence evaluation functions. Then if $\mathfrak{A}^* \not\models_{e,r} \psi$, then we are done because $\mathfrak{A}^* \models_{e,r} \psi \to \forall v_n \psi$. Now suppose $\mathfrak{A}^* \models_{e,r} \psi$. $\mathfrak{A}^* \models_{e,r} \forall v_n \psi$ if and only if $\mathfrak{A}^* \models_{e_{n,x},r} \psi$ for all $x \in |\mathfrak{A}|$. But since v_n is not free in ψ, e and $e_{n,x}$ agree on all the first order variables free in ψ, so $\mathfrak{A}^* \models_{e_{n,x},r} \psi$ by Lemma 5.1.1 and our assumption. Thus $\mathfrak{A}^* \models_{e,r} \forall v_n \psi$, and we have shown that $\mathfrak{A}^* \models_{e,r} \psi \to \forall v_n \psi$.

Part 8: $\psi \to \forall^* \boldsymbol{\phi}_n\ \psi$ where $\boldsymbol{\phi}_n$ does not occur free in ψ is valid.

Let $\mathfrak{A}^*$ be a $\mathcal{L}^*$-structure and e, r be variable, sentence evaluation functions. Then if $\mathfrak{A}^* \not\models_{e,r} \psi$, then we are done because $\mathfrak{A}^* \models_{e,r} \psi \to \forall^* \boldsymbol{\phi}_n\, \psi$. Now suppose $\mathfrak{A}^* \models_{e,r} \psi$. $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\, \psi$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\sigma}} \psi$ for all $\sigma \in WFF_{\mathcal{L}}$. But since $\boldsymbol{\phi}_n$ is not free in ψ, r and $r_{n,\sigma}$ agree on all the first order variables free in ψ, so $\mathfrak{A}^* \models_{e,r_{n,\sigma}} \psi$ by Lemma 5.1.2 and our assumption. Thus $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\, \psi$, and we have shown that $\mathfrak{A}^* \models_{e,r} \psi \to \forall^* \boldsymbol{\phi}_n\, \psi$.

Part 9: $x = x$ is valid, where the equality symbol is an element of $\mathcal{L}$.

We can apply Enderton [1] Lemma 25A because none of the machinery exclusive to $\mathcal{L}^*$ is used, and all of the machinery from $\mathcal{L}$ is included in $\mathcal{L}^*$.

Part 10: $x = y \to (\psi \to \psi')$, where ψ' is obtained by switching some occurrences of x for y in the atomic $\mathcal{L}$-formula ψ and the equality symbol is an element of $\mathcal{L}$.

We can apply Enderton [1] Lemma 25A again because this also does not use anything new from $\mathcal{L}^*$.

Part 11: $(\chi \leftrightarrow \psi) \to (\chi^s_t \leftrightarrow \psi\{t\}_s)$, where $\psi \in T_{\mathcal{L}^*}$ and t is substitutable for s in χ, is valid.

Proof: Let $\mathfrak{A}^*$ be a $\mathcal{L}^*$-structure and e, r be variable, sentence evaluation functions. If $\mathfrak{A}^* \not\models_{e,r} \chi \leftrightarrow \psi$, then we are done. Assume $\mathfrak{A}^* \models_{e,r} \chi \leftrightarrow \psi$. Then this means that $\mathfrak{A} \models_e \chi \leftrightarrow \overline{r}(\psi)$, so χ and $\overline{r}(\psi)$ are logically equivalent by Enderton [1] Corollary 25C. Then by Lemma 5.1.3 $\mathfrak{A}^* \models_{e,r} \chi^s_t$ and $\mathfrak{A}^* \models_{e,r} \psi\{t\}_s$ if and only if $\mathfrak{A}^* \models_{e_{s,t},r} \chi$ and $\mathfrak{A}^* \models_{e_{s,t},r} \psi$, respectively. Then since $\mathfrak{A}^* \models_{e_{s,t},r} \psi$ if and only if $\mathfrak{A}^* \models_{e_{s,t},r} \overline{r}(\psi)$ by definition and χ and $\overline{r}(\psi)$ are logically equivalent, so $\mathfrak{A}^* \models_{e,r} \chi^s_t \leftrightarrow \psi\{t\}_s$. □

Theorem 5.1.2. *(Soundness) Let Γ be a set of $\mathcal{L}^*$-WFF's and ψ be a $\mathcal{L}^*$-formula. Then if $\Gamma \vdash \psi$, $\Gamma \models \psi$.*

Proof. Let S be the set of all ψ such that $\Gamma \vdash \psi$ and $\Gamma \models \psi$. We will show that if $\Gamma \vdash \psi$, then $\psi \in S$. There are three cases:

- $\psi \in \Delta$: By the previous theorem if $\psi \in \Delta$ then ψ is valid, e.g. $\models \psi$. Thus every $\mathcal{L}^*$ structure models ψ, so in particular all of the ones that also model Γ do. Thus $\psi \in S$.
- $\psi \in \Gamma$: $\Gamma \models \psi$ by definition, so $\psi \in S$.
- ψ comes from Modus Ponens on $\sigma \to \psi$, where σ and $\sigma \to \psi$ are elements of S: Since $\sigma \in S$, we have $\Gamma \models \sigma$. Similarly, $\Gamma \models \sigma \to \psi$. By definition, $\Gamma \models \sigma \to \psi$ if and only if $\Gamma \not\models \sigma$ or $\Gamma \models \psi$. As $\Gamma \models \sigma$, we have $\Gamma \models \psi$ and $\psi \in S$.

Thus we have shown that S contains Δ, Γ, and is closed under Modus Ponens, so S is the set of all provable statements and we are done. □

5.2 Completeness

Throughout this section we will assume that $\mathcal{L}^*$ is a countable language, as our approach to completeness is adapted from Enderton's [1] approach to Completeness for first order logic. Check

Enderton [1] for a discussion on how to generalize this to uncountable languages. We are approaching completeness from its alternative formulation specified in Chapter 2: Any consistent set of $\mathcal{L}^*$-WFF's is satisfiable. We will not manage to prove Completeness here, but we will prove its equivalence to a technical statement about consistent sets.

Lemma 5.2.1. *Let Γ be a set of $\mathcal{L}^*$-WFF's. Then if $\Gamma \vdash \psi$ and v_n does not occur free in χ for any $\chi \in \Gamma$, it follows that $\Gamma \vdash \forall v_n \psi$.*

Proof. Let S be the set of all $\mathcal{L}^*$-WFF's such that $\Gamma \vdash \forall v_n \psi$.

If $\psi \in \Gamma$, then v_n does not occur free in ψ by assumption, so we can prove $\forall v_n \psi$ with the formal proof $\langle \psi, \psi \to \forall v_n \psi, \forall v_n \psi \rangle$, where the second entry is an axiom because v_n does not occur free in ψ and the third is obtained from Modus Ponens on the first two. Thus $\Gamma \vdash \forall v_n \psi$ and $\psi \in S$.

If $\psi \in \Delta$, then $\forall v_n \psi$ is a generalization of ψ, so it is also in Δ, and thus the one line proof $\langle \forall v_n \psi \rangle$ demonstrates that $\Gamma \vdash \forall v_n \psi$. Thus $\psi \in S$.

Let χ be such that $\chi \in S$ and $\chi \to \psi \in S$. Then if Σ is the proof of $\forall v_n \chi$ and Ψ is the proof of $\forall v_n (\chi \to \psi)$, then $\Sigma \frown \Psi \frown \langle \forall v_n (\chi \to \psi) \to (\forall v_n \chi \to \forall v_n \psi), \forall v_n \chi \to \forall v_n \psi, \forall v_n \psi \rangle$ (where $\frown$ is again concatenation) demonstrates that $\Gamma \vdash \forall v_n \psi$. (Here we use an axiom and two iterations of Modus Ponens as the last three elements of the proof.) Thus $\psi \in S$.

Thus S contains Γ and Δ and is closed under Modus Ponens, so S is the set of all statements provable from Γ by induction, which gives us the desired result. $\square$

Lemma 5.2.2. *Let Γ be a set of $\mathcal{L}^*$-WFF's. If $\Gamma \vdash \psi$ and $c \in \mathcal{C}$ does not occur in ψ, then there exists a variable v_i such that v_i does not occur in ψ and $\Gamma \vdash \forall v_i \ \psi^c_{v_i}$. There also exists a proof of $\forall v_i \ \psi^c_{v_i}$ from Γ which does not contain any formula which contains c.*

Proof. Let $\langle \chi_1, \ldots, \chi_n \rangle$ be a proof of ψ from Γ. Let $i < 0$ be the largest index such that v_i does not occur in χ_j for any $1 \leq j \leq n$. Then we shall show that $\langle (\chi_1)^c_{v_i}, \ldots, (\chi_n)^c_{v_i} \rangle$ is a proof of $\psi^c_{v_i}$ from Γ. We shall argue by induction on χ_n.

For $n = 1$, $\chi_n \in \Gamma$ or $\chi_n \in \Delta$, where Δ is the set of logical axioms. If $\chi_n \in \Gamma$, then c does not occur in χ_n by assumption, so $(\chi_n)^c_{v_i} = \chi_n \in \Gamma$, so χ_n is a valid entry of a proof. If $\chi_n \in \Delta$, then the fact that $(\chi_n)^c_{v_i} \in \Delta$ follows from how we defined first order substitution for formula quantifiers and the fact that v_i does not appear in χ_n by assumption.

Suppose there exists $k < n$ such that for all $1 \leq j \leq k$ $(\chi_k)^c_{v_i}$ is also provable from Γ, e.g. $\langle (\chi_1)^c_{v_i} \ldots (\chi_k)^c_{v_i} \rangle$ is a valid proof of $(\chi_k)^c_{v_i}$. Then consider χ_{k+1}. If $\chi_{k+1} \in \Gamma$ or $\chi_{k+1} \in \Delta$, then the argument used for the base case applies again. If χ_{k+1} is obtained from Modus Ponens on χ_a and χ_b, e.g. $\chi_b = \chi_a \to \chi_{k+1}$, then by the strong induction hypothesis $(\chi_a)^c_{v_i}$ and $(\chi_b)^c_{v_i}$ are in the proof $\langle (\chi_1)^c_{v_i} \ldots (\chi_k)^c_{v_i} \rangle$ of χ_k. But then by definition of substitution $(\chi_b)^c_{v_i} = (\chi_a \to \chi_{k+1})^c_{v_i} = (\chi_a)^c_{v_i} \to (\chi_{k+1})^c_{v_i}$. Then we can apply Modus Ponens to deduce $(\chi_{k+1})^c_{v_i}$.

Therefore we have shown that $\langle (\chi_1)^c_{v_i}, \ldots, (\chi_n)^c_{v_i} \rangle$ is a valid proof by regular induction. Now notice that since $\chi_n = \psi$ by our definition of a proof, this is a proof of $\psi^c_{v_i}$.

Then $\langle(\chi_1)^c_{v_i}, \ldots, (\chi_n)^c_{v_i}\rangle$ is a deduction of $\psi^c_{v_i}$ from a subset of Γ, namely $\Psi = \{\chi_\ell : \chi_\ell \in \Gamma\}$. Then, since by assumption v_i does not appear free in any of the elements of Ψ, $\Psi \vdash \forall v_i\ \psi^c_{v_i}$ by Lemma 5. Thus $\Gamma \vdash \forall v_i\ \psi^c_{v_i}$ because $\Psi \subseteq \Gamma$, and we see by checking the proof of Lemma 5.2.1 that the proof will not contain c. This completes the proof. □

Notice that here it is important that we added v_{-n} for $n \in \mathbb{N}$ to $\mathcal{L}^*$ and not to $\mathcal{L}$: since v_{-n} is not in any $\mathcal{L}$-WFF, it does not occur free in $\boldsymbol{\phi}_i$ for any i, so for any finite collection of $\mathcal{L}^*$-WFF's we will have a first order variable that does not occur (or occur free) in any of them. Without the new variable symbols, this would not necessarily be true. (In fact, all first order variables would be free in the $\mathcal{L}^*$-WFF $\boldsymbol{\phi}_i$.) We also obtain the following result:

Corollary 5.2.1. *If $\Gamma \vdash \psi^{v_i}_c$ and c does not occur in Γ or ψ, then there is a proof of $\forall v_i\ \psi$ from Γ not containing c.*

Proof. By Lemma 5.2.2, $\Gamma \vdash \forall v_j\ (\psi^{v_i}_c)^c_{v_j}$, where j is such that v_j is not in $\psi^{v_i}_c$. Since c is not in ψ, then we have that $(\psi^{v_i}_c)^c_{v_j} = \psi^{v_i}_{v_j}$. Since v_j does not occur in ψ, we see that $(\psi^{v_i}_{v_j})^{v_j}_{v_i} = \psi$. (This is the reason why we defined substitution to cancel out on e-terms.) $\{\forall v_j\ \psi^{v_i}_{v_j}\} \vdash \forall v_i \psi$ follows. □

Lemma 5.2.3. *Let Γ be a consistent set of $\mathcal{L}^*$-WFF's. Then if $\Gamma \cup \{\psi\}$ is inconsistent, then $\Gamma \vdash \neg\psi$.*

Proof. Let χ witness the inconsistency of $\Gamma \cup \{\psi\}$. Then the result follows from the fact that $\neg((\psi \to \chi) \to \neg(\psi \to \neg\chi)) \to \neg\psi$ is a tautology. □

Lemma 5.2.4. *(Alphabetic Variants) Let ψ be a $\mathcal{L}^*$-WFF, t a term, and v_i a first order variable. Then there is a $\mathcal{L}^*$-WFF ψ' such that ψ' is the same as ψ except ψ' has different quantified first order variables, $\psi \vdash \psi'$, $\psi' \vdash \psi$, and t is substitutable for v_i in ψ'.*

Proof. The proof given in Enderton [1] works because we have Enderton's [1] "generalization" (our Lemma 5) and "Exercise 9" (mentioned in the Corollary to Lemma 5.2.2). See Enderton [1, Section 2.4] for more information. □

Lemma 5.2.5. *Let ψ be a $\mathcal{L}^*$-WFF and χ a $\mathcal{L}$-WFF. Let $\mathfrak{A}^*$ be a $\mathcal{L}^*$-structure, e a variable evaluation function, and r a sentence evaluation function such that $r(\boldsymbol{\phi}_n) = \chi$. Then if γ is an alphabetic variant of χ substitutable for $\boldsymbol{\phi}_n$ then $\mathfrak{A}^* \models_{e,r} \psi$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\gamma}} \psi$.*

Proof. From the definition of substitutability we see that the only way that χ is not substitutable for $\boldsymbol{\phi}_n$ is if there is a term ψ containing $\boldsymbol{\phi}_n$ and $\{t\}_s$, where t is not substitutable for s in χ. But there will be at most finitely many such instances, so we can always find an alphabetic variant for which all t are substitutable for the corresponding s by Enderton [1] Theorem 24I. Thus such an alphabetic variant always exists.

Let S be the set of all $\mathcal{L}^*$-WFF's ψ such that for all alphabetic variants γ of $\overline{r}(\boldsymbol{\phi}_n)$ substitutable for $\boldsymbol{\phi}_n$ in ψ we have $\mathfrak{A}^* \models_{e,r} \psi$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\gamma}} \psi$ for all e and r such that $\overline{r}(\boldsymbol{\phi}_n) = \chi$.

If ψ is a first order formula then this is trivial because r is never used.

If $\psi = \boldsymbol{\phi}_i$, then if $i = n$ we have $\mathfrak{A}^* \models_{e,r} \psi$ if and only if $\mathfrak{A} \models_e r(\boldsymbol{\psi}_n)$, so any alphabetic variant will do by Enderton [1] Corollary 25D. If $i \neq n$, then r and $r_{n,\gamma}$ agree on all free variables in ψ for any alphabetic variant γ, so we are done by Lemma 5.1.2.

Suppose $\psi \in T_{\mathcal{L}^*}$ and $\psi \in S$. Then consider $\psi\{t\}_s$. Then let γ be an alphabetic variant of $\overline{r}(\boldsymbol{\phi}_n)$ substitutable for $\boldsymbol{\phi}_n$ in $\psi\{t\}_s$. Then $\mathfrak{A}^* \models_{e,r} \psi\{t\}_s$ if and only if $\mathfrak{A} \models_e \overline{r}(\psi)^s_t$. Then the result follows from the logical equivalence of alphabetic variants since this is purely first order. (Enderton [1] Corollary 25D)

Let $\psi \in S$. Then consider $\neg\psi$. $\mathfrak{A}^* \models_{e,r} \neg\psi$ if and only if $\mathfrak{A}^* \not\models_{e,r} \psi$. By the induction hypothesis, this is if and only if $\mathfrak{A}^* \not\models_{e,r_{n,\gamma}} \psi$ for any alphabetic variant γ. Then we have by definition that this is if and only if $\mathfrak{A}^* \models_{e,r_{n,\gamma}} \neg\psi$. Thus $\neg\psi \in S$.

Let ψ and σ be elements of S. Then consider $\psi \to \sigma$. If γ is substitutable for $\boldsymbol{\phi}_n$ in $\psi \to \sigma$, then the replacement is $\psi^{\boldsymbol{\phi}_n}_\gamma \to \sigma^{\boldsymbol{\phi}_n}_\gamma$. By Lemma 5.1.4 then $\mathfrak{A}^* \models_{e,r} \psi^{\boldsymbol{\phi}_n}_\gamma \to \sigma^{\boldsymbol{\phi}_n}_\gamma$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\gamma}} \psi \to \sigma$. This is if and only if $\mathfrak{A}^* \not\models_{e,r_{n,\gamma}} \psi$ or $\mathfrak{A}^* \models_{e,r_{n,\gamma}} \sigma$. By the induction hypothesis then, this is if and only if $\mathfrak{A}^* \not\models_{e,r} \psi$ or $\mathfrak{A}^* \models_{e,r} \sigma$ and we are done.

Let $\psi \in S$. Consider $\forall v_i\ \psi$. $\mathfrak{A}^* \models_{e,r} \forall v_i\ \psi$ if and only if $\mathfrak{A}^* \models_{e_{i,a},r} \psi$ for all $a \in |\mathfrak{A}|$. Then by the induction hypothesis if γ is an alphabetic variant of χ substitutable for $\boldsymbol{\phi}_n$ in ψ we have $\mathfrak{A}^* \models_{e_{i,a},r} \psi$ for all $a \in |\mathfrak{A}|$ if and only if $\mathfrak{A}^* \models_{e_{i,a},r_{n,\gamma}} \psi$ for all $a \in |\mathfrak{A}|$. But this is if and only if $\mathfrak{A}^* \models_{e,r_{n,\gamma}} \forall v_i\ \psi$, so $\forall v_i\ \psi \in S$.

Let $\psi \in S$. Consider $\forall^* \boldsymbol{\phi}_n\ \psi$. If $i = n$, then any alphabetic variant will do because $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\ \psi$ if and only if $\mathfrak{A}^* \models_{e,r_{i,\sigma}} \psi$ for all σ, which is if and only if $\mathfrak{A}^* \models_{e,r_{i,\gamma}} \forall^* \boldsymbol{\phi}_i\ \psi$. If $i \neq n$, then $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_i\ \psi$ if and only if $\mathfrak{A}^* \models_{e,r_{i,\sigma}} \psi$ for all σ. Then since $r_{i,\sigma}(\boldsymbol{\phi}_n) = \chi$, we can apply the induction hypothesis to get that this is if and only if $\mathfrak{A}^* \models_{e,r_{n,\gamma_{i,\sigma}}} \psi$ for all σ. This then if and only if $\mathfrak{A}^* \models_{e,r_{n,\gamma}} \forall^* \boldsymbol{\phi}_i\ \psi$ by definition. Thus $\forall^* \boldsymbol{\phi}_i\ \psi \in S$.

Thus S is the set of all $\mathcal{L}^*$-WFF's by induction, so we are done. □

These lemmas are quite powerful tools, but we still are unable to prove completeness in this thesis. We can use these lemmas, however, to reformulate completeness into a more concrete conjecture that is equivalent.

Theorem 5.2.1. *(Counterexample Conjecture) For any consistent set* Γ *of* $\mathcal{L}^*$*-WFF's* Γ*,* $\mathcal{L}^*$*-WFF* ψ*, and* $n \in \mathbb{N}$*, either there exists a* $\mathcal{L}$*-WFF* χ *such that* $\Gamma \cup \{\neg\forall^* \boldsymbol{\phi}_n\ \psi \to \neg\psi^{\boldsymbol{\phi}_n}_\chi\}$ *is consistent or* $\Gamma \cup \{\forall^* \boldsymbol{\phi}_n\ \psi\}$ *is consistent.*

We shall now show that the Counterexample Conjecture is equivalent to Completeness of our new logic.

5.2.1 Equivalence of Completeness and the Counterexample Conjecture

Theorem 5.2.2. *If the Counterexample Conjecture holds, then any consistent set is satisfiable. That is to say that the Counterexample Conjecture implies Completeness.*

The proof shall proceed as a series of proofs of smaller theorems. We will label any theorem that explicitly invokes the Counterexample Conjecture as an assumption.

First we extend $\mathcal{L}^*$ to $\mathcal{L}^{**} = \mathcal{L}^* \cup \{k_i : i \in \mathbb{N}\}$, where each k_i is a new constant symbol that does not occur in $\mathcal{C}$.

Theorem 5.2.3. *Γ is consistent as a set of $\mathcal{L}^{**}$-WFF's.*

Proof. Suppose not. Then there exists a $\mathcal{L}^{**}$-WFF ψ such that $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$. Then at most finitely many of the k_i's can occur in ψ, and none of them appear in Γ (since it is a set of $\mathcal{L}^*$-WFF's), so by Lemma 5.2.2, $\Gamma \vdash \psi'$ and $\Gamma \vdash (\neg\psi)'$, where ψ' and $(\neg\psi)'$ are $\mathcal{L}^*$-WFF's obtained from ψ and $\neg\psi$ respectively by replacing all of the k_i's that appear with variable symbols. But $(\neg\psi)' = \neg\psi'$, so $\Gamma \vdash \psi'$ and $\Gamma \vdash \neg\psi'$. Furthermore, also by Lemma 5.2.2, there is a proof of each statement that does not contain any of the k_i's. This contradicts the fact that Γ is consistent as a set of $\mathcal{L}^*$-WFF's. Thus Γ is consistent as a set of $\mathcal{L}^{**}$-WFF's. □

Now we will extend Γ such that for each $\mathcal{L}^*$-WFF ψ, the formula $\neg\forall v_i\ \psi \to \neg\psi^{v_i}_{k_j}$ for all $i \in \mathbb{Z} \setminus \{0\}$ and some $j \in \mathbb{N}$. Informally, this is saying that if $\forall v_i\ \psi$ is not true, then k_j serves as an explicit counterexample. We do this in such a way that the resulting set remains consistent.

Theorem 5.2.4. *Γ can be extended to a consistent set Γ' such that $\Gamma \subseteq \Gamma'$ and for any $\mathcal{L}^{**}$-WFF ψ and all first order variables v_i the formula $\neg\forall v_i\ \psi \to \neg\psi^{v_i}_{k_j} \in \Gamma'$ for some $j \in \mathbb{N}$.*

Proof. Let $\langle \psi_i, x_i \rangle$ be an enumeration of all possible pairs of $\mathcal{L}^{**}$-WFF's and variable symbols v_j. This is indexed by $i \in \mathbb{N}$. This is guaranteed to exist because our language is countable. Now define $\chi_i = \neg\forall x_i\ \psi_i \to \neg\psi^{x_i}_{k_{j_i}}$, where j_i is the least index such that k_{j_i} does not appear in ψ_i and k_{j_i} does not occur in χ_ℓ for any $\ell < i$.

Let $\Gamma' = \Gamma \cup \{\chi_i : i \in \mathbb{N}\}$. We wish to show that Γ' is consistent. Suppose not. Then there exists a least index n such that $\Gamma \cup \{\chi_i : 1 \leq i \leq n\}$ is inconsistent. (Since proofs are finite, if Γ' is inconsistent, then there are proofs of ψ and $\neg\psi$ using only finitely many χ_i's.) Then by assumption $\Gamma \cup \{\chi_i : 1 \leq i < n\}$ is consistent, so $\Gamma \cup \{\chi_i : 1 \leq i < n\} \vdash \neg\chi_n$ by Lemma 5.2.3. Notice that $\neg\chi_n = \neg(\neg\forall x_n\ \psi_n \to \neg{\psi_n}^{x_n}_{k_{j_n}})$ and $\neg(A \to B) \to A$ and $\neg(A \to B) \to \neg B$ are tautologies. Thus we can see that $\Gamma \cup \{\chi_i : 1 \leq i < n\} \vdash \neg\forall x_n\ \psi_n$ and that $\Gamma \cup \{\chi_i : 1 \leq i < n\} \vdash {\psi_n}^{x_n}_{k_{j_n}}$ by taking the proof of $\neg\chi_n$, appending one of the above tautologies with $A = \neg\forall x_n\ \psi_n$ and $B = \neg{\psi_n}^{x_n}_{k_{j_n}}$, and finally applying Modus Ponens. But k_{j_n} does not appear in $\Gamma \cup \{\chi_i : 1 \leq i < n\}$ by construction, so we can apply the corollary to Lemma 5.2.2 on ${\psi_n}^{x_n}_{k_{j_n}}$ to obtain $\Gamma \cup \{\chi_i : 1 \leq i < n\} \vdash \forall x_n\ \psi_n$, which contradicts the consistency of $\Gamma \cup \{\chi_i : 1 \leq i < n\}$. Thus Γ' is consistent. □

Now we will use the Counterexample Conjecture (for the only time) to extend Γ' to a new set Γ''.

Theorem 5.2.5. *(Assuming Counterexample Conjecture) Γ' can be extended to a consistent set Γ'' such that for any $\mathcal{L}^{**}$-WFF ψ and any n either there exists a $\mathcal{L} \cup \{k_i : i \in \mathbb{N}\}$-WFF χ such that $\neg\forall^* \boldsymbol{\phi}_n\ \psi \to \neg\psi^{\boldsymbol{\phi}_n}_{\chi} \in \Gamma''$ or $\forall^* \boldsymbol{\phi}_n \psi \in \Gamma''$.*

Proof. Let $\langle \psi_i, x_i \rangle$ be an enumeration of all possible pairs of $\mathcal{L}^{**}$-WFF's and variable symbols $\boldsymbol{\phi}_j$. Now define Γ'_i such that $\Gamma'_0 = \Gamma'$ and $\Gamma'_i = \Gamma'_{i-1} \cup \{\sigma_i\}$, where σ_i is the formula given by the Counterexample Conjecture applied to the consistent set Γ'_{i-1}: either $\neg\forall^*\boldsymbol{\phi}_n\ \psi \rightarrow \neg\psi^{\boldsymbol{\phi}_n}_{\chi}$ if a χ such that this is consistent exists or $\forall^*\boldsymbol{\phi}_n\ \psi$ otherwise. Then $\Gamma'' = \bigcup_{\imath\in\mathbb{N}} \Gamma'_i$ is obviously consistent. (If not, then there is a finite proof of a contradiction. Thus there exists a minimal m such that Γ'_m is inconsistent, but by out assumption one of the two possible additions makes the set consistent, which is a contradiction.) □

Definition 5.2.1. *A set of WFF's Ψ is said to maximal if for any WFF ψ in the given language either $\neg\psi \in \Psi$ or $\psi \in \Psi$, but not both.*

Theorem 5.2.6. *Any consistent set, including Γ'', can be extended to a consistent maximal set Λ such that $\Gamma' \subseteq \Lambda$.*

Proof. Let $\{\psi_n\}$ be an enumeration of all the $\mathcal{L}^{**}$-WFF's. (Again, this is possible because the language is countable. Then let $\Lambda_0 = \Gamma'$ and $\Lambda_i = \Lambda_{i-1} \cup \{\psi_i\}$ if $\Lambda_{i-1} \cup \{\psi_i\}$ is consistent and $\Lambda \cup \{\neg\psi_i\}$ otherwise. Then let $\Lambda = \bigcup_{i\in\mathbb{N}} \Lambda_i$. Then clearly Λ is maximal and $\Gamma' \subseteq \Lambda$. It is consistent, because if not, then there is a least n such that Λ_n is inconsistent. But then this implies that $\Lambda_{n-1} \cup \{\neg\psi_n\} = \Lambda_n$, since if it were $\Lambda_{n-1} \cup \{\psi_n\}$ it would be consistent by construction. But then $\Gamma_{n-1} \vdash \psi_n$ by Lemma 5.2.3, so the inconsistency of $\Lambda_{n-1} \cup \{\psi_n\}$ implies that Λ_{n-1} is inconsistent, which contradicts the minimality of n. Thus Λ is consistent. □

Theorem 5.2.7. *(Assuming the Counterexample Conjecture) Λ is satisfiable.*

Proof. Consider the set of all purely first order formulas in Λ. (In the language $\mathcal{L} \cup \{k_i : i \in \mathbb{N}\}$) Then by Godel's Completeness Theorem, since this is a maximal, consistent $\mathcal{L} \cup \{k_i : i \in \mathbb{N}\}$ set, there exists a structure $\mathfrak{A}$ and variable evaluation function e such that $\mathfrak{A} \models_e \psi$ if and only if $\psi \in \Lambda$.

Let $\{\chi_i\}$ be an enumeration of the $\mathcal{L} \cup \{k_i : i \in \mathbb{N}\}$-WFF's that are in Λ. Then let r be the sentence evaluation function such that $r(\boldsymbol{\phi}_n) = \chi_n$ if $\boldsymbol{\phi}_n \in \Lambda$ and $r(\boldsymbol{\phi}_n) = \neg\chi_n$ otherwise. Then we claim that for any $\mathcal{L}^{**}$-WFF ψ $\mathfrak{A}^* \models_{e,r} \psi$ if and only if $\psi \in \Lambda$. We shall argue by induction e-terms, and then by induction on the number of applications of $F_\neg, F_\rightarrow, F_{\forall v_i}$, and $F_{\forall^*\boldsymbol{\phi}_i}$. (That is, applications outside of first order logic. As an example, $(Px \rightarrow Py) \rightarrow \boldsymbol{\phi}_1$ would have one application because $Px \rightarrow Py$ is purely first order.)

If ψ is a $\mathcal{L} \cup \{k_i : i \in \mathbb{N}\}$-WFF, then $\mathfrak{A}^* \models_{e,r} \psi$ if and only if $\psi \in \Lambda$ by assumption.

Let $\psi = \boldsymbol{\phi}_n$. Then $\mathfrak{A}^* \models_{e,r} \boldsymbol{\phi}_n$ if and only if $\mathfrak{A} \models_e r(\boldsymbol{\phi}_n)$. $\mathfrak{A} \models_e r(\boldsymbol{\phi}_n)$ if and only if $\boldsymbol{\phi}_n \in \Lambda$ by the construction of r, so we are done with this case.

Let $\psi \in T_{\mathcal{L}^{**}}$ such that $\psi \in \Lambda$ if and only if $\mathfrak{A}^* \models_{e,r} \psi$. Now consider $\psi\{t\}_s$. By definition, $\mathfrak{A}^* \models_{e,r} \overline{r}(\psi) \leftrightarrow \psi$. Since this implies that $\overline{r}(\psi) \in \Lambda$ and $\psi \in \Lambda$ by the induction hypothesis, this means that $\overline{r}(\psi) \leftrightarrow \psi \in \Lambda$ because $A \rightarrow B \rightarrow (A \leftrightarrow B)$ is a tautology and Λ is maximal, therefore deductively closed. Thus $\overline{r}(\psi)^s_t \leftrightarrow \psi\{t\}_s \in \Lambda$. Then by the induction hypothesis applied to the $\mathcal{L} \cup \{k_i : i \in \mathbb{N}\}$-WFF $\overline{r}(\psi)$ we have $\mathfrak{A}^* \models_{e,r} \psi\{t\}_s$ if and only if $\psi\{t\}_s \in \Lambda$ since we can use either $\overline{r}(\psi)^s_t \rightarrow \psi\{t\}_s$ or $\neg\overline{r}(\psi)^s_t \rightarrow \neg\psi\{t\}_s$ and the fact that Λ is deductively closed to obtain the desired result.

Let ψ be such that $\mathfrak{A}^* \models_{e,r} \psi$ if and only if $\psi \in \Lambda$. Then $\mathfrak{A}^* \models_{e,r} \neg\psi$ if and only if $\mathfrak{A}^* \not\models_{e,r} \psi$, which is if and only if $\psi \notin \Lambda$. Since Λ is maximal, this is if and only if $\neg\psi \in \Lambda$, so we are done with this case.

Let ψ, χ be such that $\mathfrak{A}^* \models_{e,r} \psi$,$\mathfrak{A}^* \models_{e,r} \chi$ if and only if $\psi \in \Lambda$,$\chi \in \Lambda$. Then $\mathfrak{A}^* \not\models_{e,r} \psi \to \chi$ if and only if $\mathfrak{A}^* \models_{e,r} \psi$ and $\mathfrak{A}^* \not\models_{e,r} \chi$ by definition. By the induction hypothesis, this is if and only if $\psi \in \Lambda$ and $\chi \notin \Lambda$. Then this is if and only if $\psi \to \chi \notin \Lambda$, since Λ is maximal and consistent, and so must be deductively closed. This concludes this case.

Let ψ be such that for any formula χ with the same number or fewer applications of the given functions, $\chi \in \Lambda$ if and only if $\mathfrak{A}^* \models_{e,r} \chi$. (This is stronger than what we have assumed in the previous two cases, but it is necessary.) Then consider $\forall v_i\, \psi$. Then by the properties of Λ there exists a statement $\neg\forall v_i\, \psi \to \neg\psi^{v_i}_{k_j}$ for some j. Since we have "substitution" (Lemma 5.1.3) and alphabetic variants (Lemma 5.2.4), then the same argument given in Enderton [1] of Gödel's Completeness Theorem goes through by our properties of Λ and our definition of satisfaction and $\mathfrak{A}^* \models_{e,r} \forall v_i\, \psi$ if and only if $\forall v_i\, \psi \in \Lambda$. (See Enderton [1, Section 2.5])

Let ψ be such that for any formula χ with the same number or fewer applications of the given functions, $\chi \in \Lambda$ if and only if $\mathfrak{A}^* \models_{e,r} \chi$. Then consider $\forall^* \boldsymbol{\phi}_i\, \psi$. Suppose that $\forall^* \boldsymbol{\phi}_i\, \psi \notin \Lambda$. Then by construction of Λ there exists a $\mathcal{L} \cup \{k_i : i \in \mathbb{N}\}$-WFF χ substitutable for $\boldsymbol{\phi}_n$ in ψ such that $\neg\forall^* \boldsymbol{\phi}_n\, \psi \to \neg\psi^{\boldsymbol{\phi}_n}_{\chi} \in \Lambda$. Because Λ is maximal then $\neg\forall^* \boldsymbol{\phi}_n\, \psi \in \Lambda$ and Λ is deductively closed, so $\neg\psi^{\boldsymbol{\phi}_n}_{\chi} \in \Lambda$. Thus $\psi^{\boldsymbol{\phi}_n}_{\chi} \notin \Lambda$, so by the induction hypothesis $\mathfrak{A}^* \not\models_{e,r} \psi^{\boldsymbol{\phi}_n}_{\chi}$. Thus by Lemma 5.1.4 this is if and only if $\mathfrak{A}^* \not\models_{e,r_{n,\chi}} \psi$, so $\mathfrak{A}^* \not\models_{e,r} \forall^* \boldsymbol{\phi}_n\, \psi$ by definition.

Now suppose that $\mathfrak{A}^* \not\models_{e,r} \forall^* \boldsymbol{\phi}_i\, \psi$. Then there is some $\mathcal{L}$-WFF χ such that $\mathfrak{A}^* \not\models_{e,r_{i,\chi}} \psi$. Then by Lemma 5.2.5 there exists an alphabetic variant γ of χ such that γ is substitutable for $\boldsymbol{\phi}_i$ in ψ and $\mathfrak{A}^* \models_{e,r_{i,\chi}} \psi$ if and only if $\mathfrak{A}^* \models_{e,r_{i,\gamma}} \psi$. Then by Lemma 5.1.4 $\mathfrak{A}^* \not\models_{e,r_{i,\gamma}} \psi$ if and only if $\mathfrak{A}^* \not\models_{e,r} \psi^{\boldsymbol{\phi}_i}_{\gamma}$. Then by the induction hypothesis, $\psi^{\boldsymbol{\phi}_i}_{\gamma} \notin \Lambda$. Thus $\forall^* \boldsymbol{\phi}_i\, \psi \notin \Lambda$ because $\forall^* \boldsymbol{\phi}_i\, \psi \to \psi^{\boldsymbol{\phi}_i}_{\gamma}$ is a logical axiom and Λ is maximal, and thus deductively closed. This concludes this case.

Thus we have that $\mathfrak{A}^*$ satisfies every sentence of Λ by induction, so restrict $\mathfrak{A}^*$ to our original language $\mathcal{L}^*$ to finally deduce that Γ is satisfiable, and we have Completeness. (Again, given our assumption. $\square$

Thus we have shown that the Counterexample Conjecture implies Completeness by restricting our structure to the original language $\mathcal{L}^*$. Now we will show the other direction to obtain equivalence.

Theorem 5.2.8. *Completeness implies the Counterexample Conjecture.*

Proof. Let Γ be a consistent set of $\mathcal{L}^*$-WFF's. Then by completeness it is satisfiable. Let the collection $\mathfrak{A}^*$, e, and r be the witness of satisfiability. Then consider $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\, \psi$ for any n and $\mathcal{L}^*$-WFF ψ. If $\mathfrak{A}^* \models_{e,r} \forall^* \boldsymbol{\phi}_n\, \psi$, then $\Gamma \cup \{\forall^* \boldsymbol{\phi}_n\, \psi\}$ is consistent because it is satisfiable. If not, then $\mathfrak{A}^* \models_{e,r} \neg\forall^* \boldsymbol{\phi}_n\, \psi$ by definition and, by definition, there exists some $\mathcal{L}$-WFF χ such that $\mathfrak{A}^* \models_{e,r_{n,\chi}} \neg\psi$. By Lemma 5.2.5 there exists an alphabetic variant γ of χ such that γ is substitutable for $\boldsymbol{\phi}_n$ in ψ and $\mathfrak{A}^* \models_{e,r_{n,\chi}} \neg\psi$ if and only if $\mathfrak{A}^* \models_{e,r_{n,\gamma}} \neg\psi$. By Lemma 5.1.4 this is if and only if $\mathfrak{A}^* \models_{e,r} \neg\psi_{\chi}^{\boldsymbol{\phi}_n}$. Thus $\mathfrak{A}^* \models_{e,r} \neg\forall^*\boldsymbol{\phi}_n\ \psi \rightarrow \neg\psi_{\chi}^{\boldsymbol{\phi}_n}$, so $\Gamma \cup \{\neg\forall^*\boldsymbol{\phi}_n\ \psi \rightarrow \neg\psi_{\chi}^{\boldsymbol{\phi}_n}\}$ is satisfiable, and therefore consistent. □

Thus we have shown that Completeness of our new logic is equivalent to the Counterexample Conjecture, which is merely a technical statement.

5.3 Corollaries of Completeness

If Completeness holds, then we obtain the rest of our results as corollaries.

Theorem 5.3.1. *(Assuming Completeness) Let* Γ *be a set of* $\mathcal{L}^*$*-WFF's. If every finite subset of* Γ *is satisfiable, then* Γ *is satisfiable.*

Proof. Suppose every finite subset of Γ is satisfiable. Γ is consistent: If not, then there exists a proof of a contradiction. But this is a proof, and thus finite, so the set of all elements of Γ in the proof is satisfiable, and thus consistent by Soundness. Therefore it cannot prove a contradiction, so Γ cannot prove a contradiction. Then since every consistent set is satisfiable (Completeness), Γ is satisfiable. □

Lastly, we shall address the issue of how much we can prove with our new logic in relation to the old first order logic. Like the Compactness Theorem, having Completeness will show us that we cannot prove any first order statement in our new logic that we could not already prove in first order logic. We shall show informally how the argument proceeds.

(Assuming Completeness) Let Γ be a consistent set of $\mathcal{L}$-WFF's, where all but finitely many axioms are part of an axiom scheme that applies to all $\mathcal{L}$-WFF's. Let ψ be a sequence substitutions and $\neg, \rightarrow, \forall v_i$ combination operations (e.g. $F_{\neg}$, $F_{\rightarrow}$, and $F_{\forall v_i}$) that represents the axiom scheme. (For example, "$\psi = \neg(\gamma_0^{v_1} \rightarrow \neg(\forall v_1\ (\gamma \rightarrow \gamma_{Sv_1}^{v_1}))) \rightarrow \forall v_1\ \gamma$" for all $\mathcal{L}$-WFF's γ *represents* the induction scheme.) Then let Γ' be the set of $\mathcal{L}^*$-WFF's such that the axioms of Γ which are not a part of the scheme are in Γ' and $\forall^*\boldsymbol{\phi}_1\ \psi'$ is in Γ', where ψ' is the formal equivalent of ψ in $\mathcal{L}^*$. (For the induction scheme, $\psi' = \neg(\boldsymbol{\phi}_1\{0\}_{v_1} \rightarrow \neg(\forall v_1\ (\boldsymbol{\phi}_1 \rightarrow \boldsymbol{\phi}_1\{Sv_1\}_{v_1}))) \rightarrow \forall v_1\ \boldsymbol{\phi}_1$.) Then since Γ is consistent, Γ is consistent, then it is satisfiable (by Gödel's Completeness Theorem) with witness $\mathfrak{A}$ and e. Then let χ_i be an enumeration of the $\mathcal{L}$-WFF's and let $r(\boldsymbol{\phi}_i) = \chi_i$. Then it is clear that $\mathfrak{A}^*$ is also satisfies Γ'. (The only problematic part could be the axiom scheme, but $\mathfrak{A}^* \models_{e,r} \forall^*\boldsymbol{\phi}_1\ \psi$ if and only if $\mathfrak{A}^* \models_{e,r_{1,\sigma}} \psi$, which is exactly saying that $\mathfrak{A}$ satisfies every element of the scheme.) Then if $\Gamma \vdash \gamma$, then $\Gamma' \vdash \gamma$ because provability in the new logic contains all first order provability. If $\Gamma' \vdash \gamma$ for some first order γ, then $\mathfrak{A}^* \models_{e,r} \gamma$. But by definition this means that $\mathfrak{A} \models_e \gamma$, and by Gödel's Completeness Theorem again we have that $\Gamma \vdash \gamma$ as desired.

Chapter 6

Conclusions

In this thesis we built a new logic to encode first order axiom schema into single formulas in the new language. We began by reviewing propositional and first order logic, demonstrating one of the weaknesses of propositional logic that first order is meant to address.

From there we created the new logic by adding variables that are meant to represent first order formulas, notation for substitution on our new "sentence" variables, and a way to quantify over all first order formulas. We then defined analogues of satisfiability and provability in our new logic. Peano Arithmetic was used as an illustration for our thought process and our intended usage of the new logic.

In the **Results** chapter we proved various lemmas about the functionality of our new logic, verifying properties similar to those found in first order logic. We then used some of these to prove Soundness, which means that anything we prove inside the new logic is "true," or satisfied, in that logic. We then considered Completeness and reduced it to an equivalent statement. It was then shown that if Completeness holds, we have Compactness and some measure of conservation of first order provability in our new logic.

The most important question left unanswered is whether or not this new logic is complete. It may be that it is complete and we have merely failed to prove it, it may be too expressive to be Complete, like second order logic, or we may need to add a few logical axioms. Further work could also focus on ways to restrict which formulas can be quantified over, such as only allowing quantification over Σ^1_0 formulas or over recursive sets of formulas.

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