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DEPARTMENT OF MATHEMATICS

ON CUBIC PARTITIONS
AND
PARTITIONS CLASSIFIED BY SMALLEST MISSING PART

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Abstract

Integer partitions are a fascinating subject in the area of combinatorics. The basic concept and a brief history of partition theories are included in Chapter 1, as well as the notion of a generating function. Some most elementary results in partition theory are also proved in Chapter 1.

In Chapter 2, we briefly introduce q-series with some transformations that are playing an important role in the study of q-series, and lay the foundation of proving theorems in Chapter 3 and 4.

In chapter 3, we prove a congruence relation of cubic partitions in a new way, and also analyze the coefficients of its generating function which is an analog of Dyson's crank generating function.

In chapter 4, we prove the main theorem of this thesis on partitions classified by its smallest missing part. We also explore a few applications of it.

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Chapter 1

Introduction

1.1 Brief History on the Theory of Partitions

In 1674, first question about integer partitions was asked by G. W. Leibniz in his letter to J. Bernoulli [16]. He denoted $p(n)$ as the number of ways of writing n as a sum of positive integers (permutations not counted). For the first few numbers, $p(n)$ is easy to compute by hand:

$$p(1) = 1,\ p(2) = 2,\ p(3) = 3,\ p(4) = 5,\ p(5) = 7,\ \cdots$$

The first non-trivial theorem was discovered by L. Euler [8] in the 18th century. He was asked by Ph. Naudé to compute the number of ways to write an integer n as a sum of m positive integers. He discovered and proved

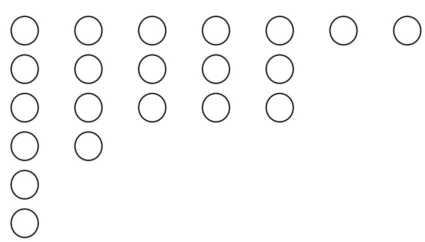
$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n\in\mathbb{Z}}(-1)^n q^{(3n-1)/2}, \tag{1.1}$$

which is known as Euler's Pentagonal Number Theorem; and he derived the recurrence relation for $p(n)$:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots,\ n > 0 \text{ and } p(0) = 1.$$

The method of generating functions that he introduced has been the most basic and powerful tool in the study of partitions.

In the 19th century, J. Sylvester [19] introduced a significant notion of representing a partition by a figure, which is called Ferrers diagram. An example of Ferrers diagram of partition $21 = 7+5+5+2+1+1$ is:



This insight lead to deep combinatorial studies of partitions and also provided combinatorial proofs to both old and new theorems.

Although great mathematicians like A. Cauchy and C. Gauss made great contributions to partitions, the most surprising result came from G. Hardy and S. Ramanujan. Hardy and Ramanujan mostly completed the exact formula for $p(n)$ [13], and it was fully completed and improved later by H. Rademacher [17]:

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)k^{\frac{1}{2}}\left[\frac{d}{dx}\frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{\frac{1}{2}}\right)}{\left(x-\frac{1}{24}\right)^{\frac{1}{2}}}\right]_{x=n}, \tag{1.2}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} e^{-2\pi i n h/k + \pi i\cdot s(h,k)}$$

with

$$s(h,k)=\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\left\lfloor\frac{\mu}{k}\right\rfloor-\frac{1}{2}\right)\left(\frac{h\mu}{k}-\left\lfloor\frac{h\mu}{k}\right\rfloor-\frac{1}{2}\right).$$

Asymptotically, as n tends to infinity,

$$p(n)\sim\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}. \tag{1.3}$$

Another contribution by Ramanujan, considered as "the most beautiful formulae" by Hardy, was 'Rogers-Ramanujan' identities [12].

The First Rogers-Ramanujan Identity is

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

and the Second Rogers-Ramanujan Identity is

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

He also discovered the following congruence relations in partitions:

$$\begin{aligned}p(5n+4)&\equiv 0 \pmod 5\\ p(7n+5)&\equiv 0 \pmod 7\\ p(11n+6)&\equiv 0 \pmod{11},\end{aligned}$$

and many other similar congruences were proved hereafter.

There are many other topics discussed later in partitions such as composition, rank and crank statistics, and high-dimensional partitions. And there are also many new topics emerging such as cubic partitions, over partitions, etc [4].

1.2 Integer Partitions

A *partition* of a positive integer n is defined as $n=\sum_{i=1}^{r}\lambda_i$ with $\lambda_1,\lambda_2,\cdots,\lambda_r$ being a finite nonincreasing sequence of positive integers. The λ_i's are called the *parts* of the partition. For example, $27=6+6+5+3+3+3+1$ is a partition of 27. The partition function $p(n)$ counts the number of partitions of n. For convention, we set $p(0)=1$, and $p(n)=0$ for negative n. The following list gives the partitions of n from 1 to 5 with the value of $p(n)$:

$$\begin{aligned}&p(1)=1:\ 1;\\ &p(2)=2:\ 2,\ 1+1;\\ &p(3)=3:\ 3,\ 2+1,\ 1+1+1;\\ &p(4)=5:\ 4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1;\\ &p(5)=7:\ 5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\\ &\qquad\qquad 2+1+1+1,\ 1+1+1+1+1.\end{aligned}$$

1.3 Generating Function

The *generating function* $f(q)$ for the sequence $a_0, a_1, a_2, \cdots$ is the power series $f(q) = a_0 + a_1 q + a_2 q^2 + \cdots = \sum_{n\geq 0} a_n q^n$. In this section, we derive the generating function for $p(n)$ in infinite product form.

Let $H = \{h_1, h_2, \cdots\}$ be a subset of $\mathbb{N}$, and $p(H, M, n)$ denotes the number of partitions of n in which each part is an element of H and no part appears more than M times. Then the generating function for $p(H, M, n)$ with fixed M is:

$$\begin{aligned}
f_{H,\,M}(q) =& \sum_{n\geq 0} p(H, M, n) q^n \\
=& \sum_{a_1=0}^{M} \sum_{a_2=0}^{M} \cdots \sum_{a_N=0}^{M} q^{a_1 h_1 + a_2 h_2 + a_3 h_3 + \cdots} \\
=& (1 + q^{h_1} + q^{2h_1} + q^{3h_1} + \cdots + q^{Mh_1}) \\
& \times (1 + q^{h_2} + q^{2h_2} + q^{3h_2} + \cdots + q^{Mh_2}) \\
& \times (1 + q^{h_3} + q^{2h_3} + q^{3h_3} + \cdots + q^{Mh_3}) \\
& \cdots \\
=& \prod_{j\in H} (1 + q^j + q^{2j} + \cdots + q^{Mj}) \qquad (1.4) \\
=& \prod_{j\in H} \frac{1 - q^{(M+1)j}}{1 - q^j}. \qquad (1.5)
\end{aligned}$$

For $| q |< 1$ and $j \in \mathbb{N}$, recall the infinite geometric series

$$\sum_{j=0}^{\infty} q^j = 1 + q + q^2 + q^3 + \cdots = \frac{1}{1-q}. \qquad (1.6)$$

Using (1.6) and letting M tend to ∞ in (1.4), we obtain generating function for partitions with each part being an element of H:

$$\begin{aligned}
f_H(q) =& \sum_{n\geq 0} p(H, n) q^n \\
=& \prod_{j\in H} \frac{1}{1 - q^j}. \qquad (1.7)
\end{aligned}$$

Set $H = \mathbb{N}$ in (1.7), and we can get the generating function for ordinary partitions:

$$\begin{aligned}
f(q) =& \sum_{n\geq 0} p(n) q^n \\
=& \prod_{j=1}^{\infty} \frac{1}{1 - q^j}. \qquad (1.8)
\end{aligned}$$

One direct application of (1.7) is the following corollary by Euler:

Corollary 1 (Euler, 1748) *The number of partitions of n into odd parts $p\left(\mathscr{O}, n\right) = p\left(\mathbb{N}, 1, n\right)$, the number of partitions of n into distinct parts (i.e. each part only appears once).*

Instead of proving Corollary 1, we prove another corollary by M. Subbarao:

Corollary 2 (Subbarao, 1971) *The number of partitions of n in which each part appears 2, 3, or 5 times equals the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12.*

Proof. Let $p_1(n)$ count the number of partitions of n in which each part appears 2, 3, or 5 times, and let $p_2(n)$ count the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12.

By (1.7),

$$\sum_{n=0}^{\infty} p_2(n)q^n = \prod_{j=0}^{\infty} \frac{1}{(1-q^{12j+2})(1-q^{12j+3})(1-q^{12j+6})(1-q^{12j+9})(1-q^{12j+10})}. \tag{1.9}$$

By (1.4),

$$\begin{aligned}
\sum_{n=0}^{\infty} p_1(n)q^n &= \prod_{j=1}^{\infty}(1+q^{2j}+q^{3j}+q^{5j}) \\
&= \prod_{j=1}^{\infty}(1+q^{2j})(1+q^{3j}) \\
&= \prod_{j=1}^{\infty} \frac{(1+q^{2j})(1-q^{2j})(1+q^{3j})(1-q^{3j})}{(1-q^{2j})(1-q^{3j})} \\
&= \prod_{j=1}^{\infty} \frac{(1-q^{4j})(1-q^{6j})}{(1-q^{2j})(1-q^{3j})} \\
&= \prod_{j=0}^{\infty} \frac{(1-q^{12j+4})(1-q^{12j+8})(1-q^{12j+12})}{(1-q^{12j+2})(1-q^{12j+4})\cdots(1-q^{12j+10})(1-q^{12j+12})} \cdot \\
&\quad \frac{(1-q^{12j+6})(1-q^{12j+12})}{(1-q^{12j+3})(1-q^{12j+6})(1-q^{12j+9})(1-q^{12j+12})} \\
&= \prod_{j=0}^{\infty} \frac{1}{(1-q^{12j+2})(1-q^{12j+3})(1-q^{12j+6})(1-q^{12j+9})(1-q^{12j+10})} \\
&= \sum_{n=0}^{\infty} p_2(n)q^n \text{ by (1.9).}
\end{aligned}$$

The uniqueness of power series of a function implies $p_1(n) = p_2(n)$. ∎

Chapter 2

q-Hypergeometric Series

2.1 Introduction

In Euler's work on generating functions for partitions, a class of series involving factors of the form

$$(1-a)(1-aq)(1-aq^2)\cdots(1-aq^n)$$

have been studied, and they are called "basic hypergeometric series" or "q-series". The first one who studied q-series systematically is E. Heine, a German mathematician in 19th century. The study of hypergeometric series has never ceased since then. Now it has been developed such that it not only is an important subject in classical analysis, combinatorics, and number theory; but it also has influence and application in Lie algebras, statistics and physics [2].

Let q be a complex constant with norm strictly less than 1, and for $n > 1$, we introduce the following notation:

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

and for convenience we write $(a)_n$ as a shorthand of $(a;q)_n$ (only when the second argument is q with power 1). For $n = 0$, we define $(a)_0 = 1$. Also the notion

$$(a)_\infty = (a;q)_\infty := \lim_{n\to\infty}(a;q)_n = \prod_{j\geq 0}(1-aq^j)$$

is used a lot in the study of q-series. Also we may define $(a)_n$ for all real numbers n by

$$(a)_n = (a)_\infty \,/\, (aq^n)_\infty \,.$$

2.2 Fundamental Transformation in q-Series

First of all, Cauchy's theorem laid the foundation of the studies in q-series [1], [10].

Theorem 1 (Cauchy) *For* $|\, t \,| < 1$ *and fixed* $a \in \mathbb{C}$*,*

$$\frac{(at)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty}\frac{(a)_n t^n}{(q)_n} \tag{2.1}$$

Proof. Denote the left hand side as a function of t, and let A_n be the coefficients of its power series expansion, i.e.

$$F(t) = \frac{(at)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty} A_n t^n, \tag{2.2}$$

and A_n exists since the infinite product is uniformly convergent inside $|\, t \,| < 1$, and therefore it defines a analytic function of t inside $|\, t \,| < 1$. Now

$$F(t) = \frac{1-at}{1-t}\cdot\frac{(atq)_\infty}{(tq)_\infty} = \frac{1-at}{1-t}\cdot F(qt).$$

Comparing the coefficients of power series on both sides

$$
\begin{aligned}
(1-t)\sum_{n=0}^{\infty} A_n t^n &= (1-at)\sum_{n=0}^{\infty} A_n q^n t^n \\
\sum_{n=0}^{\infty} A_n t^n - \sum_{n=1}^{\infty} A_{n-1} t^n &= \sum_{n=0}^{\infty} A_n q^n t^n - a\sum_{n=1}^{\infty} A_{n-1} q^{n-1} t^n \\
A_n - A_{n-1} &= q^n A_n - a q^{n-1} A_{n-1} \\
A_n &= \frac{1-aq^{n-1}}{1-q^n} A_{n-1}. \qquad (2.3)
\end{aligned}
$$

Clearly $A_0 = F(0) = 1$, then the recurrence relation gives

$$
\begin{aligned}
A_n &= \frac{(1-aq^{n-1})(1-aq^{n-2})\cdots(1-a)A_0}{(1-q^n)(1-q^{n-1})\cdots(1-q)} \\
&= \frac{(a)_n}{(q)_n} \qquad (2.4)
\end{aligned}
$$

Substitute (2.4) in (2.2), and the theorem is proved. ■

The following two special cases, by Euler [9] of Theorem 1 are found very useful:

Corollary 3 (Euler) *For* $| t |< 1$,

$$
\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty} \qquad (2.5)
$$

$$
\sum_{n=0}^{\infty} \frac{t^n q^{\frac{1}{2}n(n-1)}}{(q)_n} = (-t)_\infty \qquad (2.6)
$$

Proof. Let $a = 0$ in Theorem 1, then we get

$$
\sum_{n=0}^{\infty} \frac{(0)_n}{(q)_n} \cdot t^n = \frac{(0)_\infty}{(t)_\infty},
$$

which is just

$$
\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty}.
$$

Now replace a by a/b and t by bz in Theorem 1, then we can get

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(1-a/b)(1-qa/b)\cdots(1-q^{n-1}a/b)(bz)^n}{(q)_n} &= \frac{(az)_\infty}{(bz)_\infty} \\
\sum_{n=0}^{\infty} \frac{(b-a)(b-aq)\cdots(b-aq^{n-1})\,z^n}{(q)_n} &= \frac{(az)_\infty}{(bz)_\infty}.
\end{aligned}
$$

Set $b = 0$, $a = -1$,

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{1}{2}n(n-1)}}{(q)_n} = (-z)_\infty .$$

■

2.3 Heine's Transformation and its Application

Based on Cauchy's Theorem, Heine [14] proved the following identity that plays an instrumental role in the study of other transformations in q-series:

Theorem 2 (Heine) *For* $\mid q \mid, \mid t \mid, \mid b \mid < 1$

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(q)_n (c)_n} = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m (t)_m b^m}{(q)_m (at)_m}. \tag{2.7}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(q)_n (c)_n} &= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n} \cdot \frac{(cq^n)_\infty}{(bq^n)_\infty} \\
&= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_n t^n}{(q)_n} \cdot \frac{(c/b)_m (bq^n)^m}{(q)_m} \text{ by Theorem 2} \\
&= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(c/b)_m b^m}{(q)_m} \cdot \frac{(a)_n (tq^m)^n}{(q)_n} \\
&= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m b^m}{(q)_m} \cdot \frac{(atq^m)_\infty}{(tq^m)_\infty} \text{ by Theorem 2} \\
&= \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m (t)_m b^m}{(q)_m (at)_m}.
\end{aligned}$$

■

The following corollary is discovered by Jacobi [15] in 1829, it is so important that it is often referred to as Jacobi's triple product identity:

Corollary 4 (Jacobi) *For* $z \neq 0$,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \left(zq; q^2\right)_\infty \left(q/z; q^2\right)_\infty \left(q^2; q^2\right)_\infty . \tag{2.8}$$

Proof.

$$
\begin{aligned}
\left(zq;q^2\right)_\infty &= \sum_{m=0}^{\infty} \frac{z^m q^{m^2}}{(q^2;q^2)_m} \text{ by (2.6)} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} z^m q^{m^2} \left(q^{2m+2};q^2\right)_\infty \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2} \left(q^{2m+2};q^2\right)_\infty \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2} \sum_{r=0}^{\infty} \frac{(-1)^r q^{r^2+2mr+r}}{(q^2;q^2)_r} \text{ by (2.6)} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r z^m q^{r^2+2mr+m^2+r}}{(q^2;q^2)_r} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r z^{m+r} z^{-r} q^{(m+r)^2} q^r}{(q^2;q^2)_r} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-q/z)^r}{(q^2;q^2)_r} \sum_{m=-\infty}^{\infty} z^{m+r} q^{(m+r)^2} \\
&= \frac{1}{(q^2;q^2)_\infty (-q/z;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2} \text{ by (2.5).}
\end{aligned}
$$

Note that as m runs through the set of integers, $m+r$ also covers the entire set of integers without repetition. Therefore

$$
\left(zq;q^2\right)_\infty \left(q^2;q^2\right)_\infty \left(-q/z;q^2\right)_\infty = \sum_{m=-\infty}^{\infty} z^m q^{m^2}.
$$

■

Corollary 5

$$
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(c)_n (q)_n} = \frac{(abt/c)_\infty}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n (abt/c)^n}{(c)_n (q)_n}. \tag{2.9}
$$

Proof. Repeatedly apply Theorem 1:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n t^n}{(c)_n(q)_n} &= \frac{(b)_\infty(at)_\infty}{(c)_\infty(q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n(t)_n b^n}{(at)_n(q)_n} \\
&= \frac{(b)_\infty(at)_\infty}{(c)_\infty(q)_\infty} \cdot \frac{(c/b)_\infty(bt)_\infty}{(at)_\infty(b)_\infty} \sum_{n=0}^{\infty} \frac{(abt/c)_n(b)_n(c/b)^n}{(q)_n(bt)_n} \\
&= \frac{(c/b)_\infty(bt)_\infty}{(c)_\infty(t)_\infty} \sum_{n=0}^{\infty} \frac{(abt/c)_n(b)_n(c/b)^n}{(q)_n(bt)_n} \qquad (2.10) \\
&= \frac{(c/b)_\infty(bt)_\infty}{(c)_\infty(t)_\infty} \cdot \frac{(abt/c)_\infty(c)_\infty}{(bt)_\infty(c/b)_\infty} \sum_{n=0}^{\infty} \frac{(c/a)_n(c/b)_n(abt/c)^n}{(c)_n(q)_n} \\
&= \frac{(abt/c)_\infty}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(c/a)_n(c/b)_n(abt/c)^n}{(c)_n(q)_n}.
\end{aligned}$$

■

These identities lay the foundation for the study of q-series. Some more specific results of q-series will be discussed and proved when they are needed to use in the following chapters.

Chapter 3

Cubic Partitions

3.1 Cubic Partitions

The notion of cubic partitions raised and discussed recently. Cubic partitions refer to the 2-color partition with colors red (r) and blue (b) subject to the restriction that the color b only appears in even parts. For example, there are 4 such partitions for 3:

$$3_r,\ 2_r + 1_r,\ 2_b + 1_r,\ 1_r + 1_r + 1_r.$$

It is called cubic partition because its generating function has certain relations with Ramanujan's cubic continued fraction [6], [7]. The cubic partition generating function $a(n)$ is defined by:

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q)_\infty (q^2;q^2)_\infty} \tag{3.1}$$

3.2 Congruence Relations

We recall the following identities of partition functions from Ramanujan [18]:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Now we are searching for similar identities of cubic partition function, $a(n)$, and we have the following result by H. Chan [6]:

Theorem 3

$$a(3n+2) \equiv 0 \pmod 3 \tag{3.2}$$

We provide my new proof of this theorem:

Proof. Define $a_n(z)$ as

$$\sum_{n\geq 0} a_n(z)q^n = \frac{(q;q^2)_\infty}{(qz)_\infty (q/z)_\infty}. \tag{3.3}$$

Let ω denote a cubic root of unity,

$$\begin{aligned} \sum_{n\geq 0} a_n(\omega)q^n &= \frac{(q;q^2)_\infty}{(q\omega)_\infty (q/\omega)_\infty} \\ &= \frac{(q;q^2)_\infty (q)_\infty}{(q^3;q^3)_\infty} \\ &= \frac{(q^2;q^2)_\infty (q;q^2)_\infty (q;q^2)_\infty}{(q^3;q^3)_\infty} \text{ by Corollary 4} \\ &= \frac{1}{(q^3;q^3)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} \end{aligned}$$

Notice that $n^2 \not\equiv 2 \pmod 3$, so the coefficient of q^{3n+2} is 0, in other words, $a_{3n+2}(\omega) = 0$. We can conclude that the polynomial $(1+z+z^2)$ divides the polynomial $a_{3n+2}(z)z^k$ where z^k is multiplied to cancel the denominator of the rational function $a_{3n+2}(z)$. Thus when we pick $z = 1$, we have $3 \mid a_{3n+2}(1) = a(3n+2)$. ■

Actually in Chan's paper [6], he actually proves a generalized result:

Theorem 4 *For $k \geq 1$ and n being nonnegative integer,*

$$a(3^k n + c_k) \equiv 0 \pmod{3^{k+\delta(k)}},$$

where $\delta(k)$ takes value 1 when k is even, and 0 when k is odd.

Clearly Theorem 3 follows directly from it, but his proof of Theorem 4 requires heavy calculations and complicated analysis on Ramanujan's cubic continued fraction and q-series.

3.3 More on Cubic Generating Function

The generating function (3.3) of cubic partitions looks very similar to the crank generating function defined by G. Andrews and F. Garvan [5]:

$$\sum_{n=0}^{\infty} c_n(z) q^n = \frac{(q)_\infty}{(zq)_\infty (q/z)_\infty},$$

it is natural to look for some combinatorial explanation for (3.3) similar as the crank statistics for ordinary partitions. Unfortunately, such an interpretation has not been found yet. However, we prove a new theorem on the property of the coefficients in (3.3):

Theorem 5 *The generating function of cubic partitions*

$$\sum_{n \geq 0} a_n(z) q^n = \frac{(q;q^2)_\infty}{(zq)_\infty (q/z)_\infty}$$

has nonnegative coefficients except the one of q.

Proof. From the crank generating function for ordinary partitions in [5], we know that in

$$\frac{(q)_\infty}{(zq)_\infty (q/z)_\infty}$$

all coefficients are positive except the one of q. Now

$$\frac{(q;q^2)_\infty}{(zq)_\infty (q/z)_\infty} = \frac{(q)_\infty}{(zq)_\infty (q/z)_\infty} \frac{1}{(q^2;q^2)_\infty} = \frac{(q)_\infty}{(q^2;q^2)_\infty} \sum_{j \geq 0} \frac{z^j q^j}{(q)_j} \sum_{k \geq 0} \frac{z^{-k} q^k}{(q)_k} \text{ by (2.5).}$$

Hence the only negative coefficients possible arises for z^0, i.e. when $j = k$. Thus we need only show that $-q$ is the only negative term in

$$\begin{aligned}&\frac{(q)_\infty}{(q^2;q^2)_\infty}\sum_{j\geq 0}\frac{q^{2j}}{(q)_j^2}\\=&(q;q^2)_\infty\sum_{j\geq 0}\frac{q^{2j}}{(q)_j^2}.\end{aligned}$$

Consider the expression

$$(tq;q^2)_\infty\sum_{j\geq 0}\frac{t^jq^{2j}}{(q)_j(tq)_j},$$

clearly this is the above expression when taking $t = 1$, and we want to prove the only negative term in this expression is $-tq$.

Let $a, b \to 0$, $x \to tq^2$, $c \to tq$ in Corollary 5, we get

$$(tq;q^2)_\infty\sum_{j\geq 0}\frac{t^jq^{2j}}{(q)_j(tq)_j}=\frac{1-tq}{(tq^2;q^2)_\infty}\sum_{n\geq 0}\frac{t^{2n}q^{n^2+n}}{(q)_n(tq)_n}.$$

Now all terms on the right hand side have non-negative coefficients except the term $n = 0$. We take $n = 0$ term and $n = 1$ term:

$$\frac{1}{(tq^2;q^2)_\infty}-\frac{tq}{(tq^2;q^2)_\infty}+\frac{t^2q^3}{(1-q)(tq^2;q^2)_\infty}.$$

We notice that first term has positive terms of even powers of q; second term has negative terms of odd powers of q; third term has positive terms of powers ≥ 3 of q. So we only need to compare second and third term.

$$\begin{aligned}&\frac{t^2q^3}{(1-q)(tq^2;q^2)_\infty}-\frac{tq}{(tq^2;q^2)_\infty}\\=&\frac{t^2q^3}{1-q}\sum_{j\geq 0}\frac{t^jq^{2j}}{(q^2;q^2)_j}-tq\sum_{j\geq 0}\frac{t^jq^{2j}}{(q^2;q^2)_j}\\=&\sum_{j\geq 2}\frac{t^jq^{2j-1}}{(1-q)(q^2;q^2)_{j-2}}-\sum_{j\geq 1}\frac{t^jq^{2j-1}}{(q^2;q^2)_{j-1}}\\=&-tq+\sum_{j\geq 2}\left[\frac{t^jq^{2j-1}}{(1-q)(q^2;q^2)_{j-2}}-\frac{t^jq^{2j-1}}{(q^2;q^2)_{j-1}}\right]\\=&-tq+\sum_{j\geq 2}\frac{t^jq^{2j}(1-q^{2j-3})}{(1-q)(q^2;q^2)_{j-1}}\end{aligned}$$

Notice that $1/(q^2;q^2)_M$ expands into terms with positive coefficients, and $(1-q^N)/(1-q)$ is a sum of geometric sequence and hence it has positive coefficients. Therefore we could see that the whole expression contains all positive terms except $-tq$. Now let $t = 1$ and this completes the proof. ∎

Chapter 4

Partitions Classified by Smallest Missing Part

4.1 Introduction

Many studies in integer partitions deal with the actual summands of partitions. In this chapter, however, we focus on parts that are not summands. The smallest missing part in a partition is the smallest positive integer that does not appear in the summand. For example the smallest missing part of the partition $15 = 1 + 1 + 2 + 3 + 3 + 5$ is 4, the smallest missing part of the partition $24 = 3 + 3 + 4 + 5 + 9$ is 1. My main theorem focuses on the parity of the smallest missing part of a partition:

Theorem 6 *The number of partitions of n in which the smallest missing part is odd is always larger than or equal to the number of partitions of n in which the smallest missing part is even except the case when $n = 1$.*

We first find the generating function of the difference of these two partitions; then we prove a lemma; finally we use the lemma to prove this theorem.

4.2 Generating Function

Based on (1.6), we construct the generating function of the number of partitions of n in which the smallest integer not appearing is odd, $O(n)$, as

$$o(q) := \sum_{n\geq 1} O(n)q^n = \sum_{j=0,\ j\ even}^{\infty} \frac{q^{1+2+3+\cdots+j}}{\prod_{i=1,\ i\neq j+1}(1-q^i)}. \tag{4.1}$$

Also for even case, $E(n)$:

$$e(q) := \sum_{n\geq 1} E(n)q^n = \sum_{j=0,\ j\ odd}^{\infty} \frac{q^{1+2+3+\cdots+j}}{\prod_{i=1,\ i\neq j+1}(1-q^i)}. \tag{4.2}$$

We want to prove $O(n) \geq E(n)$ for all $n > 1$, so we look at $o(q) - e(q)$ and prove that all its coefficients are non-negative except for p^1.

$$\begin{aligned} o(q) - e(q) &= \sum_{j=0,\ j\ even}^{\infty} \frac{q^{1+2+3+\cdots+j}}{\prod_{i=1,\ i\neq j+1}(1-q^i)} - \sum_{j=0,\ j\ odd}^{\infty} \frac{q^{1+2+3+\cdots+j}}{\prod_{i=1,\ i\neq j+1}(1-q^i)} \\ &= \sum_{j=0,\ j\ even}^{\infty} \frac{(1-q^{j+1})q^{1+2+3+\cdots+j}}{\prod_{i\geq 1}(1-q^i)} - \sum_{j=0,\ j\ odd}^{\infty} \frac{(1-q^{j+1})q^{1+2+3+\cdots+j}}{\prod_{i\geq 1}(1-q^i)} \\ &= \frac{1-2q^1+2q^{1+2}-2q^{1+2+3}+2q^{1+2+3+4}-\cdots}{(q)_\infty}. \end{aligned} \tag{4.3}$$

4.3 Proof of Theorem 6

First of all, we prove the following lemma that will be used in the proof of Theorem 6:

Lemma 1

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n^2}=\frac{1}{(q)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{j(j+1)/2};$$
$$\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q)_{n-1}(q)_n}=\frac{1}{(q)_\infty}\sum_{j=1}^{\infty}(-1)^{j+1} q^{j(j+1)/2}.$$

Proof. For the first part, set $a=b=1/z$, $c=q$, $t=q^2z^2$ in (2.10), we have

$$\sum_{n=0}^{\infty}\frac{(1/z)_n(1/z)_n z^{2n}q^{2n}}{(q)_n(q)_n}=\frac{(qz)_\infty(q^2z)_\infty}{(q)_\infty(q^2z^2)_\infty}\sum_{n=0}^{\infty}\frac{(1/z)_n(q)_n(qz)^2}{(q)_n(q^2z)_n}$$

As $z\to 0$, then the left hand side

$$\begin{aligned}\lim_{z\to 0}\sum_{n=0}^{\infty}\frac{(1/z)_n(1/z)_n z^{2n}q^{2n}}{(q)_n(q)_n}&=\lim_{z\to 0}\sum_{n=0}^{\infty}\frac{(1-1/z)^2(1-q/z)^2\cdots(1-q^{n-1}/z)^2z^{2n}q^{2n}}{(q)_n^2}\\&=\lim_{z\to 0}\sum_{n=0}^{\infty}\frac{(z-1)^2(z-q)^2\cdots(z-q^{n-1})^2q^{2n}}{(q)_n^2}\\&=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n^2},\end{aligned}$$

also the right hand side

$$\lim_{z\to 0}\frac{(qz)_\infty(q^2z)_\infty}{(q)_\infty(q^2z^2)_\infty}\sum_{j=0}^{\infty}\frac{(1/z)_j(q)_j(qz)^2}{(q)_j(q^2z)_j}=\frac{1}{(q)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{j(j+1)/2}$$

Thus

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n^2}=\frac{1}{(q)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{j(j+1)/2}.$$

For the second part, set $t=q^3/(ab)$, $c=q^2$, and let $a,b\to\infty$ in (2.10), we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q)_n(q^2)_n}=\frac{1}{(q^2)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{j(j+3)/2}$$
$$\frac{q}{1-q}\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q)_n(q^2)_n}=\frac{q}{1-q}\cdot\frac{1}{(q^2)_\infty}\sum_{j=1}^{\infty}(-1)^{j+1} q^{(j-1)(j+2)/2}$$
$$\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q)_n(q)_{n-1}}=\frac{1}{(q)_\infty}\sum_{j=1}^{\infty}(-1)^{j+1} q^{j(j+1)/2}.$$

∎

Now we are ready to prove Theorem 6. Applying Lemma 1 to the expression (4.3)+q, we get:

$$q+\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n^2}-\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q)_{n-1}(q)_n}. \tag{4.4}$$

Now we prove that (4.4) has nonnegative coefficients to show that (4.3) has $-q$ as its only term with negative coefficient. Define the partial sum of (4.4) as

$$S(k):=q+\sum_{n=0}^{k}\frac{q^{n^2+n}}{(q)_n^2}-\sum_{n=1}^{k}\frac{q^{n^2}}{(q)_{n-1}(q)_n}. \tag{4.5}$$

We prove that $S(k)$ has positive coefficients for all $k\in\mathbb{N}$. Firstly,

$$S(1)=1+\frac{q^3}{(1-q)^2}$$
$$S(2)=1+\frac{q^3}{1-q}+\frac{q^8}{(1-q)^2(1-q^2)^2}.$$

Suppose that

$$S(k)=A_k(q)+\frac{q^{(k+1)^2-1}}{(q)_k^2},$$

where $A_k(q)$ has positive coefficients. We want to prove that

$$S(k+1)=A_{k+1}(q)+\frac{q^{(k+2)^2-1}}{(q)_{k+1}^2},$$

where $A_{k+1}(q)$ has positive coefficients.

$$\begin{aligned}
S(k+1)&=S(k)+\frac{q^{(k+1)^2+(k+1)}}{(q)_{k+1}^2}-\frac{q^{(k+1)^2}}{(q)_k(q)_{k+1}}-\frac{q^{(k+2)^2-1}}{(q)_{k+1}^2}+\frac{q^{(k+2)^2-1}}{(q)_{k+1}^2}\\
&=A_k(q)+\frac{q^{(k+1)^2-1}}{(q)_k^2}+\frac{2q^{k^2+3k+2}-q^{k^2+2k+1}-q^{k^2+4k+3}}{(q)_{k+1}^2}+\frac{q^{(k+2)^2-1}}{(q)_{k+1}^2}\\
&=A_k(q)+\frac{q^{k^2+2k}\left(1-q\right)\left(1-q^{k+1}\right)^2}{(q)_{k+1}^2}+\frac{q^{(k+2)^2-1}}{(q)_{k+1}^2}\\
&=A_k(q)+\frac{q^{k^2+2k}\left(1-q\right)}{(q)_k^2}+\frac{q^{(k+2)^2-1}}{(q)_{k+1}^2}
\end{aligned}$$

Notice that

$$\frac{q^{k^2+2k}\left(1-q\right)}{(q)_k^2}=\frac{q^{k^2+2k}}{(q)_k(q^2;q)_{k-1}}$$

has positive coefficients, so it ends the prove.

Since (4.5), the partial sums of (4.4), has positive coefficients, (4.4) has positive coefficients for all powers of q, and this proves Theorem 6. ∎

4.4 Partial Proof of Theorem 1 Using Theorem 6

Apart from its intrinsic interest, Theorem 6 also provides a partial proof of Theorem 1. we want to show that $-q$ is the only negative term in

$$\begin{aligned}&(q)_\infty \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} \frac{q^{2j}}{(q)_j^2}\\ =&(q;q^2)_\infty \sum_{j\geq 0} \frac{q^{2j}}{(q)_j^2}.\end{aligned}$$

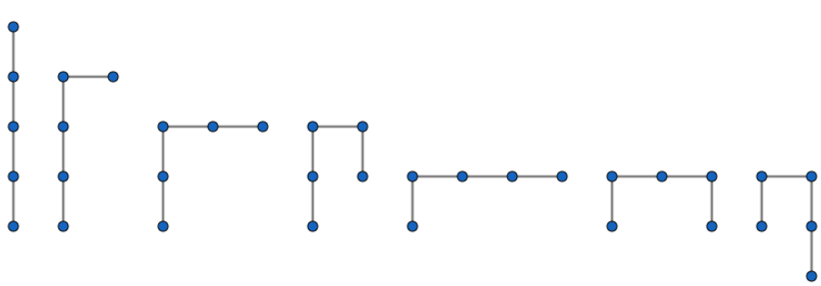
Let $a, b \to 0,\ t = q^2,\ c = q$ in Theorem 2, we get

$$\begin{aligned}(q;q^2)_\infty \sum_{j\geq 0} \frac{q^{2j}}{(q)_j^2} &= \frac{1}{(q)_\infty^2} \sum_{s\geq 0} q^{s(s+1)/2}(1-q^{1+s})\\ &= \frac{1}{(q^2;q^2)_\infty (q)_\infty}((1-q) - q(1-q^2) + q^3(1-q^3) - q^6(1-q^4) + \cdots)\\ &= \frac{1}{(q^2;q^2)_\infty (q)_\infty}(1 - 2q + 2q^3 - 2q^6 + 2q^{10} - \cdots)\\ &= \frac{1}{(q^2;q^2)_\infty}\left(\frac{2(1-q+q^3-q^6+q^{10}-\cdots)-1}{(q)_\infty}\right)\end{aligned}$$

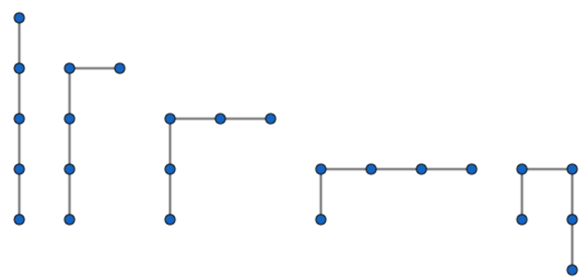
Clearly the expression in the bracket is just (4.3), hence it proves even powers have positive coefficients. It remains to show the odd parts except 1 have positive coefficients.

4.5 Relation to Spiral, Self-Avoiding Walks

Our main theorem is not only concerning the missing part of partitions, it is also directly related to a special kind of lattice walks. In [11], Guttmann and Hirschhorn introduce the concept of Concatenatable Spiral, Self-Avoiding Walks. Spiral self-avoiding walks are lattice walks that satisfying (1) the first step is toward positive y-axis; (2) each step after is either following the same direction or turn $90°$ to the right; (3) the walk should be self-avoiding, meaning that it cannot intersect itself at any point. For example, all length 4 spiral self-avoiding walks are:



Concatenatable spiral self-avoiding walks (=CSSAWs) have extra restrictions that it can go on forever. For example all length 4 CSSAWs are:



In [3], G. Andrews proves that

Lemma 2 *The number of CSSAWs of length n with an odd number of turns equals the number of partitions of n in which the smallest integer not appearing is odd; the number of CSSAWs of length n with an even number of turns equals the number of partitions of n in which the smallest integer not appearing is even.*

Using Lemma 2 and Theorem 6 we can easily deduce

Corollary 6 *The number of CSSAWs of length n with an odd number of turns is always larger than or equal to the number of CSSAWs of length n with an even number of turns except the case when $n = 1$.*

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