THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF MATHEMATICS

SPHERE EVERSION: ANALYSIS OF A VERIDICAL PARADOX

MATTHEW LITMAN
SPRING 2017

A thesis
submitted in partial fulfillment
of the requirements
for a baccalaureate degree
in Mathematics
with honors in Mathematics

Reviewed and approved* by the following:

Sergei Tabachnikov
Professor of Mathematics
Thesis Supervisor

Nathanial Brown
Professor of Mathematics
Associate Head for Equity & Diversity
Honors Adviser

*Signatures are on file in the Schreyer Honors College.

Abstract

In this honors thesis, we investigate the topological problem of everting the 2-sphere in 3-space, i.e. turning a sphere inside out via continuous change allowing self-intersection but not allowing tearing, creasing, or pinching. The result was shown to exist by an abstract theorem proven in the 1950s, but the first explicit construction was not published until almost a decade later. Throughout the past 60 years, many constructions have been made, each providing their own insight into the theory behind the problem. In the following pages, we study the history surrounding the problem, the theory that made it possible, and a myriad of explicit examples giving a solid foundation in descriptive topology.

Table of Contents

List of Figures

Acknowledgements

First off, I would like to dedicate this thesis to the people who did not think I was capable of making it to this point in my mathematical career. Among these are the Penn State representative who told me I would not be able to get accepted in this school, the people who thought I was just plain crazy because of my looks and my flowing hair, and my brother who thought I could not pass a simple actuarial test.

On a more serious note, I would truly like to thank my parents for making me the person I am today. Without their guidance through my younger years, I would have never had the opportunity to sit behind this keyboard and write these sentimental words. I would like to thank Sergei Tabachnikov for his guidance throughout this process and teaching the MASS class which inspired me to write this thesis. A special mention and extension of gratitude needs to be made to Simranjeet Singh as well, without his constant jokes and springboard for good ideas (and bad ones) I would have given up a long time ago. There are plenty of people who have helped me grow as a person and as a mathematician over the past few years, and I thank you all. If you believe you fall in this group, give yourself a big pat on the back and remind me later to get you coffee for dealing with me. If not, well too damn bad... Now sit back, prepare your brain to be stretched in ways you didn't think it could be before, and enjoy the paper.

Chapter 1

Introduction

The topological problem of everting a sphere by a regular homotopy has been a topic of much research in the past 58 years. To state the problem more explicitly, to evert a sphere by regular homotopies is to turn it inside out by a continuous deformation that is allowed to pass through itself but does not puncture, rip, crease, or pinch the surface. The study was initiated by an abstract theorem that proved its existence but did not give an explicit construction. In this thesis, we investigate the history of the problem, the theorem that proved its existence, and four ways in which one can perform such an eversion.

In **Chapter 2**, we learn about the history surrounding this problem and the key players involved. This includes a background on the problem as well as a literature review. Attention here is put on the people involved in the rich history of this problem as the methods in which they devised are touched on in later chapters.

Chapter 3 introduces us to the necessary definitions and notions needed to understand how it is possible to evert a sphere. We will look at simple examples of homotopies, regular homotopies, and isotopies to see the topological ideas behind the methods to follow in the later chapters. This section is integral in understanding the machinery used not only to give an explicit construction, but to prove the existence in the first place.

In **Chapter 4**, we investigate Smale's original work to understand the broader ideas and impacts of his methods. We state the main results he arrived at as well as some insight into how he proved them.

In the remaining chapters, we look at 4 different methods of sphere eversion: the roll and pull method (**Chapter 5**), the tobacco pouch method (**Chapter 6, Section 2**), the use of minimal surfaces (**Chapter 6, Section 3**), and Thurston's corrugation technique (**Chapter 7**). Each method has a style all its own, thus the combination provides a solid foundation in descriptive topology.

In particular, the roll and pull method makes use of small perturbations on the surface and is the easiest to visually get a grasp on. The next two methods, the tobacco pouch method and the use of minimal surfaces, rely heavily on the idea of *halfway models*, which are immersed spheres that are halfway inside out in certain sense that is covered in **Chapter 6, Section 1**. The tobacco pouch eversions sport a rotational and aesthetic symmetry, and make use of the symmetries of halfway models to construct an explicit eversion. The minimal surfaces approach utilizes halfway models in a different sense; starting with a halfway model, we aim to relax it to a sphere by gradient descent with respect to the *Willmore energy*. Thus if such a descent is possible, we start with the immersed sphere, send it to a halfway model by gradient ascent, apply a rigid rotation sending the halfway model back to itself, then apply gradient descent to send it to an inside out immersed sphere. Thurston's corrugation technique makes use of a so called "regularity procedure". Using Thurston's technique, one corrugates the surface of the sphere so that at any point in which a homotopy becomes illegal with respect to our rules, the corrugation smooths out that area so in fact we are not tearing, puncturing, or creasing the surface. This method is very natural in its essence as will be seen **Chapter 7**.

Chapter 2

The Paradoxical Problem of Everting the Sphere

2.1 Mathematical Paradoxes

Mathematics presents a myriad of beautiful and sometimes surprising results. Such phenomena can be found in every field, with more emphasis put on those examples that are as interesting as they are informative. From Peano's construction of a space-filling curve in point-set topology to the Banach-Tarski Paradox in measure & set theories, the underlying beauty of the systems studied in mathematics present themselves when looked in the right light.

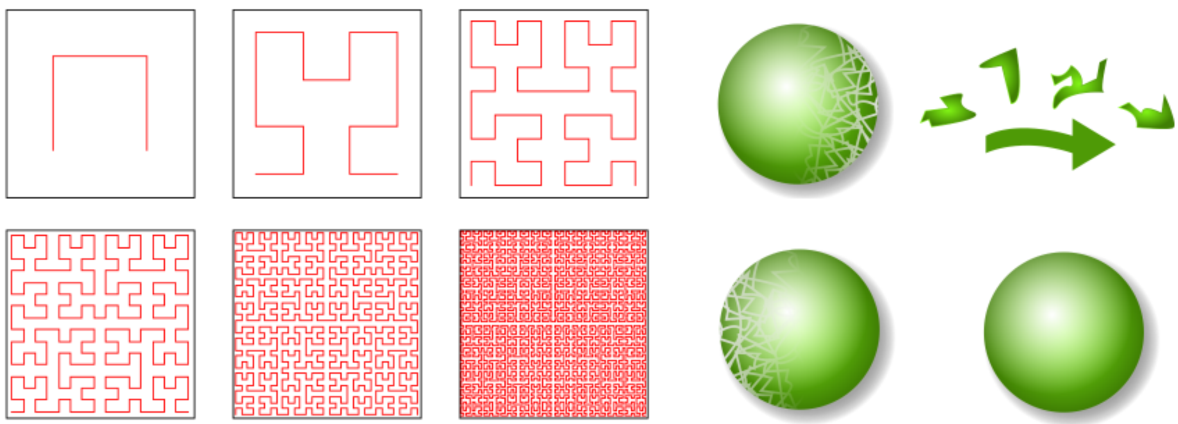


Figure 2.1: (Left) Iterations in creating Peano's space filling curve. (Right) Deconstruction of one sphere into two spheres of exactly the same size, showing the essence of the Banach-Tarski paradox.

In this thesis, we will be studying a paradox that is not given much attention in the undergraduate classroom - the problem of turning a sphere inside out. Clearly, given any spherical object in real life, there is no way to turn it inside out without puncturing the surface. This leads our intuition to believe that *any* sphere cannot be turned inside out. But what if we allow the surface to pass through itself? Does this thwart our intuition and make it possible to do such a feat? In the following pages, we will theoretically and explicitly show our natural intuition is wrong and with this one physical caveat averted, we are able to perform the desired eversion.

Before we start the mathematical theory behind the problem, it is important to define our problem in just the right way. We start now with the problem in layman's terms and equivalently reformulate it in Chapter 3 once the necessary machinery is provided.

Question: Given a mathematical sphere which can pass through itself and be stretched without tearing, is it possible to turn it inside out in 3 dimensions without tearing the surface or pinching it infinitely thin?

The short answer to this question is yes, it is possible to turn such a sphere inside out. But the answer is not the end of the story for us. What we will be investigating is not only why it is possible to turn a sphere inside out (through abstract mathematical methods), but how to do it (through visual mathematical methods).

2.2 Literature Review

A natural place to start our review is with the person who showed that sphere eversion was even possible and the series of papers that started the whole rat race towards explicit sphere eversion constructions. In 1958, Stephen Smale released a paper with R.K. Lashof titled "On the Immersions of Manifolds in Euclidean Space" [LaSm], whereas they delved into the structure of maps from a k - dimensional manifold into a $k+\ell$ - dimensional manifold and what information can be gathered about the spaces by looking at the maps. This paper is mentioned as it inspired the work titled "The classification of immersions of spheres in Euclidean space", the paper in which Smale surprised the world and showed that everting the sphere was possible. While his methods did not give an explicit construction, with a lot of work, his methods *could* provide an explicit construction. A more detailed exposition about the stronger results he proved will be covered in **Chapter 3**.

As with most ideas that push the boundaries of what our minds are capable of, an explicit example really helps. To this end, Smale's doctoral advisor, Raoul Bott, requested/asked the mathematical community for/challenged those willing to try to create an explicit eversion so as to visualize the phenomena. Due to the intricate and non-intuitive nature of self-intersecting surfaces, expressing such a construction has its own visual boundaries present. Thus, to rise to Bott's challenge not only takes the mathematical prowess to understand continuous deformations of the sphere but an uncanny ability to visualize the changing structure of a surface as it passes through itself. We now go through a time-line of events following Smale's discovery.

In 1960, Arnold Shapiro was the first to rise to the challenge. He introduced the use of halfway models into the problem of everting a sphere, the method discussed in **Chapter 6**, as well as using standard topological constructions only. While many regard him as being the first to construct an eversion, he never published his work and his methods were not widely known until 1979 when George Francis and Bernard Morin wrote an article in *The Mathematical Intelligencer* explaining his method, 17 years after Shapiro's death [FrMo]. The next mathematician to perform the task was Anthony Phillips. In 1966, Phillips used Boy's surface as a halfway model, relying on the fact that the sphere was a proper double cover of the immersed projective plane. To visualize his method, he described the process in sections, drawing each step in the eversion and providing the necessary argument to validate each step in the construction.

Later in the 1960s, 1968 to be exact, Bryce De Witt provided another pictorial argument for a sphere eversion. In a similar manner to Shapiro, De Witt did not publish his approach and explicit details behind his method have still not made their way to mainstream mathematics. Around 1970, Charles Pugh began making a series of chicken-wire models showing the stages of an eversion constructed a few years earlier by topologist Bernard Morin, a French mathematician who had been blind since his childhood. Pugh completed the models which were displayed in the Berkeley mathematics department until they were stolen less than a decade later. In 1977, Nelson Max released the animation "Turning a Sphere Inside

Out", an effort that took 6 years to make whereas Max digitalized Pugh's chicken-wire models with the coordinates calculated by hand from the wire models and entered directly as data into the computer. This compilation took the mathematics community by storm and was a huge international success.

Again at Berkeley, but this time in 1974, William Thurston began work on an eversion using ideas of the Dirac "belt-trick", a method which states that a strip can be given a full-twist by either twisting the two ends or passing them through each-other along a straight line. This resulted in an eversion using the idea of corrugation and relying on the intrinsic symmetries present at each step of the eversion, a method we will study in **Chapter 7**. Thurston's approach was compiled by Silvio Levy, Delle Maxwell, and Tamara Munzner in 1994 into the animation "Outside In", a film that has been an international success and shown both mathematicians and non-mathematicians the beauty behind this problem.

In 1987, George Francis released "A Topological Picturebook", a topology book that studied sphere eversions as one of its main facets [Fr2]. Many approaches are discussed, including Shapiro's original approach, halfway models, and tobacco pouch eversions. Five years later in 1992, Francǫis Apéry with the help of Bernard Morin, explicitly described an algebraic halway model. Before their work, computer animations relied on gathering points from a database, a timely and not always pleasantly visual task. The introduction of algebraic equations makes rendering a lot easier as well as a more timely and geometrically pleasing result. In 1995, Francis along with the help of John Sullivan and Bob Kusner devised a construction using Willmore energy, built off previous results of Kusner and Robert Bryant. The resulting minimax eversion was a geometrically optimal eversion in that it minimized the Willmore energy and gave a beautifu; geometric construction. 3 years later in 1995, Francis, Sullivan, and Stuart Levy compiled these minimax ideas into the animation "The Optiverse", an approach we will investigate in **Chapter 6**.

The next approach was created by Eric de Neve in 1996 but was not animated into a movie until 2015 by Chris Hills. This approach utilizes small perturbations on the surface of a sphere and is covered in more detail in **Chapter 5**. 50 years after being introduced, Shapiro's methods were used once again in 2010 by Iain Aitchison in his film "The Holiverse" [Ai]. The rendered images provide insight into Shapiro;s method but with an added element of simplicity. Most recently, in 2015 Arnaud Chéritat released a video of his construction, utilizing ideas of Whitney's that come from the 2 dimensional analogue of the problem (whether a circle embedded in the plane can be turned inside out) [Ch]. This list is most likely not exhaustive as there were surely communications between these mathematicians and others that have not made their way to the limelight.

The history of the problem shows the different approaches and methodologies taken by those rising to the task, and gives a solid background for us to begin our investigation of everting the sphere in 3-space.

Chapter 3

Homotopies and Regular Homotopies of the Sphere

3.1 Definitions

In this section we provide the necessary tools to study the problem at hand. Here we assume all spaces we are dealing with (X, Y, etc.) are Hausdorff.

Definition 3.1.1. *Let f and g be continuous maps from X to Y. A map $H : X \times [0,1] \to Y$ is a homotopy if it satisfies the following:*

1. *If $x \in X$, then $H(x,0) = h_0(x) = f(x)$ and $H(x,1) = h_1(x) = g(x)$.*
2. *$H(x,t) = h_t(x)$ is continuous in both x and t.*

If there exists a homotopy between maps f and g, we say f and g are *homotopic* and write $f \simeq g$. It can easily be checked that in fact, homotopy of maps is an equivalence relation. Thus, if there exists a homotopy taking one map to another, or in more generality one space to another, we say the two maps (or spaces) are *homotopically equivalent*. There are some interesting properties that one can derive quite easily from this definition. The following is an example of one such property.

Lemma 3.1.2. *Let $X \subset \mathbb{R}^n$ be a convex domain and let f and g be continuous maps from another topological space Y to X. Then $f \simeq g$.*

Proof. We prove this by giving an explicit homotopy between f and g. Define $H : Y \times I \to X$ by $H(y,t) = tg(y) + (1-t)f(y)$, namely this is the homotopy that takes the image of each $y \in Y$ under f to that of g by the straight line connecting $f(y)$ to $g(y)$. As X is convex, every straight line connecting two points of X is also in X, thus H is a valid homotopy and $f \simeq g$. □

Now in an effort to restrict our attention to everting a sphere, we need the following definitions.

Definition 3.1.3. *The unit sphere S^2 is the set of points that lie at a distance of 1 from a fixed central point, i.e. $S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$.*

Definition 3.1.4. *An immersed sphere in $\mathbb{R}^3$ is the image of a smooth map $F : S^2 \to \mathbb{R}^3$ such that the Jacobian matrix DF is everywhere of rank 2.*

Definition 3.1.5. *A sphere is embedded if the immersion map F is also one-to-one.*

Definition 3.1.6. *A map F_t, $t \in I$, through immersions is a regular homotopy if $t \to DF_t$ is also continuous. We can equivalently define a regular homotopy as a homotopy through immersions. Additionally, a regular homotopy through embeddings is an isotopy.*

Definition 3.1.7. *An eversion of the sphere is a regular homotopy from the identity inclusion $S^2 \to \mathbb{R}^3$ to the antipodal map of S^2 into $\mathbb{R}^3$.*

Remark 3.1.8. *If we imagine taking the identity inclusion sphere and coloring the outside surface red and the inside blue, then the antipodal embedding represents the sphere with the outside blue and the inside red. Thus, a regular homotopy taking the identity to the antipodal sphere is a continuous deformation of the sphere which does not pinch, crease, or tear the surface, i.e. an eversion as introduced earlier in this paper.*

First thing to note, not all immersions are embeddings. In fact, a sphere cannot be everted by an isotopy as any map taking the sphere inside out must take the surface of the sphere through itself, and at any place on the surface where we have a double line, triple point, quadruple point, etc., we lose injectiveness of the map taking S^2 to that immersion. In fact, it has been shown that every eversion of the sphere has a quadruple point [Hu]. To see why regular homotopies are of importance in the process of everting a sphere, we must note that otherwise (if we only required a homotopy), the problem would be trivial. This is formulated as the following.

Lemma 3.1.9. *Let f and g be two immersions of S^2. Then $f \simeq g$.*

Proof. Define $H : S^2 \times I \to \mathbb{R}^3$ by $H(x,t) = tf(x) + (1-t)g(x)$. To see this is a homotopy, since f and g are continuous with respect to x, H is continuous in x. Continuity in t follows from linear functions being continuous. Thus $f \simeq g$. □

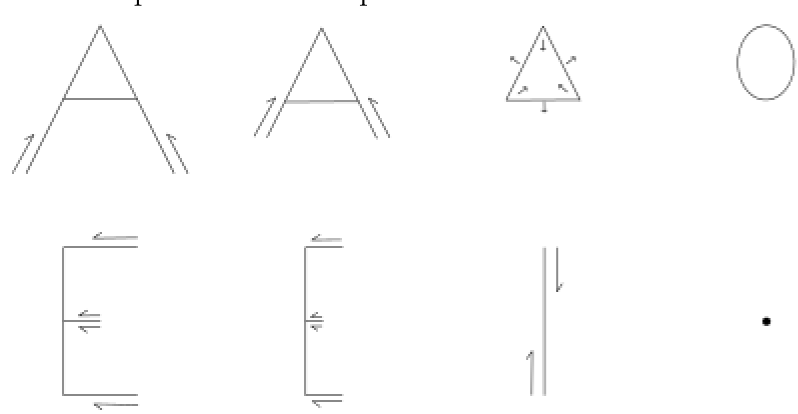
As all immersions are homotopic, trying to turn a sphere inside-out by a usual homotopy is no big feat, simply slide every point $f(x)$ to $g(x)$ with constant speed and you are done. Doing this without pinching, tearing, or creasing the surface is where the trouble comes into the mix. In the next section we go through some visual examples of homotopies, regular homotopies, and isotopies to get ourselves familiar with the objects we are dealing with.

3.2 Examples

The first few examples we show deal with simple ideas just to get the feel for homotopy theory. We then move to transformations of the sphere. In the end, we show that the linear homotopy from the immersed sphere to the inside-out sphere by switching antipodal points is not a regular homotopy. Additionally, we show the transformation of pushing the north pole through the south pole is not a regular homotopy as well.

Example 3.2.1 (Categorization of the alphabet)**.** *The English alphabet can be partitioned into the following classes of letters which are homotopically equivalent to each other:* $\{A,D,O,P,Q,R\}$, $\{B\}$, $\{C,E,F,G,H,I,J,K,L,M,N,S,T,U,V,W,X,Y,Z\}$.

Proof. Here we show two pictorial homotopies and leave the rest for the interested reader.



A very similar approach can be done for each letter in the prescribed equivalence classes to show the assertion is correct. □

Another important notion in homotopy theory is the idea of contractibility. We say a space X is *contractible* if the identity map on X is homotopic to the constant map on the space taking everything to a point.

Example 3.2.2. *Any convex set in $\mathbb{R}^n$ is contractible.*

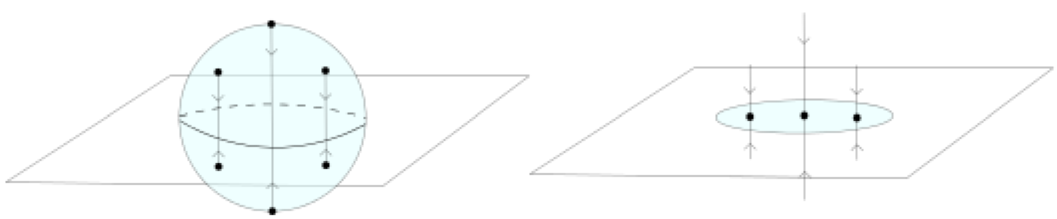
Proof. Let x_0 be an arbitrary point in our convex set C. By defining f to be the identity on C and g to be the constant function sending everything to x_0, the result follows directly from **Lemma 3.1.2**. □

Example 3.2.3. *The north pole-south pole switching map is not an eversion, where the north pole-south pole switching map is the linear map that swaps the z coordinate of the standard immersion.*

Proof. Let $f : S^2 \to \mathbb{R}^3$ be the standard inclusion, and $-f$ be the antipodal map. Define $F_t(X, Y, Z) = (X, Y, (2t-1)Z)$. This gives rise to the Jacobian

$$DF_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2t-1 \end{pmatrix}$$

This is not an eversion as it fails to be an immersion at $t = 1/2$ along $Z = 0$. Geometrically, we get this picture



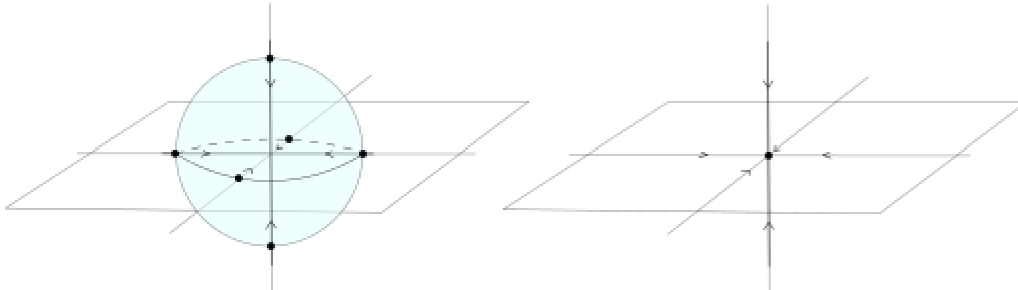
□

Example 3.2.4. *The linear antipode switching map is not an eversion.*

Proof. Let $f : S^2 \to \mathbb{R}^3$ be the standard inclusion, and $-f$ be the antipodal map. Define $F_t(X, Y, Z) = ((2t-1)X, (2t-1)Y, (2t-1)Z)$. Just as before, we look at the Jacobian

$$DF_t = \begin{pmatrix} 2t-1 & 0 & 0 \\ 0 & 2t-1 & 0 \\ 0 & 0 & 2t-1 \end{pmatrix}$$

This is not an eversion as it fails to be an immersion at $t = 1/2$ along $X, Y, Z = 0$. Geometrically, we get

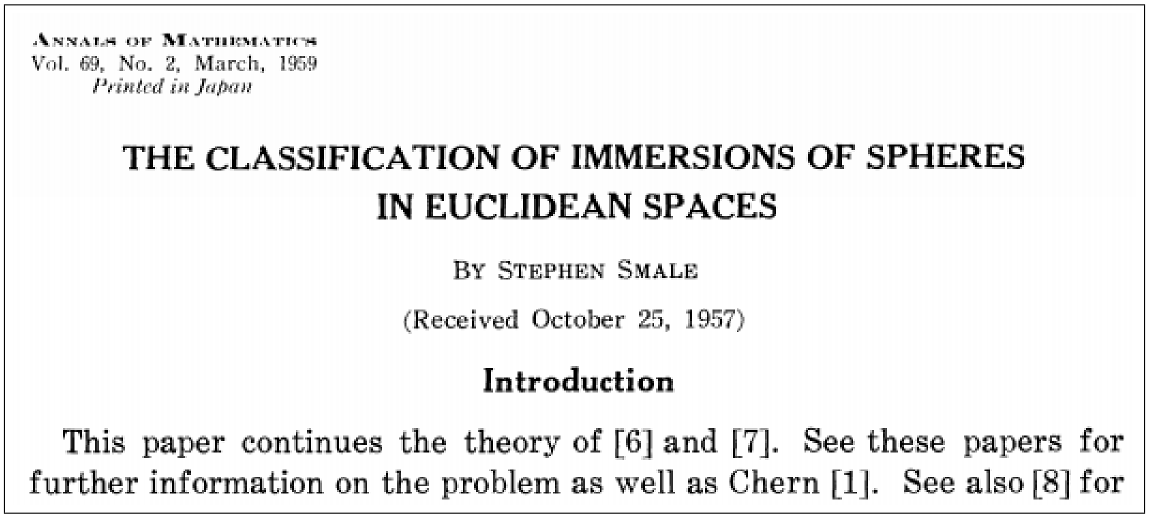


This problem is just a generalization of **Example 3.2.3** where all three coordinates get reversed. $\square$

Chapter 4

Smale's Proof of Existence

4.1 Smale's Main Result & its Consequences


ANNALS OF MATHEMATICS
Vol. 69, No. 2, March, 1959
Printed in Japan

THE CLASSIFICATION OF IMMERSIONS OF SPHERES IN EUCLIDEAN SPACES

BY STEPHEN SMALE

(Received October 25, 1957)

Introduction

This paper continues the theory of [6] and [7]. See these papers for further information on the problem as well as Chern [1]. See also [8] for

Figure 4.1: Excerpt from the title page of *The Classification of Immersions of Spheres in Euclidean Space.*

In Stephen Smale's 1959 (written in 1957) paper *The Classification of Immersions of Spheres in Euclidean Space* [Sm], Smale proved an abstract theorem guaranteeing the existence of a sphere eversion but did not give an explicit construction. This resulted in Raoul Bott's request of an explicit eversion, a task that would not be met for a few years, but this has already been discussed in the **Literature Review** section of **Chapter 2**. In this chapter, we go over the statements of Smale's theorems and discuss the implications of the revolutionary ideas present in his paper.

Let's start off with the statement of one of the many theorems in his paper and see what we can take from it.

Theorem 4.1.1 (Smale (1959) Theorem B)**.** *Two C^∞ immersions of S^k in $\mathbb{E}^n$ are regularly homotopic when $n \geq 2k+1$.*

First off, we need to clarify what a few terms in this statement mean. A C^∞ function is a map that is infinitely often differentiable, a term that many mathematicians refer to as being "smooth". While it may sound similar to a function being analytic, this assertion is weaker than analyticity as every analytic map is smooth but not every smooth map is analytic. Here by $\mathbb{E}^n$, we simply mean n-dimensional Euclidean space, i.e. $\mathbb{R}^n$ equipped with the standard inner product. Now that we have the notational background, we can explore the implications of this theorem.

The result above essentially states that if you have two smooth immersions of a k-dimensional sphere in n-dimensional Euclidean space, there exists a regular homotopy taking one immersion to the other if n is of a reasonable size. Another equivalent formulation is that the space of all smooth immersions of S^k into $\mathbb{R}^n$, when $n \geq 2k+1$, is connected. If we try directly applying this result to the problem where $k = 2$ and $n = 3$, we see this theorem does not guarantee anything about everting the 2-sphere in 3-space. Nevertheless, this theorem is very powerful and is not a clause of necessity and sufficiency. Luckily for us, Smale actually proved an additional statement in his paper that gives us what we are looking for.

Theorem 4.1.2 (Smale (1959))**.** *The space of smooth immersions of S^2 into $\mathbb{R}^3$ is connected.*

From this, we deduce the following.

Corollary 4.1.3. *There exists a path in the space of smooth immersions of S^2 into $\mathbb{R}^3$ that connects the standard embedding, f, of the sphere to the antipodal embedding, $-f$.*

So there it is, the statement that inspired half a century of tumultuous work and miraculous examples of descriptive topology. Smale's proof of **Theorem 4.1.2** involves identifying the classes of smooth immersions of the sphere with the homotopy group of the Stiefel manifold, an object that plays a big role in differential topology & geometry. We mention it here for completion of explanation but for a more in depth study of the use of the Stiefel manifold and its relation to sphere eversions, see [Sm].

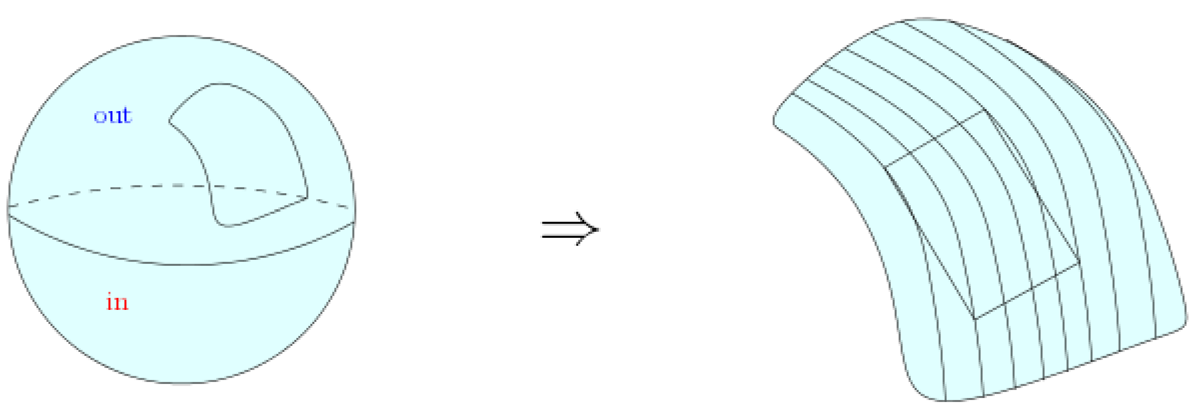
Chapter 5

“Roll and Pull” Method

5.1 Explanation & Execution

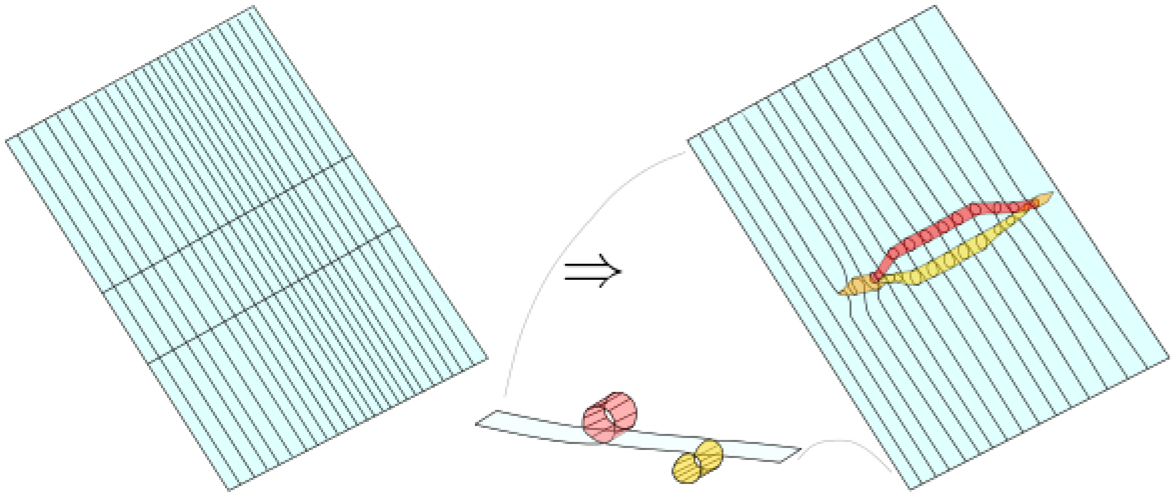
The "roll and pull" method (R&P method) is the simplest method of sphere eversion to visualize, making it a strong first choice of methods to investigate. Instead of making use of a small number of topological events, it deals with small perturbations on the surface which create a "ring" that can be pulled to the equator. Once this ring is present, simply pass the poles through one another and the eversion is complete. As with most methods investigated in this thesis, there is a nice animation showing this eversion. The video, titled "Simple Sphere eversion", can be found at https://www.youtube.com/watch?v=FL4JoWlVj98. Named after its creators, an alternative name for this method is the deNeve/Hills eversion.

While other arguments in this paper rely on heavy topological tools, this approach is very visual and geometric. Since this approach can be visualized and followed by the imaginative and willing reader, we actually go step by step through this eversion. This is unlike the other methods in this paper where we describe the steps that take place but do not give a topological argument at each stage explaining why the step indeed is regularly homotopic. Note though that at each step, one can check that the actions that are happening are not violating the rules of a valid sphere eversion. Hence this step by step approach is actually a valid proof to the existence of a sphere eversion once one argues that each step is valid; but as this is not difficult, it is left to the reader to fill in as an exercise. Notationally, we use S^2_{o} to denote the standard embedding of the sphere and S^2_{i} for the reverse embedding. We now walk through the steps of the R&P eversion to give insight into the beauty behind this problem.



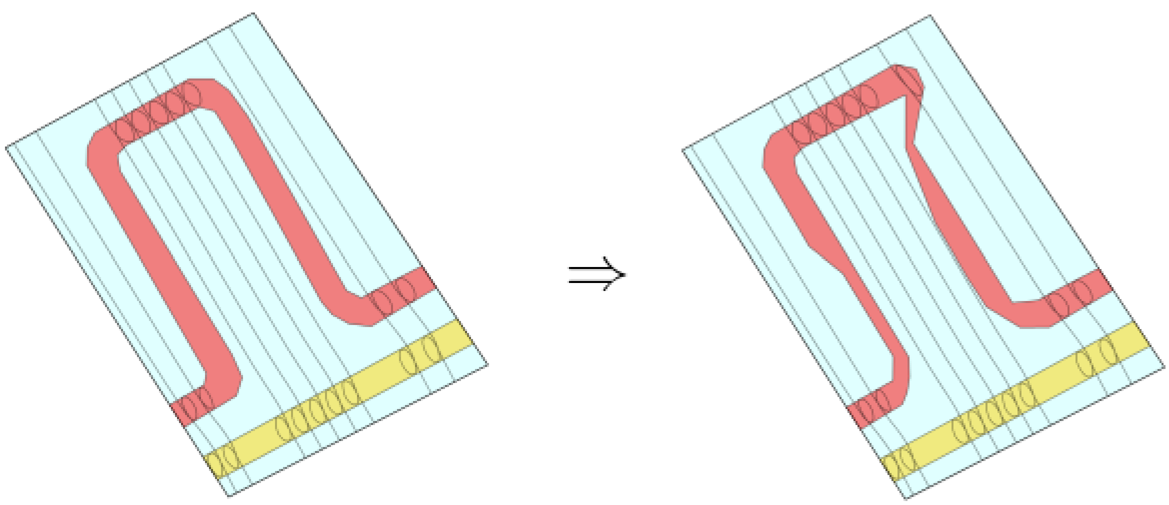
Step 0 Start with the sphere S^2_{o} with the exterior surface colored blue and the interior surface colored orange.

Step 1 Zoom in on a small section of the surface until the resulting structure looks like $\mathbb{R}^2$.



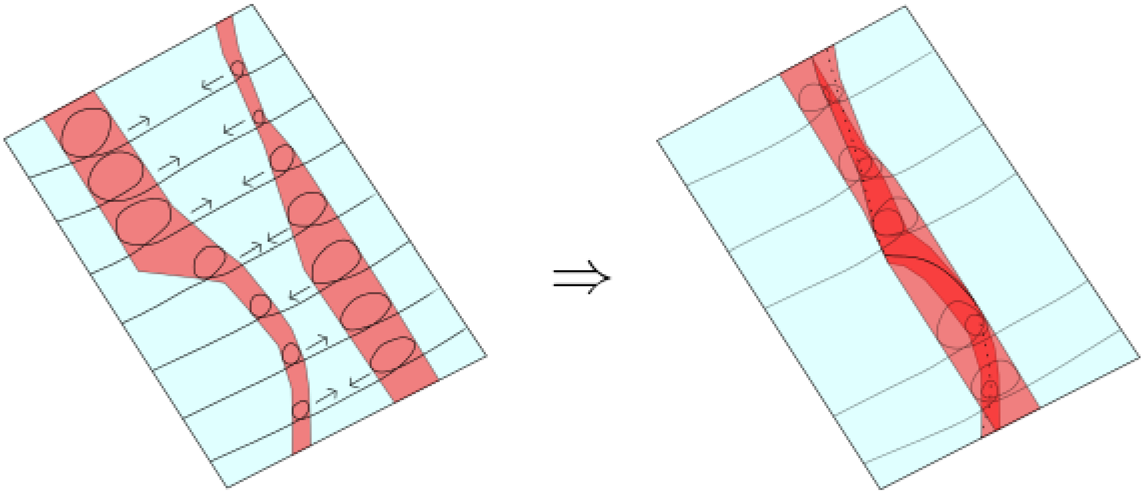
Step 2 Rule the surface of the sphere so any perturbations to the surface are easily visible.

Step 3 Starting with the lower line from **Step 2**, slide it up and under the upper line creating two rolled “tubes” on the sphere, one below the surface and one above it. We call the red tube “tube R” and the gold tube “tube G”.



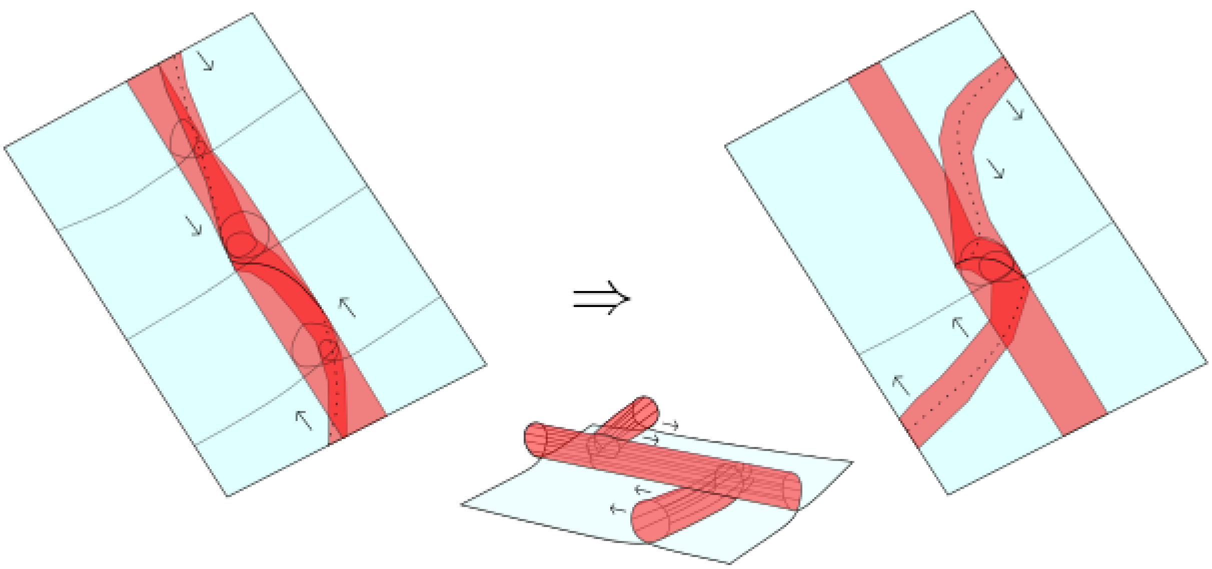
Step 4 Slide the middle section of tube R away from tube G. Tube R should have a box car function shape to it. We call this configuration the arch.

Step 5 Pinch the lower side of the left vertical tube comprising tube R as well as the upper part of the right vertical tube. Additionally, slide the upper right corner further right away from the arch.



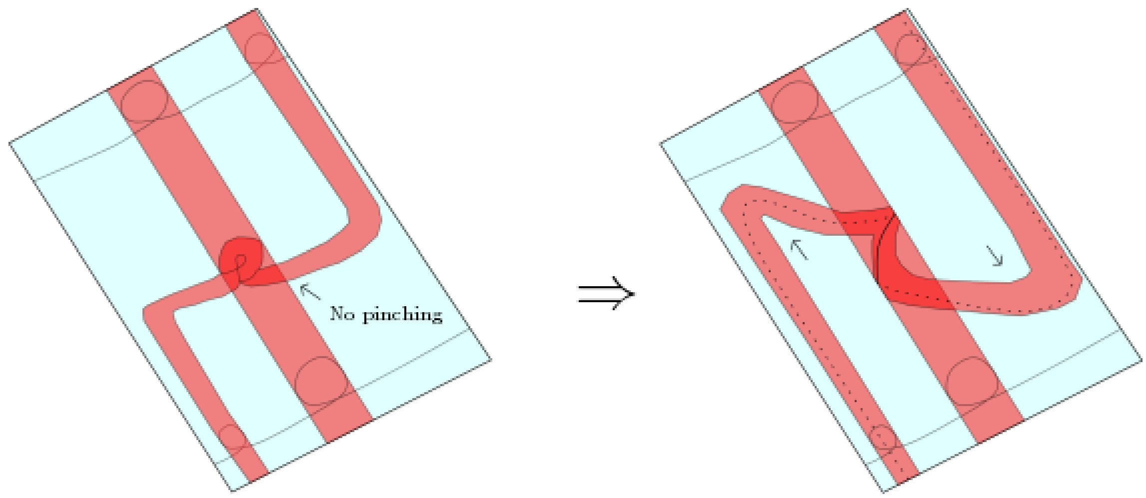
Step 6 Slide the two vertical tubes together, making the small part of the left tube line up with the thick part of the right tube and vice-versa.

Step 7 Construct a curve that lies on the intersection of the two tubes surfaces.



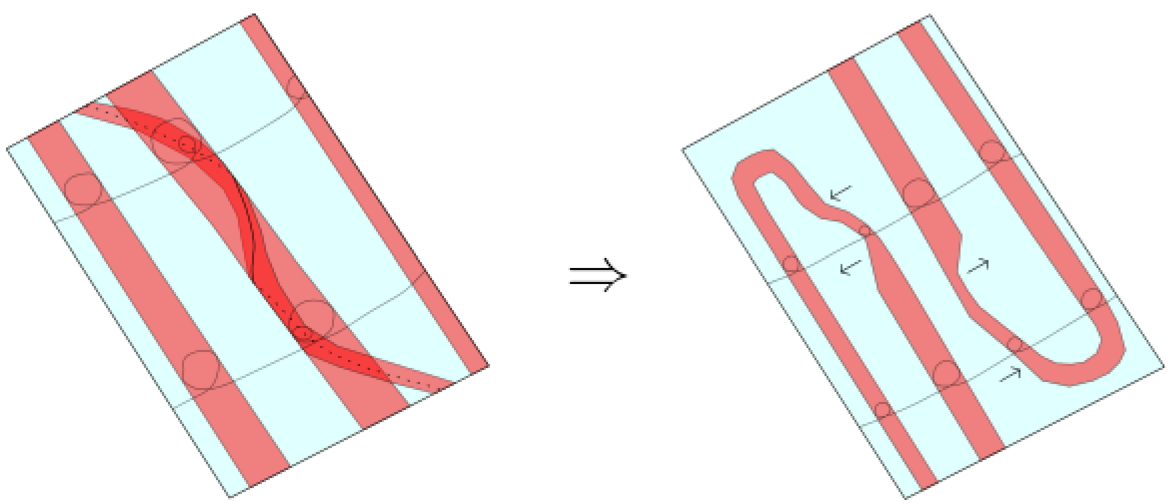
Step 8 Start pulling the small section of the right tube down around the tube intersection while pulling the small section of the left tube up.

Step 9 Continue the process from **Step 8**. Note that while we are sliding these two small sections of tube along their intersection, no pinching or tearing of the surface occurs.



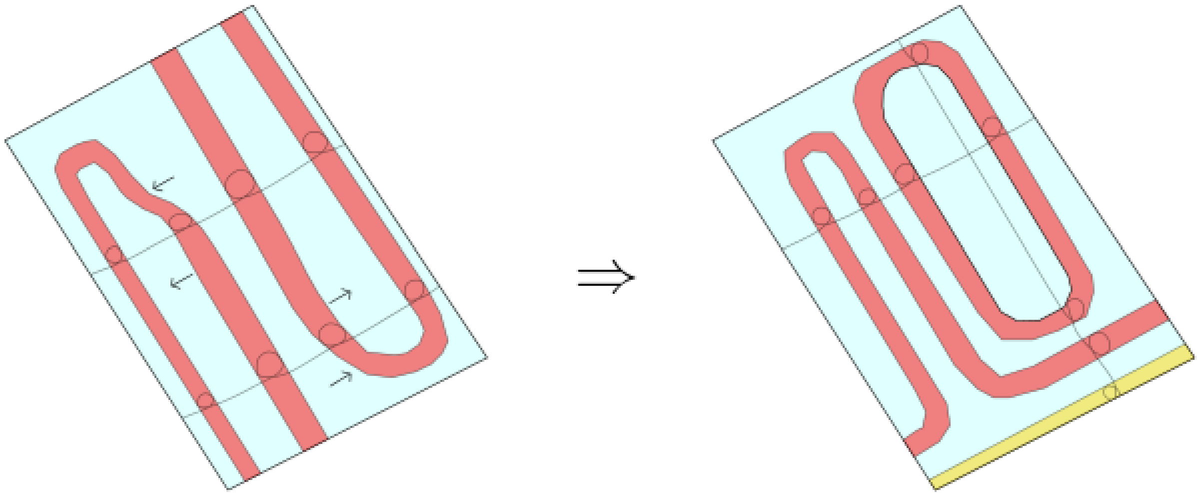
Step 10 While sliding the two small tubes along the intersection, they will eventually pass each other. If the small tube thickness is less than half of the large tubes thickness, no pinching will happen.

Step 11 Continue moving the left tube up and the right tube down.



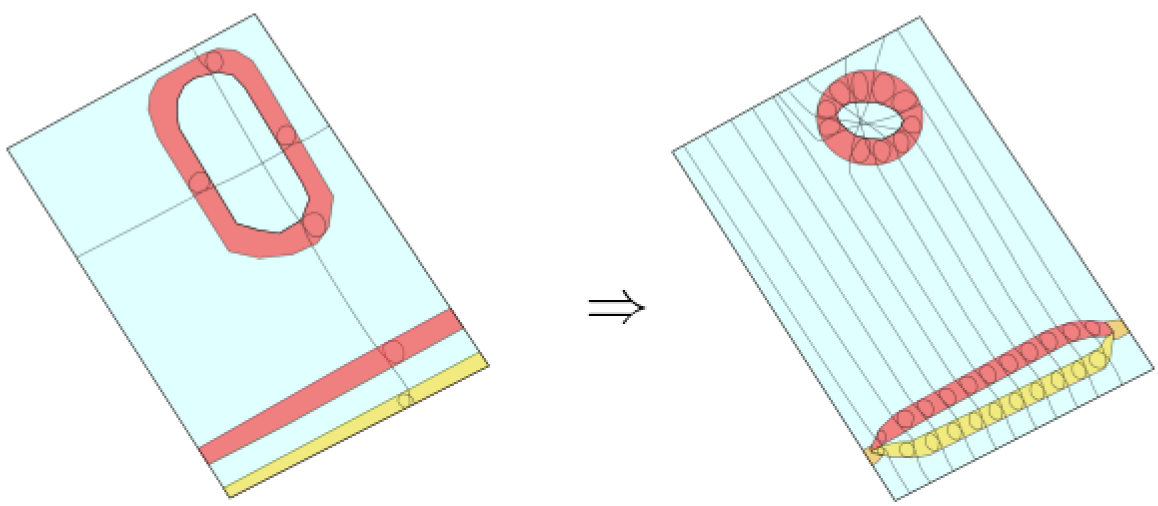
Step 12 As the left tube is moved up and the right tube is moved down, we are left with the original tube intersection configuration essentially reflected about the hyperplane passing vertically through the intersection tube.

Step 13 Just as the tubes were coalesced to create the intersection tube, we can separate them leaving two disjoint red tubes, the lower tube, tube R_1, and the upper tube, tube R_2.



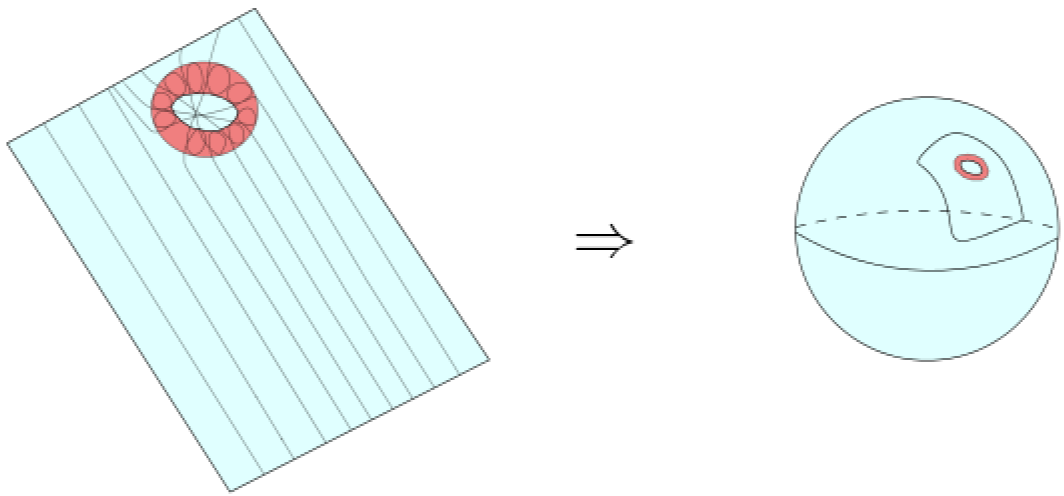
Step 14 Re-thicken the areas of the tubes that were shrunk in **step 5**.

Step 15 Relax the tubes back into a more geometrically sound position. What can be noted now is that tube R_1 is just tube R we started with and tube R_2 is a free closed tube on the surface.



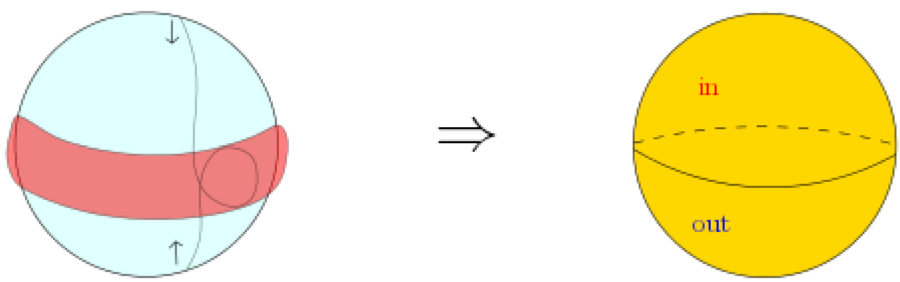
Step 16 Relax tube R_1 back to a horizontal tube and start pulling tube R_2 into a circular shape.

Step 17 Start unrolling tube R_1 and tube G while finishing making tube R_2 a torus on the surface of S^2_o.



Step 18 Finish unrolling tubes R_1 and G from each other.

Step 19 Start increasing the size of the torus tube R_2.



Step 20 Pull the tube to the equator of S^2_{o} and pass the north and south poles through each other.

Step 21 Finish pulling the poles through process and relax the shape into the everted sphere, S^2_{i}. This step concludes the R&P method of everting the sphere.

Chapter 6

Halfway Model Approaches

6.1 Halfway Models

In this section, we begin with the definition of a halfway model and then work through some examples to uncover the beauty and simplicity behind its structure.

Definition 6.1.1. *A halfway model is an immersed sphere which is halfway inside-out in the sense that it has a symmetry that interchanges the two sides of the surface. In particular, for an appropriate parameterization, antipodal points on the abstract ambient sphere will get mapped by the immersion to points which are related by this symmetry.*

At first glance, this definition gives us that a halfway model is a "kind of inside-out sphere" with certain symmetry properties. These symmetries interchange the sides of the surface, whereas we mean in the halfway model, certain rigid motions of space will swap the inside of the sphere with the outside of the sphere. This idea is most easily understood by example.

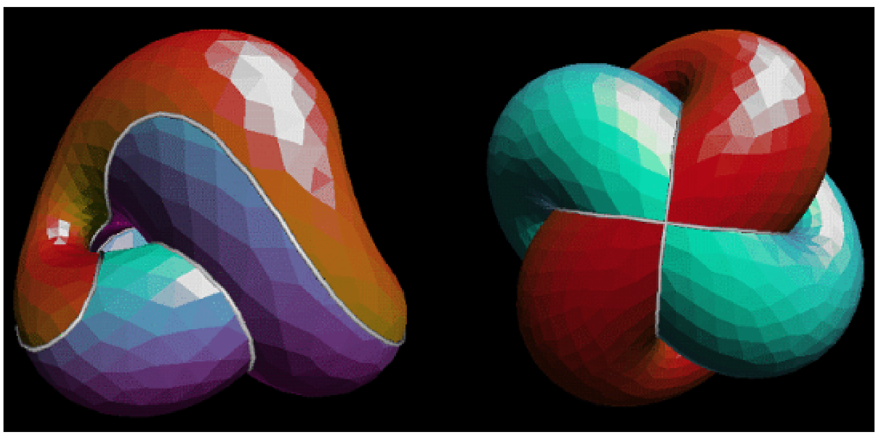


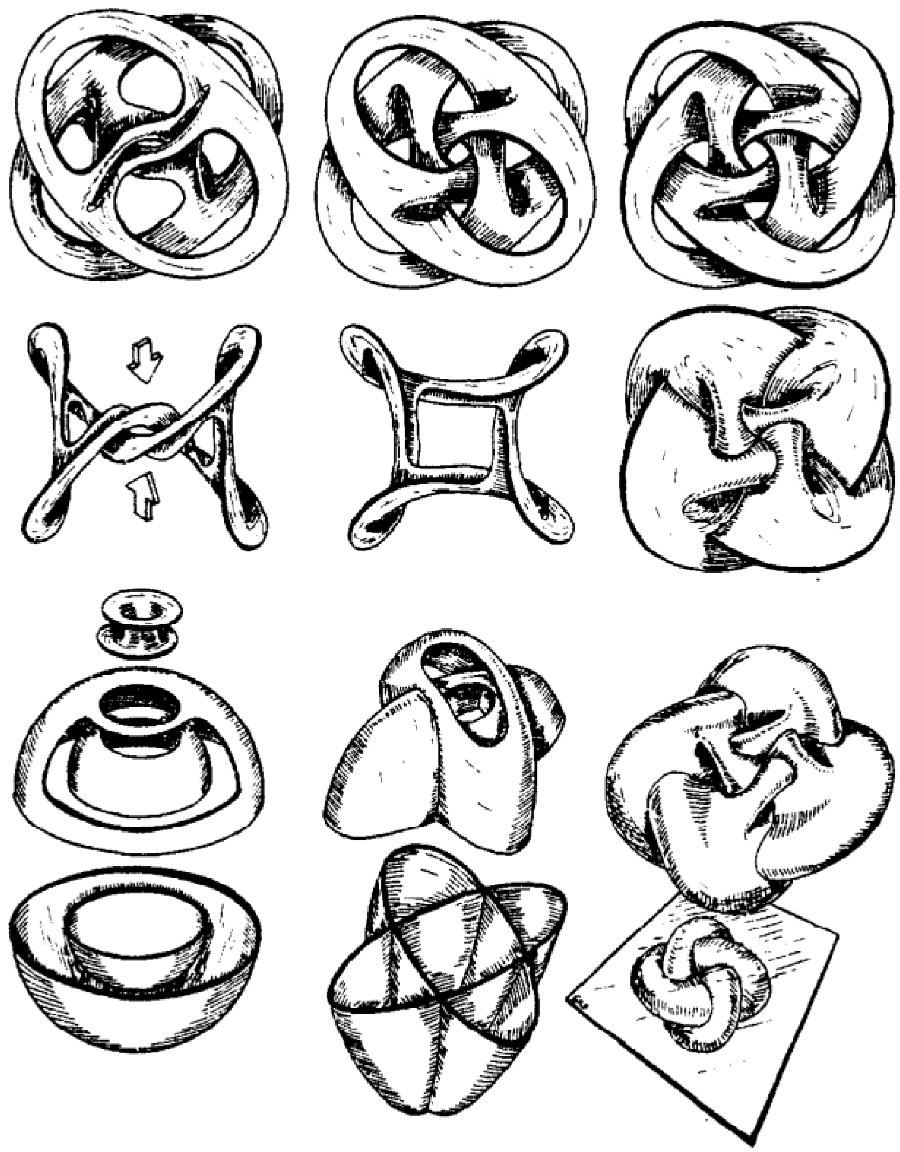
Figure 6.1: (Left) A halfway model sporting a three-fold symmetry. This halfway model is an immersion of the projective plane with a single triple point. (Right) A halfway model sporting four-fold rotational symmetry, an example of a *Morin model*. Note that each rotation by $\frac{\pi}{2}$ about the visible quadruple point leaves the model invariant with the sides of the surface reversed, [Su1].

In the above examples, we are given 2 immersed surfaces that act as halfway models for different eversions. The key idea behind the use of halfway models is that if we can relax our halfway model to an immersed sphere with no intersections, namely our canonical immersed sphere or the inside out sphere, then we can recover a full eversion. To see this, start with the immersed sphere, reverse the relaxation process to obtain the halfway model, apply a rigid motion to the surface to reverse orientation, then relax back to an immersed sphere, which will be our original sphere but inside out.

This idea of "relaxing" the surface has many flavors, but we will only be discussing two of them, the tobacco pouch method and gradient descent with respect to Willmore energy, dubbed the minimax eversion. The first method we look at gets its name from its visual relation to French tobacco pouches and are particularly intricate. The method of gradient descent comes from the introduction of an energy associated to the immersed spheres and gives geometrically pleasing constructions, computed using different software programs engineered at different geometry centers around the country.

6.2 Tobacco Pouch Method

In this section, we present the background for a countably infinite set of eversions of the sphere, indexed by a number $n \in \mathbb{Z}_{\geq 2}$. These so called tobacco pouch eversions sport a surprising number of symmetries and are named for their likeness to the common French tobacco pouch *blague automatique* [Fr1]. We go through this method of eversion, utilizing the illustrations and arguments of George Francis to give pictorial arguments for the eversion in the case of $n = 2$. The following figure (found in [Fr1]) will be referred to throughout this argument, where the number system (ij) refers to the picture in the ith row and jth column, e.g. (43) is the reflection of the Morin model in the mirror, also shown in **Figure 6.1**.



To start the eversion, isotopically deform the sphere into the shape of a gastrula, a sphere with a dimple pushed into it. This shape consists of two concentric spheres attached by a neck that is reminiscent of the interior of a torus. This has been separated into separate pictures in (31) and (41). (11) is the neck of the gastrula once a turning procedure has been applied, with 4 windows cut into it so we can see what is going on within the surface. This could be done by turning both concentric spheres into ellipsoids with perpendicular major axes as in (32) and (42). It is important to note that the intersecting ellipsoidal shape collection has two-fold symmetry, a rotation by π about the axis straight through the top of the surface leaves the surface invariant.

If we now shift our attention to the ribbon neighborhood (21) of the contour, a valid regular homotopy which takes the lobes pointed to by the arrow through each other gives us the four-cusped hypocycloid found in (22). Again, note that (22) and (42) both sport a four-fold symmetry. If the opposite sides of the sphere were colored, a quarter turn of (22) or (42) would give us the same surface just with the colors swapped. By looking at (12), this four-fold swapping symmetry is not present at for the whole surface we have at this stage. To get four-fold symmetry throughout, push the right and left side of (12) through each other to obtain (13). Removing the windows from (13), we obtain (23), an immersed annulus. Putting (23) and (42) together, we obtain a version of the Morin surface (33), whose reflection in (43) is what we have seen above in the first section of this chapter. Now to finish this eversion, apply the reverse of this process to other pair of opposite segments in (22).

This method can be generalized to all symmetries of even order. The contour in the middle stage (the halfway model for tobacco pouch eversions), the (generalization of the) Morin surface, can be generalized to a hypocycloid with $2n$ cusps. The surface remains invariant by a $1/2n$ turn about the vertical axis. For n odd, this method yields a different type of eversion, with an odd number of ends twisted into the neck of the gastrula. For a lengthy discussion of the odd case, see [Fr1]. The next section investigates another method utilizing halfway models, this time from a geometric, energy minimizing standpoint.

6.3 The Minimax Eversion

To see the insight behind the method involving minimal surfaces, lets start by imagining a stiff metal wire. The elastic bending energy of a stiff wire is the integral of the squared curvature over the whole wire. This measures how far from minimal the new position of the wire is. The same idea can be applied to surfaces in space. At each point on a 2-dimensional surface, there are two principal curvatures κ_1, κ_2 which are determined by the maximum and minimum of the normal curvature at that point. Additionally, there is the average of these two called the mean curvature, $H = \frac{1}{2}(\kappa_1 + \kappa_2)$, which again measures how far the surface is from being minimal. Lastly, there is the Gaussian curvature which is the product of the two principals, $\kappa = \kappa_1 \cdot \kappa_2$. From these observations, we formulate the following definition.

Definition 6.3.1. *For a 2-dimensional surface Σ, the Willmore energy of Σ is*

$$W = \int_\Sigma H^2 dA - \int_\Sigma \kappa dA = \int_\Sigma H^2 dA - 2\pi\chi(\Sigma)$$

where H is the mean curvature, κ is the Gaussian curvature, and $\chi(\Sigma)$ is the Euler characteristic of Σ. The second equality holds by the Gauss-Bonnet Theorem.

Here, as a reminder, we restate Gauss-Bonnet for manifolds with no boundary.

Theorem 6.3.2 (Gauss-Bonnet)**.** *Suppose Σ is a compact two-dimensional Riemannian manifold with empty boundary. Let κ be the Gaussian curvature for Σ and $\chi(\Sigma)$ be the Euler characteristic of Σ. Then we have the equality*

$$\int_{\Sigma} \kappa dA = 2\pi\chi(\Sigma)$$

There a few facts we will utilize while trying to understand this method of sphere eversion.

- Of all closed surfaces, the round sphere minimizes W. Thus we can normalize W such that $W_{S^2} = 1$.
- Any self-intersecting surface with a k-tuple point yields $W \geq k$, [Ku, LiYa].
- As mentioned in **Chapter 3**, every sphere eversion has at least one stage where a quadruple point is encountered, hence we have $W \geq 4$ [Hu].
- The term *gradient descent* refers to a variational optimization method used to find local minima of a function by taking steps along the function proportional to the negative of the gradient. Here, the function we are looking at is W in the (infinite-dimensional) space of all 2 dimensional surfaces in $\mathbb{R}^3$ with the gradient defined on this space.
- In the 1980's, Robert Bryant was able to classify all critical points of the Willmore energy for immersed spheres; even more so, they all have integer energy values greater than or equal to 4 (except for the round sphere which has value 1) [Br].

We now have all the necessary tools to explore the sphere eversions utilizing the idea of Willmore energy.

Given an immersion of the sphere, we can calculate its Willmore energy, which is guaranteed to be an integer value greater than or equal to 4. By gradient descent, we can ride down from this local minima to the point where the Willmore energy is 1, namely the round sphere. Thus, to complete a sphere eversion using a halfway model, we start with the canonical embedded sphere, go over one of these local minima corresponding to a halfway model, and then ride the gradient descent back down to the inside-out round sphere. The easiest model to use for this process is one found by Rob Kusner whereas the energy of the halfway model is exactly 4. Since there is only one minimum below it, gradient descent will take us directly to the round sphere.

One may be a bit critical of this approach because it seems plausible that the process of gradient descent could lead us to a surface with pinches or creases. Currently there is not enough theory known behind fourth order partial differential equations to formally guarantee this will not happen, where fourth order PDEs are brought up here because some of the fundamental equations of the minimax eversion are fourth order PDEs. Luckily, the process of eversion by gradient descent could be computed to verify the correctness of the assertion.

One of the first programs used to check this process was created by Ken Brakke and simply named *evolver*. This program was created to solve variational problems, which made it an excellent candidate for computing sphere eversions. When Sullivan, Francis, and Kusner used *evolver* in 1995 to test some of their results, not only did they verify that gradient descent was a valid approach, but that the eversion that was rendered was topologically optimal in the sense that it used the least number of topological events *and* it was geometrically optimal in that it that it require the least amount of bending energy; whereas by topological event we mean a moment in time where the topology of the surface changes, i.e. the creation or annihilation of a double curve or triple point. These results have been compiled into a internationally renowned video titled "The Optiverse". In this presentation of everting a sphere, they show examples from a countably infinite family of eversions using different halfway models indexed by the number of "lobes" necessary to perform the eversion. In the two halfway models shown in **Figure 6.1**, the corresponding lobe index is 3 and 4 for the immersed projective plane and Morin model respectively.

This method of sphere eversion is natural in that it requires the least amount of bending energy and that the visualizations have a large amount of symmetries. The next approach is natural in a more intuitive and constructive way, as we will now explore.

Chapter 7

Thurston's Corrugation Technique

7.1 Introduction

In 1974, Bill Thurston had the idea to implement the "belt-trick" to try to construct a sphere eversion. The belt-trick refers to a method of twisting a long-strip, which can be done either by twisting it at each end or by pushing the two ends through one another by parallel transport along the straight line connecting the two ends. This one key idea is what lead to a whole new approach of everting the sphere. One may ask "what is the connection between twisting long strips and turning a sphere inside-out?", and this is natural question to ask after being introduced to the belt-trick. In the next section, we will answer this question and show some of the beautiful geometric objects that are present in this method of eversion.

7.2 Corrugations and the Belt-Trick

Aside from the use of the belt-trick, the most crucial part of Thurston's eversion is the use of *corrugations*. His idea was to take any continuous motion of the sphere to its antipodal embedding and make it *regular* by adding corrugations in a controlled manner (when possible). By regular, we simply mean taking out any singularities such as pinches or creases. These added corrugations make the surface more flexible and easier to perform self-intersections without forming pinches or creases.

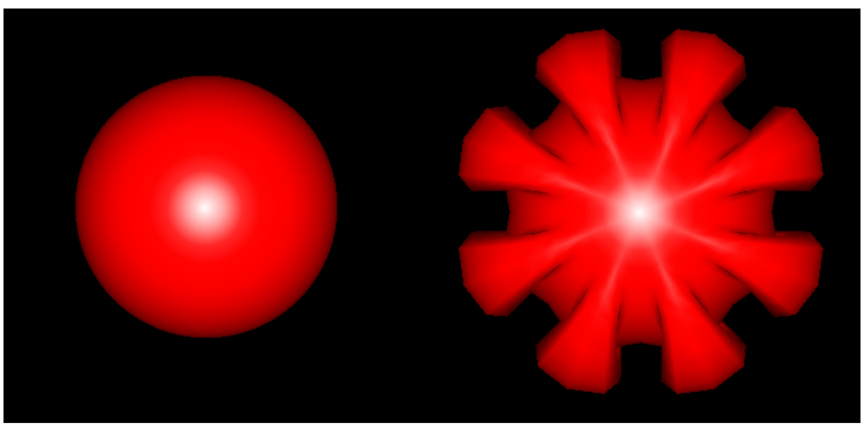


Figure 7.1: In this figure we have two immersed spheres: the left one represents the standard embedding of S^2 and the right is a sphere with corrugations, both viewed from above. These pictures were created using the software *sphereEversion*.

The introduction of the idea of strips comes into play if we imagine the sphere being a barrel with a cap on each end. If we decompose the barrel into thin vertical strips, we can just think of a sphere as a collection of vertical strips with caps connecting the ends of the strips. As mentioned above, a strip can be twisted by means of passing the ends through each other, then rotating the ends of the strips. This is shown below in **Figure 7.2**.

Figure 7.2: Three steps showing the belt-trick applied to one of the strips comprising the surface of the sphere.

Thus, Thurston's technique boils down to corrugating the surface of the sphere, pinpointing strips which will serve as guides for the eversion, applying the belt-trick to the guide strips, and relaxing the surface back to its normal shape. This approach can be summed up in the following picture series.

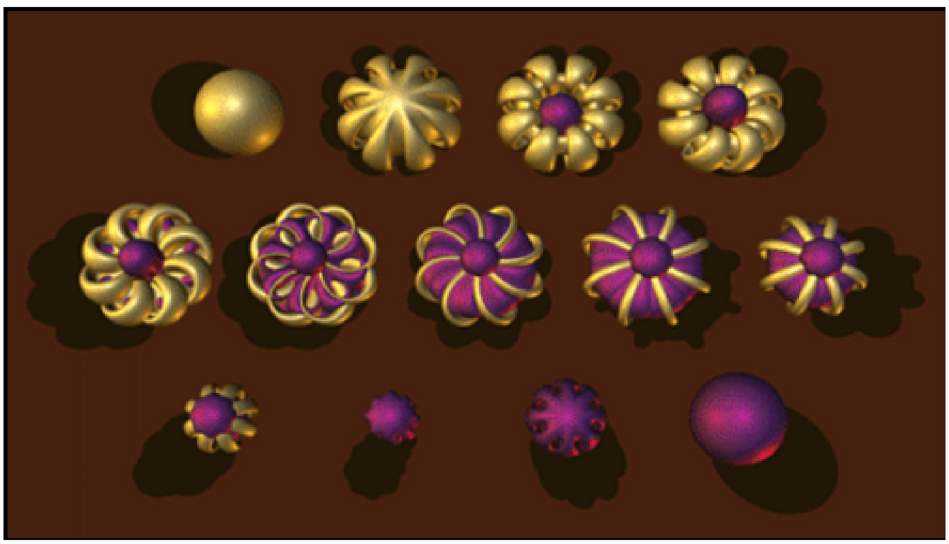


Figure 7.3: Step by step approach of the sphere eversion method devised by Thurston. These still shots are from the video "Outside In".

Thurston's method of sphere eversion is more visually stunning than the previous ones discussed and contains many more pleasing symmetries. This approach was compiled by the Geometry Center at the University of Minnesota into the movie "Outside In", a film that has become a world-wide sensation and introduced the masses to the problem of everting a sphere. A lengthy discussion of what happens at each stage of the Thurston eversion is given in the film as well as historical content and the relationship between everting the sphere and the Whitney-Graustein theorem.

This concludes our study of the topological problem of everting the sphere. Many different methods of solving the same problem were shown, giving insight into the intricacy behind this "paradox". The interested reader can follow the links present throughout this thesis to explore each approach in its own right and to gain a deeper appreciation of the past 60 years of descriptive topology.

Bibliography

[Ai] I. Aitchison. *The 'Holiverse': holistic eversion of the 2-sphere in* $\mathbb{R}^3$, (2010) https://arxiv.org/pdf/1008.0916.pdf.

[Br] R. Bryant. *A Duality Theorem for Willmore Surfaces*, Journal of Differential Geometry, 20 (1984), pp. 23–53.

[BrChFrHaKuSu] K. Brakke, G. Chappell, G. Francis, C. Hartman, R. Kusner, J. Sullivan. *The Minimax Sphere Eversion*, Visualization and Mathematics (1997), pp. 3-20.

[Ch] A. Chéritat, *Another Sphere Eversion*, John H. Hubbard 70th Birthday Conference in Bremen (2015), https://www.math.univ-toulouse.fr/ cheritat/Exposes/BremenJHH70.pdf.

[Hu] J. Hughes, *Another Proof that Every Eversion of the Sphere has a Quadruple Point*, Amer. J. Math. 107(1985), pp. 501–505.

[Fr1] G. Francis. *Drawing Surfaces and Their Deformations: The Tobacco Pouch Eversions of the Sphere*, Mathematical Modelling, 1 (1980), pp. 273–281.

[Fr2] G. Francis. *A Topological Picturebook*, Springer-Verlag Inc., New York, 1987.

[FrMo] G. Francis and B. Morin. *Arnold Shapiro's Eversion of the Sphere*, The Mathematical Intelligencer, 2 (1979), pp. 200–203.

[FrSu] G. Francis, J. Sullivan. *Visualizing a Sphere Eversion*, IEEE Transactions on Visualization and Computer Graphics, 10 (2004), pp. 509–515.

[Ku] R. Kusner. *Conformal Geometry and Complete Minimal Surfaces*, Bulletin of the American Mathematical Society, 17 (1987), pp. 291-295.

[LaSm] R.K. Lashof, S. Smale. *On the Immersions of Manifolds in Euclidean Space*, Annals of Mathematics, 68 (1958), pp. 562–583.

[LiYa] P. Li and S.T. Yau. *A New Conformal Invariant and its Applications to the Willmore Conjecture and the First Eigenvalue of Compact Surfaces*, Inventiones Mathematicae, 69 (1982), pp. 269–291.

[Mc] M. J. McGuffin. *sphereEversion* program, http://www.dgp.toronto.edu/ mjmcguff/eversion/.

[Sm] S. Smale, *The Classification of Immersions of Spheres in Euclidean Spaces*, Annals of Mathematics, 69 (1959) pp. 327–344.

[Su1] J. Sullivan, *Sphere Eversions: From Smale Through "The Optiverse"*, http://www.math.uiuc.edu/ jms/Papers/isama/eversions.pdf.

[Su2] J. Sullivan, *A History of Sphere Eversions*, http://torus.math.uiuc.edu/jms/Papers/isama/color/opt2.htm.

Matthew Litman

Pennsylvania State University
Department of Mathematics
McAllister Building
University Park, PA 16801

Email: `mlitman422@gmail.com`
Erdős Number: 4

Education

B.S. Mathematics w. Honors, Schreyer Honors College, Penn State University
Minors: Statistics, Geophysics
Thesis: *Sphere Eversion: Analysis of a Veridical Paradox*
Advisor: Sergei Tabachnikov
Phi Kappa Phi

MASS Program 2015 Graduate. Department of Mathematics, Penn State University

Research Experience

Individual Research under Zhiren Wang, Penn State University
August 2016 - Present

Project in Dynamical Systems & Number Theory: *Mobius Disjointness for Skew Products on the Heisenberg Nilmanifold*

Muhlenberg College, Department of Mathematics and Computer Science REU
May - July 2016

Project in Topological Graph Theory: Classifying what graphs are realizable as intersection graphs of neighborhoods in $\mathbb{Z}$ endowed with the Furstenburg topology (Mentor: Eugene Fiorini)

Project in Algebraic Number Theory: Guaranteeing the existence of finite fields with adjacent primitive nth roots of unity (Mentor: Joshua Harrington)

William G. Pritchard Fluid Mechanics Laboratory, Department of Mathematics, Penn State University
December 2015 - May 2016

Studied pattern formation of a reactive micellar interface in a Hele-Shaw cell

Work conducted under Andrew Belmonte

Penn State University/Peking University REU
July - August 2015

Worked on collaborative projects in Differential Geometry (Symplectic Geometry, in particular)

Publications

Journal Articles

1. *On Consecutive Primitive nth Roots of Unity Modulo q*, joint with T. Brazelton, J. Harrington, and S. Kannan, *Journal of Number Theory* 174 (2017) pp. 494-504.

2. *Flow Instabilities Due to Interfacial Surfactant/Polar Oil Material in a Hele-Shaw Cell*, joint with A. Belmonte and Z. Niroobakhsh (2016). Submitted, *Physical Review Letters E* (2017).

Presentation

Young Mathematicians Conference, Ohio State University, August 2016. *On Consecutive Primitive nth Roots of Unity Modulo q*.

Mid-Atlantic Seminar on Numbers I (MASON I), Towson University, October 2016. *On Consecutive Primitive nth Roots of Unity Modulo q*.

Joint Mathematics Meetings, Atlanta, GA, January 2017. Poster Presentation: *Consecutive Elements of Order n in* $\mathbb{F}_q$.

Undergraduate Exhibition, Penn State University, April 2017. Poster Presentation: *Möbius Disjointness for Skew Products on the Heisenberg Nilmanifold*.

Mathematical Outing for Undergraduates, Penn State University, April 2017. *Continued Fractions and Symbolic Coding via the Moduli Space* $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.

Awards and Positions

Awards

Spring 2017 Mary Lister McCammon Award, Department of Mathematics, Penn State University

2016-2017 Undergraduate Research Grant on Mobius Disjointness, Eberly College of Science

2016-2017 Class of 1922 Memorial Scholarship

Fall 2015 Overall Outstanding Performance, MASS Program

2015-2017 Remmey Family Trustee Scholarship

2015-2017 Aldine Cech Honors Scholarship

2013-2017 Penn State Academic Grant

2013-2015 Strawbridge Memorial Trustee Scholarship

Positions

2016-2017 Secretary for Math Club, Penn State University