THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENTS OF MATHEMATICS AND FINANCE

APPROXIMATING PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO FINANCIAL OPTION PRICING

SIMRANJEET SINGH
SPRING 2017

A thesis
submitted in partial fulfillment
of the requirements
for baccalaureate degrees
in Mathematics and Finance
with Interdisciplinary Honors
in Mathematics and Finance

Reviewed and approved* by the following:

Anna Mazzucato
Professor of Mathematics
Thesis Supervisor

Brian Davis
Clinical Associate Professor of Finance
Thesis Supervisor

Leonid Vaserstein
Professor of Mathematics
Honors Adviser

*Signatures are on file in the Schreyer Honors College.

Abstract

Herein, we first consider the basic assumptions and derivation of the original Black-Scholes model via a generalized portfolio replication argument. Thereafter, we analyze the Dyson-Taylor commutator method for short-time expansions of the heat kernel (typically referred to as the Green's function), and apply it to the Constant Elasticity of Variance (CEV) local volatility model, in order to find closed-form approximate solutions for the pricing kernel.

Table of Contents

Acknowledgements

A great deal of thanks goes to a select group of individuals when it comes to the culmination of both this thesis as well as my undergraduate years. First and foremost, I would like to give an incredible amount of thanks to Dr. Anna Mazzucato for her immense patience, kindness, and understanding. Without her guidance, this thesis would certainly not have been attempted, let alone possible. Dr. Brian Davis, for *maybe* reading this thesis (if at all), you rock.

There are three friends in particular that deserve special mention: Sarah "Sushi" Sherif El Marsafawi, Jonathan "Papi" Arguello, and Matthew "The Man Who Shows My Beard the Love" Litman. Sushi, for being the older sister I never had (and causing me many a headache; she should certainly know better), will always have my thanks. Needless to say, my journey would not have been as amusing or bearable without her companionship. Jonathan, for being an honest man beyond compare, and a brother across the multiple dimensions of spacetime, has my immense gratitude. Matthew, for always being there to provide completely inappropriate humor (as if any other kind would suffice), memorable times, and mathematical enlightenment, you have my sincerest appreciation. Your friendship will always be valued beyond measure (of the generalized Hausdorff kind).

The various professors at Penn State Lehigh Valley are owed a great debt for arming me with a sense of hope and passion that has shaped me to this day. Namely, Professor Bob Wolfe (pardon the Wikipedia entries), Dr. Barb Cantalupo (were the two roads really one?), Mrs. Instructor Dr. Coordinator Liz Flaherty (the columns will always be Corinthian), and Dr. Todd Retzlaff (for all those saddle points, vector fields, and much much more).

Last but not least, I would like to thank my parents and older brother for ingraining me with the lessons that have come to define who I am as a person (and mathematician), for teaching me the value of sacrifice and dedication, and most importantly, for their unwavering love and support. May I always do them proud.

Chapter 1

Introduction and Literature Review

1.1 Analyzing Assumptions

In order to construct the original model proposed by Fisher Black and Myron Scholes, we first establish and analyze a list of crucial assumptions taken directly from [1], which the model is contingent upon:

(a) The short-term interest rate (typically referred to as the *risk-free* rate) is known and is constant through time.

(b) The price of a stock follows a random walk in continuous time (Geometric Brownian Motion), with a variance rate proportional to the square of the price. Consequently, the distribution of possible stock prices at the end of any finite interval is *log-normally* distributed. The variance rate of the return on the stock is constant.

(c) The stock pays no dividends or other cash flows.

(d) The option is *European*, and so, can only be exercised at maturity.

(e) There are no transaction costs in buying or selling the stock or the option.

(f) It is possible to borrow any fraction of the price of a security to buy it or hold it, at the short-term interest rate.

(g) There are no penalties to short selling. Any seller that does not own a security will accept the price of the security from a buyer, and agree to settle with the buyer on some future date by paying an amount equal to the price of the security on the date.

In actual practice, it is clear that the above assumptions hold tenuously at best, and are blatantly violated at worst. Consider (a) for example. A variety of models ranging from Chen's stochastic three-factor model to the Black-Derman-Toy lattice model suggest that the short-rate is anything but constant. The assumption of constant variance in (b) is done away with by practitioners through the use of local and stochastic volatility models. Further, we have that [15] suggests the notion of normally-distributed returns is undermined by the occurrence of tail events on a more frequent basis than would be suggested by a simple normal curve. Moreover, there exists a large variety of dividend-paying stocks (in violation of (c)), while transaction costs are certainly present, ranging from a few basis points to a more significant percentage of the transaction amount. We see in [10] that there can be significant constraints for short sales, despite their role in helping control the overpricing of securities.

How closely we attempt to enforce the above assumptions is a matter best left to practitioners. Nonetheless, we can use these points to establish the original Black-Scholes model, which can in turn be extended to the CEV model.

1.2 Underlying Principles

For further exploration of the key definitions and theorems that follow, many of which are standard in the field of financial engineering and risk management, the interested reader should consult the work of McDonald [6] and Hull [9]. For a more advanced treatment there is also the work of Gatheral [7] and lectures by Emanuel Derman.

Definition 1. *(Trading Strategy)*
Given n assets with values $S_1(t), \ldots, S_n(t)$ at time t, the standard definition of a *trading strategy* is an n-dimensional stochastic process $w_1(t), \ldots, w_n(t)$ that represents the allocation of capital into each respective asset at time t.

Hence, we have that the value of the portfolio at time t is given by $\sum_{i=1}^{n} w_i(t)S_i(t)$

Definition 2. *(Self-financing)*
A trading strategy is said to be *self-financing* if the change in portfolio value is strictly due to changes in the value of the assets, and not due to the inflow or outflow of funds.

This implies that a strategy is self-financing if

$$\begin{aligned} d\Pi(t) &= d\left(\sum_{i=1}^{n} w_i(t)S_i(t)\right) \\ &= \sum_{i=1}^{n} w_i(t)dS_i(t). \end{aligned} \tag{1.1}$$

Definition 3. *(Complete Market)*
A market model is a *complete market* if any financial derivative admits a self-financing strategy with a value greater than zero, which can also replicate the payout of the derivative as well.

Definition 4. *(Arbitrage Opportunity)*
An *arbitrage opportunity* is a self-financing trading strategy that has the following properties with respect to the portfolio value:

$$\Pi(t) \leq 0,$$
$$P[\Pi(T) > 0] = 1.$$

That is, the initial portfolio value is zero or negative, and the value of the portfolio at time T will be positive with absolute certainty. Since the strategy is self-financing, no external funds are required to produce this wealth, which effectively eliminates any associated risk of holding the portfolio.

Definition 5. *(Filtration)*
Given a measurable space $(\Omega, \mathcal{F})$, a *filtration* is a sequence of $\sigma-$algebras $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{F}$, where $t \in \mathbb{R}$, and $t_1 \leq t_2 \Rightarrow \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$.

Definition 6. *(Adapted Process)*

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an index set I with a total order $(\leq)$, and a measurable space (S, Σ), a stochastic process $X : I \times \Omega \to S$ is said to be *adapted* to the filtration $\mathcal{F}_i$ if $X_i : \Omega \to S$ is a $(\mathcal{F}_i, \Sigma)$-measurable function for each particular $i \in I$.

Perhaps a bit more informally, we can consider an adapted process as one where for every realization at time $t \geq 0$, X_t is only known at time t.

Definition 7. *(Continuous-Time Martingale)*

A real-valued continuous-time stochastic process $\{X_t\}_{t\geq 0}$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is called a *continuous-time martingale* if

(i) $\mathbb{E}(X_t) < \infty, \ \ \forall t \geq 0,$

(ii) $\mathbb{E}(X_t|\mathcal{F}_s) = X(s)$ almost surely $\forall s \in (0, t]$.

Lemma 1. *(Itô's Lemma)*

Assume X_t is a drift-diffusion process that satisfies the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dB_t, \tag{1.2}$$

where B_t is a standard Brownian motion (or Weiner Process). Let $f(X_t, t) \in \mathcal{C}^2$. Then

$$df = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial X_t} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial X_t^2} \right) dt + \sigma_t \frac{\partial f}{\partial X_t} dB_t. \tag{1.3}$$

This immediately implies that $f(X_t, t)$ is itself a drift-diffusion process.

Theorem 1. *(First Fundamental Theorem of Asset Pricing)*

A market is free of arbitrage if and only if there is a probability measure $\mathbb{Q}$, with $\mathbb{Q}(\omega) > 0$, $\forall \omega \in \Omega$, such that every discounted price process is a $\mathbb{Q} - martingale$. Such a measure is called an equivalent martingale measure (EMM), or *risk-neutral* probability measure.

Theorem 2. *(Second Fundamental Theorem of Asset Pricing)*

An arbitrage-free market is complete if and only if there exists a unique risk-neutral probability measure $\mathbb{Q}$.

We now consider a particular type of contingent claim that was used in the original derivation.

Definition 8. *(Call Option)*

A *call option* is a contingent claim whose payoff is a function of the underlying risky asset. Such contracts are typically designated as either *European* or *American*. The former grants the holder of the contract the right to exercise the option, and purchase a security at a predetermined strike price only *on* a predefined expiry date. An American option, however, allows the holder to exercise at *any time* up to and including the expiry date.

1.3 Derivation via Replication (Hedging)

What follows now is a derivation of the original Black-Scholes model as discussed in [1] and [4]. Consider a portfolio comprised of one share of the underlying risky asset, and θ shares of the option. Denoting the option value as $V = V(t)$, we have that the portfolio can be represented as $\Pi(t) = \theta V + S_t$. Recalling the assumption that the dynamics of S_t follow a Geometric Brownian Motion, we have: $dS_t/S_t = \mu dt + \sigma dB_t$. Combining this with Itô's Lemma yields

$$\begin{aligned} d\Pi &= \theta dV + dS_t \\ &= \left(\theta\frac{\partial V}{\partial t} + \theta\mu S_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\theta\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} + \mu S_t\right)dt + \left(\theta\sigma S_t\frac{\partial V}{\partial S_t} + \sigma S_t\right)dB_t. \end{aligned} \tag{1.4}$$

In order for the portfolio to be perfectly riskless (i.e. perfectly hedged), we set $\theta = -\left(\frac{\partial V}{\partial S_t}\right)^{-1}$. Substituting this into (1.4), we also take into account

$$d\Pi = r\Pi dt = r[\theta V + S_t]dt, \tag{1.5}$$

since a riskless asset will grow at the *constant* short-term rate r. Canceling the μS_t term, we obtain

$$\left(\theta\frac{\partial V}{\partial t} + \frac{1}{2}\theta\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2}\right)dt = r[\theta V + S_t]dt. \tag{1.6}$$

Dividing by dt, and gathering all the terms to one side gives us the Black-Scholes equation for the option value V, which we recognize as a second-order parabolic PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} + rS_t\frac{\partial V}{\partial S_t} - rV = 0, \tag{1.7}$$

which can be interpreted as the measuring the difference between the hedged option portfolio, and the return on the riskless asset. For there to be no arbitrage, we must have this difference equal to zero, in the case of European options (although this condition need not hold for American options). The associated solution to (1.7), with the boundary condition $V(S_t, T) = \max\{S_T - K, 0\}$, is given by

$$V(S_t, T) = S_t\mathbf{\Phi}(d_1) - Ke^{-r(T-t)}\mathbf{\Phi}(d_2), \tag{1.8}$$

where $\mathbf{\Phi}(\cdot)$ is the normal cumulative distribution function (CDF), and

$$d_1 = \frac{\log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

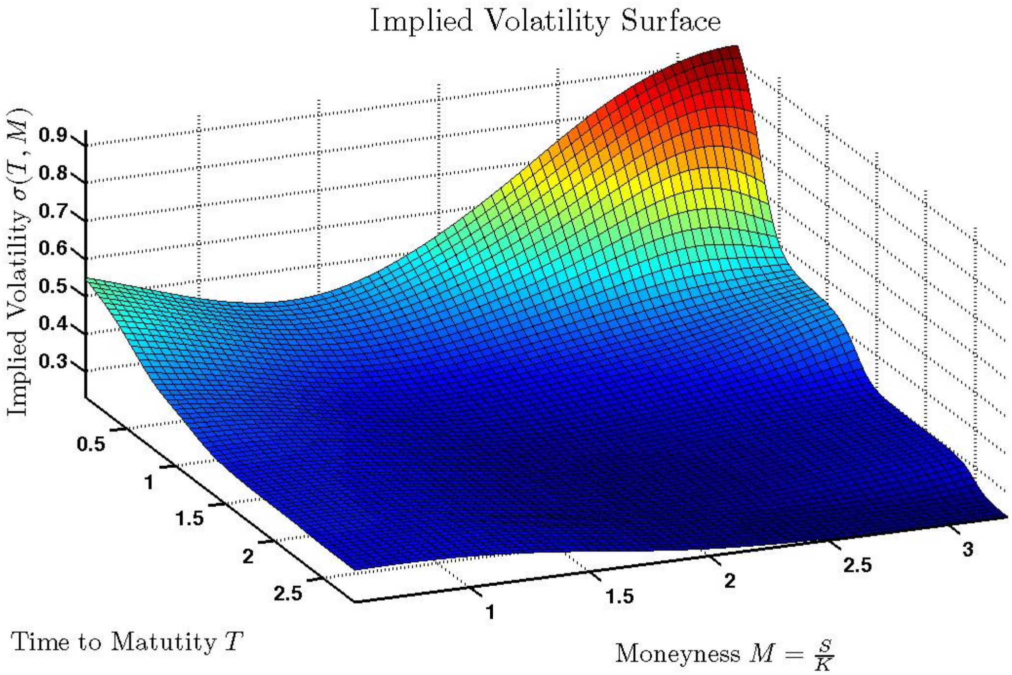

Figure 1.1: A plot of the market implied volatility from [14], as a function of the time to expiry for the underlying option, as well as the ratio of how *in the money* the option is, i.e. its moneyness.

1.4 Introducing the CEV Model

Up until now, we have used the assumption of constant volatility σ (refer to assumption (b)). However, practical examples such as *Black Monday* (October 1987), strongly undermine the idea of constant volatility, and instead, suggest something called a volatility smile. That is, when using the Black-Scholes formulation to back-out the volatility *implied* by market prices, we observe a skewed curve, rather than a constant function. An example of this can be seen in Figure 1.1. As a consequence of the volatility smile observed in markets, a local volatility model is better suited to describe the associated dynamics. Thus, we arrive at CEV model (see [3] for the original consideration by Cox and Ross) where L is given by

$$L = \frac{1}{2}\sigma(S_t)^2 S_t^2 \frac{\partial^2}{\partial S_t^2} + rS_t \frac{\partial}{\partial S} - r, \tag{1.9}$$

wherein $\sigma(S_t) = \sigma S_t^{\beta-1}$, and σ, β, and r are constant. When $\beta < 0$ we have the *leverage effect*, where decreases in market prices tend to cause a sharp peak in volatility (typically seen in equity markets). We also have the *reverse* leverage effect for $\beta > 0$, where the volatility of the asset's price increases as the price itself increases (suggested for modeling commodities markets).

1.5 Alternative Models for Thought

A more complex model for *stochastic volatility* was proposed by Heston in [5]. The associated dynamics are described by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_1, \tag{1.10}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_2, \tag{1.11}$$

where V_t is the variance of the asset S_t, κ is the *mean reversion rate* for the variance, θ is the long run *average variance*, and σ is the *volatility of variance*. In order to account for the leverage effect mentioned at the conclusion of Section 1.4, the respective Weiner processes W_1, W_2 are assumed by Heston to be correlated: $dW_1 \cdot dW_2 = \rho dt$, while the variance of the process in (1.11) is always positive.

A further analysis of Heston's model in [12] by Mikhailov and Nögel discusses how if $2\kappa\theta > \sigma^2$, then the stochastic processes for the variance cannot reach zero. Moreover, we have that the deterministic part of (1.11) is asymptotically stable if $\kappa > 0$, with an equilibrium point at $V_t = \theta$. Applying Itô's lemma and an arbitrage argument similar to the one in the derivation of Section 1.3, Mikhailov and Nögel obtain the following PDE (with the time-dependence subdued):

$$\begin{aligned}\frac{\partial C}{\partial t} + \frac{S^2 V}{2}\frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - (r-q)C[\kappa(\theta - V) - \lambda V]\frac{\partial C}{\partial V}& \\ + \frac{\sigma^2 V}{2}\frac{\partial^2 C}{\partial V^2} + \rho\sigma S V\frac{\partial^2 C}{\partial S \partial V} = 0&,\end{aligned} \tag{1.12}$$

where r is the interest rate, q is the dividend yield of the asset S, and λ is the market price of *volatility risk*. The challenge with using such model, as is the case with the CEV model from the previous section, is that closed-form solutions involve complex expressions for the associated pricing kernel. For example, the CEV model has an exact solution in the case of $\beta = 1$, where it is simply the original Black-Scholes model, while fractional values of β require expressions in terms of chi-square distributions (see [11] for a more detailed analysis). Thus, it is valuable to have closed-form approximates for the pricing kernel that are easily and quickly computable. This is primary motivation for the method introduced in [2], and discussed herein.

Chapter 2

Approximating the Green's Function

2.1 Taylor Expansion

From this point forward, we consider a generalized form of the operator (1.9). Although [2] considers variable coefficients in both spatial and temporal variables, where we treat the stock price S_t as the spatial variable, this work is limited to looking at coefficients only dependent on the spatial variable:

$$L := \frac{1}{2}a^2(x)\partial_x^2 + b(x)\partial_x + c(x). \tag{2.1}$$

The *Dyson-Taylor commutator method* discussed in [2], which we follow for the remainder of the section, can be used used to obtain small-time asymptotic expansions for the initial value problem

$$\begin{cases} \partial_t u(x,t) - Lu(x,t) = 0, & t \in (0,T),\ \ x > 0 \\ u(x,0) = f(x) & x > 0, \end{cases} \tag{2.2}$$

where L is given by (2.1). The method's name is derived from the Taylor expansion of the operator discussed in this section, and the *Dyson series* obtained from application of both Duhamel's Principle and commutators in Section 2.2 From [2] and [13] we have that the error for the nth order approximation evaluated at time $t > 0$ is on the order of $t^{(n+1)/2}$. That is, the approximation is effective for only *short-time* expansions. However, we can bootstrap the construction and obtain reasonable errors for larger time expansions as well. To firmly understand the method, we first recall from [8] that under certain restrictions on the operator L, and the initial state f, there exists a function $\mathcal{G}_t$ such that the solution to (2.2) can be written as

$$u(x,t) = \int_0^\infty \mathcal{G}_t(x-y)f(y)dy, \tag{2.3}$$

where $\mathcal{G}_t$ is referred to as the the fundamental solution or Green's function for (2.2). What the Dyson-Taylor commutator method allows us to do is calculate an approximation for $\mathcal{G}_t$ based on parabolic rescaling, the Taylor expansions of the variable coefficients of L, as well as commutator expansions.

We now consider parabolic rescaling which underlies the method. Let $z \in \mathbb{R}$ be an arbitrary base point, and $s > 0$ a parameter. Given a function $f(x,t)$, we denote

$$f^{s,z}(x,t) := f(z + s(x-z), s^2 t) \tag{2.4}$$

as the parabolic rescaling by s of the function f about z. For a function independent of time, we have $f^{s,z}(x) = f(z + s(x-z))$. In a similar way, we can define the rescaled version of the operator (2.1) as

$$L^{s,z} := \frac{1}{2}a^{s,z}(x)^2\partial_x^2 + sb^{s,z}(x)\partial_x + s^2 c^{s,z}(x), \tag{2.5}$$

which solves the associated rescaled problem

$$\begin{cases} \partial_t u^{s,z}(x,t) - L^{s,z}u^{s,z}(x,t) = 0, \\ u^{s,z}(x,0) = h^{s,z}(x), \end{cases} \tag{2.6}$$

The method we use relies on computing the Green's function of the rescaled problem (2.6) for $t = 1$. To do so, we use the Taylor expansion of the rescaled operator $L^{s,z}$ in (2.5), with respect to s about $s = 0$, up to a particular order n. That is, we simply expand the variable coefficients of $L^{s,z}$, and gather all the terms of the same order in s. We note that the rescaled operator can be written as

$$L^{s,z} = \sum_{k=0}^{n} s^k L_k^{s,z} + s^{n+1}\mathcal{R}, \tag{2.7}$$

where $L_k^{s,z}$ is the term corresponding to the k^{th}-derivative of the expansion, and we collect all the remainder terms from the Taylor expansion of the coefficients together in $\mathcal{R}$. Given a rescaled function $f^{s,z}(x)$ (as in 2.4), for $n = 2$, the appropriate Taylor expansion is

$$f^{s,z}(x) = f(z) + s(x-z)f'(z) + s^2(x-z)^2\frac{f''(z)}{2} + \mathcal{R}, \tag{2.8}$$

where $\mathcal{R}$ is the collection of all higher-order terms. In this way, we can find the terms of the Taylor expansion of $L^{s,z}$ in s at $s = 0$ for various degrees:

$$\begin{aligned} L_0^{s,z} &= \frac{1}{2}a(z)^2\partial_x^2, \\ L_1^{s,z} &= a(z)a'(z)(x-z)\partial_x^2 + b(z)\partial_x, \\ L_2^{s,z} &= \frac{1}{2}((a'(z))^2 + a(z)a''(z))(x-z)^2\partial_x^2 + b'(z)(x-z)\partial_x + c, \\ &\dots \end{aligned} \tag{2.9}$$

More compactly, we write

$$L^{s,z} = L_0^{s,z} + sL_1^{s,z} + s^2L_2^{s,z} + \mathcal{R}, \tag{2.10}$$

where $\mathcal{R}$ is the remainder term. We note that due to the iterative application of the *chain rule* in expanding the coefficients of $L^{s,z}$ above, the kth term in (2.10) will contain polynomial coefficients with an order $\leq k$ in $(x - z)$. Moreover, we have that the Green's function for $L_0^{s,z}$ is explicitly calculated in Section 2.2 ahead.

2.2 Applying Duhamel's Principle and Introducing Commutators

Given $\mathcal{G}_t^L$, the solution to (2.2), we can separate L into a second-order operator with constant coefficients (L_0), and a second-order operator with variable coefficients (V):

$$L = L_0 + V. \tag{2.11}$$

We use Duhamel's Principle as found in [8], to obtain the integral representation

$$\mathcal{G}_t^L = e^{tL_0} + \int_0^t e^{(t-\tau_1)L_0} V(\tau_1) \mathcal{G}_{\tau_1}^L d\tau_1. \tag{2.12}$$

Moreover, we can iteratively apply this principle in order to derive a recursive formulation for $\mathcal{G}_t^L$ as follows:

$$\begin{aligned}
\mathcal{G}_t^L &= e^{tL_0} + \int_0^t e^{(t-\tau_1)L_0} V(\tau_1) e^{\tau_1 L_0} d\tau_1 \\
&+ \int_0^t \int_0^{\tau_1} e^{(t-\tau_1)L_0} V(\tau_1) e^{(\tau_1-\tau_2)L_0} V(\tau_2) e^{\tau_2 L_0} d\tau_2 d\tau_1 + \dots \\
&+ \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k+1}} e^{(t-\tau_1)L_0} V(\tau_1) e^{(\tau_1-\tau_2)L_0} V(\tau_2) \dots V(\tau_{k+1}) \mathcal{G}_{\tau_{k+1}}^L d\tau_{k+2} d\tau_{k+1} \dots d\tau_1.
\end{aligned} \tag{2.13}$$

Using the iterative process above, we perform a similar manipulation to $\mathcal{G}_t^{L^{s,z}}$, the solution to the rescaled problem (2.6). If we let L_0 be $L_0^{s,z}$, the first term of the Taylor expansion of $L^{s,z}$, we then have $V^{s,z} = L^{s,z} - L_0^{s,z}$. Applying (2.13) at $t = 1$ we have that

$$\begin{aligned}
\mathcal{G}_1^{L^{s,z}} &= e^{L_0^{s,z}} + s \int_0^1 e^{(1-\tau_1)L_0^{s,z}} L_1^{s,z} e^{\tau_1 L_0^{s,z}} d\tau_1 \\
&+ s^2 \left(\int_0^1 \int_0^{\tau_1} e^{(1-\tau_1)L_0^{s,z}} L_1^{s,z} e^{(\tau_1-\tau_2)L_0^{s,z}} L_1^{s,z} e^{\tau_2 L_0^{s,z}} d\tau_2 d\tau_1 + \int_0^1 e^{(1-\tau_1)L_0^{s,z}} L_2^{s,z} e^{\tau_1 L_0^{s,z}} d\tau_1 \right) \\
&+ \mathcal{R},
\end{aligned} \tag{2.14}$$

where $\mathcal{R}$ represents all the higher-order terms and is included in the remainder. Now in order to obtain an approximation for the Green's function $\mathcal{G}_t^L$ of the original problem (2.2), we need to compute the kernels of the operators contained within (2.14), which relies on using the semigroup property of e^{tL_0} in order to evaluate the integrals above. Thus, we introduce commutators and some associated useful notation.

Given two operators A and B, we let $[A, B] := AB - BA = -[B, A]$ denote their commutator. Further, we define $ad_A(B) := [A, B]$. For a given integer k, we have the following recursion

$$ad_A^k(B) := ad_A(ad_A^{k-1}(B)). \tag{2.15}$$

Using an analogous Baker-Campbell-Hausdorff identity (from [13]) specific to this case, we obtain

$$e^{\theta L_0^{s,z}} P = \left(P + \sum_{i}^{\infty} \frac{\theta^i}{i!} ad^i_{L_0^{s,z}}(P) \right) e^{\theta L_0^{s,z}} =: Q(P,\theta,x,z,\partial) e^{\theta L_0^{s,z}} \tag{2.16}$$

for any $\theta \in (0,1)$ and differential operator $P = P(x,\partial)$ that has polynomial coefficients in x.

We note here that Q can be explicitly calculated as the series in (2.16) terminates in finitely many terms. To observe this, let P_k be a second-order differential operator containing polynomial coefficients of maximum degree k. It then follows that $ad^m_{L_0^{s,z}}(P_k) = 0$ for $m > k$. Specifically, we have that $ad^2_{L_0^{s,z}}(L_1^{s,z}) = 0$ and $ad^3_{L_0^{s,z}}(L_2^{s,z}) = 0$. Using (2.16) to compute the integrals in (2.14), we find

$$\begin{aligned} \int_0^1 e^{(1-\tau_1)L_0^{s,z}} L_1^{s,z} e^{\tau_1 L_0^{s,z}} d\tau_1 &= \int_0^1 (L_1^{s,z} + (1-\tau_1)[L_0^{s,z}, L_1^{s,z}]) e^{L_0^{s,z}} d\tau_1 \\ &= \left(L_1^{s,z} + \frac{1}{2}[L_0^{s,z}, L_1^{s,z}] \right) e^{L_0^{s,z}}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\int_0^1 \int_0^{\tau_1} e^{(1-\tau_1)L_0^{s,z}} L_1^{s,z} e^{(\tau_1-\tau_2)L_0^{s,z}} L_1^{s,z} e^{\tau_2 L_0^{s,z}} d\tau_2 d\tau_1 \\ &= \int_0^1 \int_0^{\tau_1} (L_1^{s,z} + (1-\tau_1)[L_0^{s,z}, L_1^{s,z}])(L_1^{s,z} + (1-\tau_2)[L_0^{s,z}, L_1^{s,z}]) e^{L_0^{s,z}} d\tau_2 d\tau_1 \\ &= \left(\frac{1}{2}(L_1^{s,z})^2 + \frac{1}{3} L_1^{s,z}[L_0^{s,z}, L_1^{s,z}] + \frac{1}{6}[L_0^{s,z}, L_1^{s,z}] L_1^{s,z} + \frac{1}{8}[L_0^{s,z}, L_1^{s,z}]^2 \right) e^{L_0^{s,z}}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\int_0^1 e^{(1-\tau_1)L_0^{s,z}} L_2^{s,z} e^{\tau_1 L_0^{s,z}} d\tau_1 \\ &= \int_0^1 \left(L_2^{s,z} + (1-\tau_1)[L_0^{s,z}, L_2^{s,z}] + \frac{(1-\tau_1)^2}{2}[L_0^{s,z}, [L_0^{s,z}, L_2^{s,z}]] \right) e^{L_0^{s,z}} d\tau_1 \\ &= \left(L_2^{s,z} + \frac{1}{2}[L_0^{s,z}, L_2^{s,z}] + \frac{1}{6}[L_0^{s,z}, [L_0^{s,z}, L_2^{s,z}]] \right) e^{L_0^{s,z}}. \end{aligned} \tag{2.19}$$

Thus, we have that (2.14) can be written as

$$e^{L^{s,z}} = (1 + sQ_1 + s^2 Q_2) e^{L_0^{s,z}} + \mathcal{R}, \tag{2.20}$$

where

$$\begin{aligned} Q_1 &= L_1^{s,z} + \frac{1}{2}[L_0^{s,z}, L_1^{s,z}], \\ Q_2 &= \frac{1}{2}(L_1^{s,z})^2 + \frac{1}{3} L_1^{s,z}[L_0^{s,z}, L_1^{s,z}] + \frac{1}{6}[L_0^{s,z}, L_1^{s,z}] L_1^{s,z} + \frac{1}{8}[L_0^{s,z}, L_1^{s,z}]^2 \\ &\quad + L_2^{s,z} + \frac{1}{2}[L_0^{s,z}, L_2^{s,z}] + \frac{1}{6}[L_0^{s,z}, [L_0^{s,z}, L_2^{s,z}]]. \end{aligned} \tag{2.21}$$

2.3 Evaluating Commutators

Hence, to derive the second-order approximation of $\mathcal{G}_t^{L^{s,z}}$, we must have evaluate the commutators above. Here we take a moment to recall that all of the functions in the expansion above are evaluated at $(0, z)$. That is, $a = a(0, z), a' = a'(0, z)$, etc. We find

$$
\begin{aligned}
&[L_0^{s,z}, L_1^{s,z}] = a^3 a' \partial_x^3, \\
&[L_0^{s,z}, L_1^{s,z}]^2 = a^6 a'^2 \partial_x^6, \\
&L_1^{s,z}[L_0^{s,z}, L_1^{s,z}] = a^4 a'^2 (x-z)\partial_x^5 + ba^3 a' \partial_x^4, \\
&[L_0^{s,z}, L_1^{s,z}]L_1^{s,z} = a^4 a'^2 (x-z)\partial_x^5 + (b + 3aa')a^3 a' \partial_x^4.
\end{aligned}
\tag{2.22}
$$

In order to compute commutators of the form $[AB, C]$, we note that by construction we have

$$
\begin{aligned}
A[B, C] + [A, C]B &= A(BC - CB) + (AC - CA)B \\
&= ABC - ACB + ACB - CAB \\
&= ABC - CAB \\
&= [AB, C].
\end{aligned}
\tag{2.23}
$$

Specific to our case, we find that $[\partial_x^2, x - z] = 2\partial_x$ and $[\partial_x^2, (x - z)^2] = 2 + 4(x - z)\partial_x$. When considering the case of iterated commutators, we make use of the following from [2] and [13]

Lemma 2. For integers $i, j \geq 1$ we have

$$
\partial_x^i (x-z)^j u(x) = \sum_{k=0}^{\min(i,j)} \binom{i}{k}\binom{j}{k} k!(x-z)^{j-k}\partial_x^{j-k} u(x). \tag{2.24}
$$

Thus, we have

$$
\begin{aligned}
[L_0^{s,z}, L_2^{s,z}] &= a^2(a'^2 + aa'')(x-z)\partial_x^3 + a^2\left(b' + \frac{a'^2}{2} + \frac{aa''}{2}\right)\partial_x^4, \\
[L_0^{s,z}, [L_0^{s,z}, L_2^{s,z}]] &= a^4(a'^2 + aa'')\partial_x^4, \\
(L_1^{s,z})^2 &= (aa'(x-z))^2\partial_x^4 + 2(a^2a'^2 + aa'b)(x-z)\partial_x^3 + (aa'b + b^2)\partial_x^2.
\end{aligned}
\tag{2.25}
$$

From here, we observe that the approximation kernel of $\mathcal{G}_1^{L^{s,z}}$ is obtained by applying a differential operator containing polynomial coefficients, to the Green's function of e^{tL_0}. Given a smooth function ϕ, the convolution operator with ϕ is defined as $C_\phi f(x) := \phi \star f(x) = \int \phi(x - y)f(y)dy$. From our earlier statement in Section 2.1, we have that the distribution kernel of $e^{L_0^{s,z}}$ is

$$
e^{L_0^{s,z}}(x, y) = \frac{1}{\sqrt{2\pi a^2}} \exp\left(\frac{-|x-y|^2}{2a^2}\right), \tag{2.26}
$$

and so, it is a convolution operator. Noting that $\partial_x C_\phi = C_{\partial_x \phi}$, and that the distribution kernel of C_ϕ is $C_\phi(x,y) = \phi(x-y)$ allows us to compute the terms of the distribution kernel for the approximation we are aiming to compute: $G_0(x,y;z) = e^{L_0^{s,z}}$ (from (2.26)), G_1, and G_2. Explicitly, we find that

$$\begin{aligned} G_1(x,y;z) &= \left(\left(L_1 + \frac{1}{2}[L_0, L_1]\right) e^{L_0}\right)(x,y) \\ &= \frac{1}{\sqrt{2\pi a^2}} e^{\frac{-|x-y|^2}{2a^2}} \left[\left(\frac{3aa' - 2b}{2a^2}\right)(x-y) - \frac{a'}{2a^3}(x-y)^3 + (x-z)\left(\frac{(x-y)^2 - a^2}{a^3}\right)\right]. \end{aligned} \tag{2.27}$$

Moreover, for the CEV model described in Section 1.4, we have $a = \sigma z^\beta, b = rz$, and $z = z(x,y)$, giving us

$$\begin{aligned} G_1^{CEV}(x,y;z) = \frac{1}{\sigma z^\beta \sqrt{2\pi}} \Bigg[&\left(\frac{3\beta\sigma^2 z^{2(\beta-1)} - 2r}{2\sigma z^{\beta-1}}\right)\left(\frac{x-y}{\sigma z^\beta}\right) - \left(\frac{\beta\sigma z^{\beta-1}}{2}\right)\left(\frac{x-y}{\sigma z^\beta}\right)^3 \\ &+ \left(\frac{x-z}{\sigma z^\beta}\right)\left(\left(\frac{x-y}{\sigma z^\beta}\right)^2 - 1\right)\Bigg] e^{\frac{-|x-y|^2}{2\sigma^2 z^{2\beta}}}. \end{aligned} \tag{2.28}$$

We perform the kernel expansion specifically at $z = x$, since z can be any function from $\mathbb{R}^{2N} \to \mathbb{R}^N$ such that $z(x,x) = x$ and all of its derivatives are bounded. Doing so gives us the price of a European call option in closed form. Moreover, we note that the convolution with the approximate Green's function can be calculated precisely, yielding the price of the call option in closed form. Generalizing the $t = 1$ assumption gives us that

$$\mathcal{G}_t(x,y) = \sum_{k=0}^{n} t^{k/2} \mathcal{G}_t^{[k]}(x,y,;z) + \mathcal{R}, \tag{2.29}$$

where $\mathcal{G}_t^{[0]} = \frac{1}{\sqrt{2t\pi a^2}} e^{\frac{-|x-y|^2}{2ta^2}}$. We use the main result from [2] for the nth order approximate kernel, where we have that $\mathcal{F}^n$ are the functions computed using the Dyson-Taylor Commutator method:

$$\mathcal{G}_t^{[n]}(x,y) = \frac{1}{\sqrt{t}} \mathcal{F}^n\left(z, z + \frac{x-z}{\sqrt{t}}, \frac{x-y}{\sqrt{t}}\right) \mathcal{G}_t^{[0]}\left(\frac{x-y}{\sqrt{t}}\right), \tag{2.30}$$

to compute the nth-order approximate price for a European call option with a strike price of K, and time to expiry t:

$$U^{[n]}(t,x) = \int_0^\infty \mathcal{G}_t^{[n]}(x,y) \max(0, y-K) dy. \tag{2.31}$$

Chapter 3

Maple Code

3.1 Maple Code for the Taylor Expansion of $L^{s,z}$

First, we define the rescaled operator, with only spatial dependence, using an arbitrary function $u(x)$:

> $L := \frac{1}{2}a^2(z + s\cdot(x-z))\cdot$ `diff(`$u(x)$`, [x$2]) +` $s\cdot b(z+s\cdot(x-z))\cdot$ `diff(`$u(x), x$`) +` $s^2\cdot c(z+s\cdot(x-z))$`;`

Next we initialize an empty array to store terms of the Taylor expansion. In this case, we limit ourselves to a second-order approximation.

```
> Terms := Array(1 ..3)
```

$$Terms := [\,0\;0\;0\,]$$

We write a simple *for-loop*, in order to implement the sum in (2.7). The *simplify(f)* function reduces the expression f to its most simplest form, with the given parameters. The *eval(*$f, x = \cdot$*)* function evaluates f at the value x. Lastly, the *diff(*$f, [x\$n]$*)* function calculates the nth derivative of f with respect to x.

```
> for i from 0 to 2 do
  Terms[i + 1] := (s^i)· simplify(eval(diff(L, [s$i]), s = 0))
  end do;
```

$$Terms_1 := \tfrac{1}{2}a(z)^2 D^{(2)}(u)(x)$$

$$Terms_2 := s\left(a(z)D^{(2)}(u)(x)D(a)(z)(x-z) + b(z)D(u)(x)\right)$$

$$Terms_3 := s^2\left((x-z)^2\left(a(z)D^{(2)}(a)(z) + D(a)(z)^2\right)D^{(2)}(u)(x) + 2D(b)(z)(x-z)D(u)(x) + 2c(z)\right)$$

The $D(a)(z)$ and $D^{(2)}(a)(z)$ notation used in the output above is simply used to denote the first and second derivative of the arbitrary function a, evaluated at z. Note however, that $D(a)(z)^2$ represents the derivative of a, evaluated at z, squared: $(a'(z))^2$. Now we can simply add each of the components of the *Terms* array as well as a remainder $\mathcal{R}$, which incorporates higher-order terms.

```
> add(Terms) + R
```

$$\tfrac{1}{2}a(z)^2 D^{(2)}(u)(x) + s\left(a(z)D^{(2)}(u)(x)D(a)(z)(x-z) + b(z)D(u)(x)\right)$$

$$+s^2\left((x-z)^2\left(a(z)D^{(2)}(a)(z) + D(a)(z)^2\right)D^{(2)}(u)(x) + 2D(b)(z)(x-z)D(u)(x) + 2c(z)\right) + R$$

which coincides with the second order approximation obtained in [2]. Further research is currently being conducted, using Maple, to compute the closed-form approximates of the commutators from Section 2.3

Bibliography

[1] Black, Fischer, and Myron Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81.3 (1973): 637-54. *JSTOR.* Web.

[2] Cheng, Wen, Nick Costanzino, John Liechty, Anna Mazzucato, and Victor Nistor. "Closed-Form Asymptotics and Numerical Approximations of 1D Parabolic Equations with Applications to Option Pricing." *SIAM Journal on Financial Mathematics 2.1* (2011): 901-34. Web.

[3] Cox, John C., and Stephen A. Ross. "The Valuation of Options for Alternative Stochastic Processes." Journal of Financial Economics 3.1-2 (1976): 145-66. Science Direct. Web.

[4] Cox, John C., Stephen A. Ross, and Mark Rubinstein. "Option Pricing: A Simplified Approach." Journal of Financial Economics 7.3 (1979): 229-63. Web.

[5] Heston, Steven L. "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options." Review of Financial Studies 6.2 (1993): 327-43. Web.

[6] McDonald, Robert L. Derivatives Markets. Harlow: Pearson Education Limited, 2014. Print.

[7] Gatheral, Jim. The Volatility Surface: A Practitioner's Guide. Hoboken, NJ: John Wiley & Sons, 2006. Print.

[8] Haberman, Richard. *Applied Partial Differential Equations: With Fourier Series and Boundary Value Problems.* Boston: Pearson, 2013. Print.

[9] Hull, John. Options, Futures, and Other Derivatives. Upper Saddle River, NJ: Pearson/Prentice Hall, 2006. Print.

[10] Lamont, Owen A. "Short Sale Constraints and Overpricing." Ed. Frank J. Fabozzi. Short Selling : Strategies, Risks, and Rewards (2004): n. pag. Web.

[11] Linetsky, V. and R. Mendoza (2010). The Constant Elasticity of Variance Model. In R. Cont (Ed.), Encyclopedia of Quantitative Finance. Chichester, West Sussex: Wiley.

[12] Mikhailov, Sergei, and Ulrich Nogel. "Heston's Stochastic Volatility Model: Implementation, Calibration, and Some Extensions." Wilmott Magazine July 2003: 74-79. Web.

[13] R. Costantinescu, N. Costanzino, A. L. Mazzucato, and V. Nistor, *Approximate Solutions to Second Order Parabolic Equations I: Analytic Estimates*, J. Math. Phys., 51 (2010)

[14] Sitter, Rodolphe. Volatility Surface. Digital image. MathWorks. MathWorks, 31 Mar. 2009. Web.

[15] Taleb, Nassim. The Black Swan: The Impact of the Highly Improbable. London: Allen Lane, 2011. Print.

SIMRANJEET SINGH

Email: singh.simranjeet.356@gmail.com

EDUCATION

The Pennsylvania State University, ***Schreyer Honors College (2017)*** University Park, PA
MATHEMATICS (B.S.), FINANCE (B.S.); MINOR: STATISTICS

- Honors:
  - Schreyer Honors Scholar in Mathematics & Finance
  - Mary Lister McCammon Award in Mathematics (Spring 2016, Spring 2017)
  - Excellence in Mathematics Award (Spring 2015)
  - Emerging Leader of the Year Award (Spring 2014)
  - Excellence in Public Speaking Award (Spring 2014)
- Pursued research in closed-form approximate solutions of the generalized heat equation using the Dyson-Taylor commutator method, with applications in option pricing. This culminated in an interdisciplinary thesis in Mathematics and Finance.

EXPERIENCE

Securities Division Intern, **The Goldman Sachs Group, Inc.** New York, NY
JUN 2016 – AUG 2016

- Engaged in a ten-week rotation within the Securities Division across three different desks:
  - International Shares Trading
  - Equity Derivatives Trading
  - Prime Brokerage Risk
- Created programmatic solutions in Excel, R, and Python to assist with projects on each desk.
- Presented for each of the desks to showcase skills acquired during the rotation.

Worldwide Data & Insight Intern, **Dun & Bradstreet, Inc.** Center Valley, PA
MAY 2015 – AUG 2015

- Responsible for programming algorithms to assist in data matching, while also implementing and revising certain methodologies used in the data matching and methods of analysis.
- Analyzed and extracted data tables using Oracle-based SQL, to gain insight into data matching.
- Created Access databases and Excel spreadsheets to explore large data fields, and present data in a professional and impactful way.

Learning Center Tutor, **The Pennsylvania State University** Center Valley, PA
JAN 2014 – MAY 2015

- Provided academic assistance to fellow students in various fields of study.
- Tutored in the following areas (up to 20+ hours / week):
  - Mathematics: Calculus I, Calculus II, Vector Analysis & Multivariable Calculus, Matrices, Ordinary & Partial Differential Equations, Introduction to Probability & Stochastic Processes
  - Sciences: Classical Mechanics, Electricity & Magnetism, Engineering Thermodynamics, Thermal Physics, Chemical Principles I
  - Business: Microeconomics, Macroeconomics, Accounting Principles, Finance

LEADERSHIP

- **Math Club**
  President (APR 2016 – MAY 2017)
- **Honors Society**
  President (AUG 2014 – MAY 2015)
- **Student Government Association**
  Senator, Vice President, Chair of Academic Affairs Committee (JAN 2014 – MAY 2015)

SKILLS

- Fluent in English, Punjabi, and Hindi; proficient in Latin
- Experienced in Python, R; Familiar with Mathematica, MATLAB, Maple