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A Model of Bank Behavior Under Market, Credit, and Liquidity Risk

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Abstract

In this paper we will investigate a model of how banks allocate assets to manage market, credit, and liquidity risk. Many models consider the role of reserve supply in the bank's optimization problem, here we include a shock in reserve supply to implement liquidity risk. The solution is completed numerically, utilizing value function iteration with Quasi-Monte Carlo to handle stochastic quantities. The solution method is able to model liquidity risk with a wide range of feasible shocks to give a more realistic outcome than previous models. The algorithm created is general enough to allow for easy parallelization to increase computation speed, as well as applications to other high dimensional stochastic dynamic programming problems.

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Chapter 1

Introduction

1.1 Motivation

The onset of the global financial crisis in 2007 left about one in nine commercial banks in failure or at a high risk of failure[1]. The causes of the failures faced during this time period are complex, however, it is clear that the risk of systemic bank failure was underestimated before the crisis. Among the causes of this high risk of bank failure were credit and liquidity shocks that occurred during the crisis[2].

We therefore aim to create a model of the banking firm wherein the probability of bank failure arises endogenously as a result of the bank's allocation of assets. Given the significance of credit and liquidity shocks during the financial crisis, it is clear that these shocks, as well as market risk, must be incorporated into our model. We base our investigation on the work of Gonzalez-Hermosillo Li 2008 [3], a model which incorporates all of these elements. This model takes place in a dynamic programming framework wherein the bank attempts to maximize the utility from dividends paid out to shareholders. In comparison to the existing model, we will incorporate a more realistic liquidity shock and make use of quasi-Monte Carlo to more efficiently compute our solution.

In the next section, we will discuss how the Gonzalez-Hermosillo Li model fits into the current literature on banking and risk in the banking sector.

1.2 Comparison to Existing Models

The goal of this paper is to investigate a model of how banks manage risk that arises from several different sources. Namely, we aim to expand on the model utilized in Gonzalez-Hermosillo Li 2008 [3] to include more complex liquidity risk, and seek a more efficient numerical solution. In this model, we consider a bank that gains utility from paying out dividends to it's shareholders while managing it's risk. The three risk factors that will be included in our analysis are market, credit, and liquidity risk. Market risk is regarded as the risk that exists when investing in tradable securities such as stocks. Credit risk is the risk that is generated when the bank loans money, and face the possibility that the payer will default on their loan. Banks also face the possibility that depositors will withdraw their money from the bank, this is called liquidity risk.

There is a rich literature studying separately the risks that banks face. Among these are Merton, 1974[4]. In this paper, Merton studies credit risk and uses his general theory to derive asset pricing methods for corporate debt with a probability of default. This fits into the wider Black-Scholes-Merton framework which was developed for use on a wider class of assets.

The seminal paper studying how liquidity shocks arise is that of Diamond and Dybvig, 1983[5]. They study the microeconomic origins of bank runs, namely modeling the scenario as a game of incomplete information. In their paper, they discuss how banks, and the government, can form contracts to mitigate these shocks. This paper would be the first in a long line of models studying bank instability and optimal government intervention[6]. Compared to the work of Diamond and Dybvig, we take assume the presence of liquidity shocks and instead study how the bank can optimally respond to this risk and the others it faces. We also study this problem in the context of a dynamic programming problem from the bank's point of view, rather than a game-theoretic scenario.

The model of Gonzalez-Hermosillo Li that we study is most closely related to the unpublished work of Buckinsky and Yosha, 1997 [7]. Similar to the model we study, Buckinsky and Yosha create a dynamic programming problem from the point of view of the bank. They allow for market and credit shocks and incorporate an endogenous probability of bank survival that effects how much consumers are willing to deposit in the bank. In comparison, out new model adds a shock to reserve supply to represent liquidity risk. In addition, the model of Buckinsky and Yosha allows for a non-zero reserve supply in the event that the bank is sure to fail. Gonzalez-Hermosillo Li correct for this unrealistic facet of their model.

1.3 Contribution

Among other factors, the amount of money that depositors keep in the bank depends on the interest rate that the bank provides relative to the risk-free rate, as well as the probability that the bank will fail, which is itself a function of the allocation of the bank's capital. Thus the optimal decision problem the bank faces is how to optimally allocate their capital among tradable assets, loans, and government bonds, set the interest rate on deposits, as well as pay out dividends in order to maximize their utility. The probability of failure is an endogenous factor that makes the problem somewhat more challenging to solve than traditional dynamic programming problems.

In the past model, the shock due to liquidity was a binomial variable. Considering data from 2009 to present on demand deposits, we can see that this shock is somewhat idealized. In the original paper this was used to make computation easier, but the current framework is general enough to accommodate a realistic shock. Below is a histogram of the percent change in demand deposits.

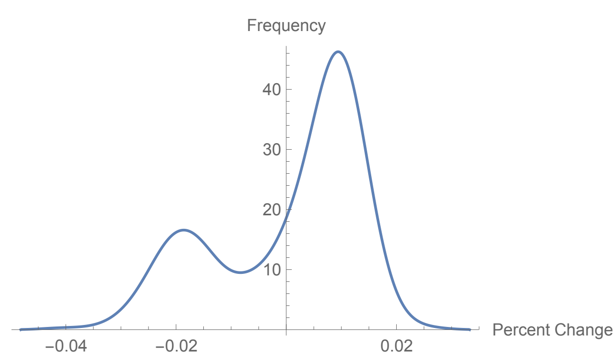

Figure 1.1: Percent Change In Demand Deposits

We can notice that the shock is bimodal, with a high peak at 0.01, and a low peak at -0.02. The data suggests that in general a positive shock is more likely, but when a negative shock happens it is more drastic.

There are several results of this analysis that we are interested in studying. First, we are interested in how the probability of survival changes with the size of the bank. This could have policy implications as breaking up banks could lead to a lower survival probability for each of the individual banks. We are also interested in how the interest rate on deposits will change with the size of the bank, as small banks with a higher probability of failure would most likely have to offer much higher interest rates to attract depositors.

Chapter 2

Problem Formulation

2.1 Problem Formulation

We consider a bank that is infinitely lived and makes decisions in discrete time periods about their capital allocation and interest rate on deposits. They gain utility from the dividends that they give to shareholders in each time period, d_t. Thus for some concave, continuous utility function u, the bank would like to maximize the following with respect to some constraints.

$$\mathbb{E}_t \sum_{k=t}^{\infty} \beta^k u(d_k) \tag{2.1}$$

Let M_t be the amount of capital available to the bank in time t. There will be three random variables that represent the, market, credit, and liquidity risk. These will be denoted Z^a_{t+1}, Z^l_{t+1}, Z^s_{t+1} respectively. Z^a_{t+1} and Z^l_{t+1} are lognormally distributed and defined as the value in the next period of one unit of capital invested in the market or loans respectively. $S_t(r_t, q_t, Z^s_{t+1})$ is defined as the supply of deposits. Taking the interest rate on deposits, r_t, the probability the bank survives until the next period, q_t, and a random component Z^s_{t+1}, the depositors decide how much money they will hold in the bank. The the bank must allocate their capital and the supply of deposits between loans, l_t, tradable securities, a_t, government bonds b_t, and dividends, d_t. This idea gives us the following relation.

$$b_t = M_t + S_t(r_t, q_t, Z^s_{t+1}) - l_t - a_t - d_t \tag{2.2}$$

Realistically, there is regulation on the amount of money that the bank must keep on hand in riskless liquid assets, b_t. This required reserve ratio is denoted $\lambda \in (0, 1)$, and in our model regulation will necessitate that $b_t \geq \lambda S_t(r_t, q_t, Z^s_{t+1})$. We know that the investment b_t will grow at a riskless rate that will be denoted r_b. We also know that the money invested in risky assets will generate returns subject to their random variables, and that the bank must pay out their interest rate on deposits, r_t, on the money deposited in the bank. Thus we have all the necessary pieces to write down the law of motion for the bank's capital. In our case, S_t will be a linear function of the random component, thus we will say $S_t(r_t, q_t, Z^s_{t+1}) = Z^s_{t+1} s(r_t, q_t)$.

$$M_{t+1} = (M_t + S_t(r_t, q_t, Z^s_{t+1}) - l_t - a_t - d_t)r_b + a_t Z^a_{t+1} + l_t Z^l_{t+1} - S_t(r_t, q_t, Z^s_{t+1})r_t \tag{2.3}$$

This is conditional on the fact that the bank actually survives until the next period. We would like to further analyze how q_t is determined as a function of the bank's investment decisions. Suppose that in some scenario the bank reaches zero capital, i.e. $M_{t+1} < 0$, then it is still possible for the bank to short sell it's tradable assets for some funnds to keep the bank afloat. We will denote this amount as $A(M_t)$, and it will be an increasing function of the amount of capital available to the bank in time t. Thus the bank truly fails in time $t + 1$ if

$$M_{t+1} - A(M_t) < 0$$

The probability of survival is then

$$q_t = 1 - P(M_{t+1} - A(M_t) < 0) \tag{2.4}$$

We will introduce a new function $F(m; c_t, q_t)$, F is the CDF of the random variable $M_{t+1} - A(M_t)$ as a function of the policy chosen $c_t = (a_t, l_t, d_t, r_t)$ and also depends on the probability of survival q_t. Thus we can see that the above equation (4) can be reformualted as follows.

$$q_t = 1 - F(0; c_t, q_t) \tag{2.5}$$

The ability to calculate F will be imperative in calculating admissible values of q_t. We have touched on all the critical aspects of the model, from here we will introduce a few more constraints as we prepare to estimate a solution.

$$0 \leq d_t \leq M_t - A(M_t) \tag{2.6}$$

The bank cannot give out dividends that exceed it's total available capital.

$$a_t \geq -A(M_t) \tag{2.7}$$

The amount the bank receives from short selling tradable assets cannot exceed the amount that the bank owns.

$$a_t \geq 0,\ l_t \geq 0,\ r_b < r_t < \bar{r} \tag{2.8}$$

The amount invested in risky assets cannot be negative, and the chosen interest rate must lie above the risk free rate, and below some maximum rate $\bar{r}$. These constraints, along with the law of motion for capital, and the reserve requirement, are all the constraints on our optimization problem.

Chapter 3

Solution

3.1 Solution

Recall that the bank would like to solve the following maximization problem. In order to solve this problem, we will utilize policy function iteration. Policy function iteration is a common technique used to solve many dynamic programming problems. We will begin with a review of this method and a proof that it applies to this specific optimization problem.

3.1.1 Policy Function Iteration

We'll begin by considering a generic dynamic programming problem that we would like to solve. This will provide the framework for our problem. Consider a state $x_t \in \mathbb{R}$, and a stationary transition function $x_{t+1}(\omega) = f(x_t, c_t) + Z_{t+1}(\omega)$, where $c_t \in \phi(x_t) \subseteq \mathbb{R}^n$ is a vector of control variables constrained to some region $\phi(x_t)$. Let $\omega \in \Omega$ where $(\Omega, \mathscr{F}, \mu)$ is a probability space, and Z_t is measurable in Ω with respect to the sigma algebra $\mathscr{F}$. Let $\mathscr{F}_t$ be a filtration adapted to the stochastic process Z_t. At each period $t = 0, 1, 2, ...$, The decision maker seeks a plan $\{c_k\}_{k=t}^{\infty}$ to maximize the following:

$$\mathbb{E}\Big[\sum_{k=t}^{\infty} \beta^k u(c_k) \Big| \mathscr{F}_t\Big]$$

Clearly this maximization is subject to the constraints given and the state, x_t, of the system at that time t. In this case, u is some payoff function and $\beta \in (0, 1)$ is a discount factor. We can now define the value function as the optimal value of this maximization problem.

$$V(x_t, \omega) := \max_{c_t} \mathbb{E}\Big[\sum_{k=t}^{\infty} \beta^k u(c_k) \Big| \mathscr{F}_t\Big](\omega)$$

In our case, we will assume that the random variable Z_t is i.i.d. for each t. The previous states give no information about later states and thus the conditional expectation $\mathbb{E}(\cdot|\mathscr{F}_t)$ is the same as the unconditional expectation and thus V does not depend on ω(it will therefore be excluded). It can be shown that a value function for the above problem also solves the Bellman equation shown below[8].

$$V(x_t) := \max_{c_t \in \phi(x_t)} \Big\{ u(c_t) + \beta \mathbb{E}\big[V(f(x_t, c_t) + Z_{t+1}(\omega))\big] \Big\}$$

We can view the right hand side of the Bellman equation as a contraction on the Banach space of continuous functions, $(C(\mathbb{R}), ||\cdot||_\infty)$. Let $\Lambda : C(\mathbb{R}) \to C(\mathbb{R})$ denote this mapping, then it is defined by the following rule:

$$\Lambda : V(\cdot) \mapsto \max_{c_t} \Big\{ u(c_t) + \beta \mathbb{E}\big[V(f(x_t, c_t) + Z_{t+1}(\omega))\big] \Big\}$$

It is well known that for (which class of functions), that the map Λ is in fact a contraction. Although this result is standard, we will review it here. To begin, we consider Blackwell's theorem[9] which gives us sufficient conditions for Λ to be a contraction.

Theorem 1. *(Blackwell) Let $\Omega \subseteq \mathbb{R}^k$ and $B(\Omega)$ the space of bounded functions $\Omega \to \mathbb{R}$. Let $\Lambda : B(\Omega) \to B(\Omega)$ be an operator. Then if the following conditions hold, Λ is a contraction.*

1. *For any* $V_1, V_2 \in B(\Omega)$ *such that* $V_1(x) \leq V_2(x)$ *for all* $x \in \Omega$*, we have that* $\Lambda(V_1)(x) \leq \Lambda(V_2)(x)$ *for all* x *in* Ω*.*

2. *There exists* $\beta \in (0,1)$ *such that* $\Lambda(V+a)(x) \leq \Lambda(V)(x) + \beta a$*, for all* $V \in B(\Omega)$*,* $a \geq 0$*, and* $x \in X$*.*

Corollary 1. *Let* u *be continuous, The Bellman equation given as follows is a contraction.*

$$\Lambda : V(\cdot) \mapsto \max_{c_t} \Big\{ u(c_t) + \beta \mathbb{E}\big[V(f(x_t, c_t) + Z_{t+1}(\omega))\big] \Big\}$$

Proof. We must satisfy the conditions for Blackwell's theorem given above. First, let V_1 and V_2 be two bounded functions on $\mathbb{R}$. Then for any $c \in \mathbb{R}^4$ and $x \in \mathbb{R}$ we have that $u(c) + \beta \mathbb{E}(V_1(f(x,c) + Z(\omega))) \leq u(c) + \beta \mathbb{E}(V_2(f(x,c) + Z(\omega)))$, this implies:

$$\max_{c \in \phi(x)} \Big\{ u(c) + \beta \mathbb{E}\big[V_1(f(x_t, c_t) + Z(\omega))\big] \Big\} \leq \max_{c \in \phi(x)} \Big\{ u(c) + \beta \mathbb{E}\big[V_2(f(x_t, c_t) + Z(\omega))\big] \Big\}$$

Therefore, $\Lambda(V_1)(x) \leq \Lambda(V_2)(x)$. Next we must check the second condition. Consider $a \geq 0$.

$$\Lambda(V+a)(x) = \max_{c \in \phi(x)} \Big\{ u(c) + \beta \mathbb{E}\big[V(f(x_t, c_t) + Z(\omega))\big] + a\beta \Big\}$$

Thus we see that $\Lambda(V+a)(x) = \Lambda(V)(x) + \beta a$ for any given x. By the assumption that $\beta \in (0,1)$ we see that the conditions are satisfied. □

Using these facts, we can show that policy function iteration converges to a solution of the Bellman equation. One step of policy function iteration proceeds as follows.

1. Begin with a guess of the derivative of the value function $V_n'(x)$
2. Optimize to solve for a policy function $x \in \Omega \mapsto \pi_n(x) \in \phi(x)$.
3. Letting $V_{n+1} = \Lambda V_n$, update, using $V_{n+1}'(x) = \frac{\partial}{\partial x}(\Lambda V_n)(x)$.
4. Go to (1).

Theorem 2. *(Policy Iteration Convergence) Let* u *be differentiable,* f *be jointly differentiable,* $\Omega \subseteq \mathbb{R}$*, and* $\phi(x)$ *be compact. Beginning with an arbitrary differentiable value function* $V_0(x)$*, policy function iteration converges in the sense where for each* x*, the sequence of policy candidates* $\pi_n(x)$ *has a subsequence which converging to a value* π *which is optimal for that* x*. i.e.:*

$$V(x) = u(\pi) + \beta \mathbb{E}(V(f(x, \pi) + Z(\omega))$$

Where V *is the true value function.*

Proof. Fix an arbitrary step $n \in \mathbb{N}$. Let π_n be the optimal policy for this value function, solved for using V_n'. We know that $(\Lambda V_n)(x) = u(\pi_n(x)) + \beta \mathbb{E}(V_n(f(x, \pi_n(x)) + Z(\omega)))$.

Using the first envelope theorem in [10], we know that V_{n+1} is differentiable. Since Λ is a contraction we know that $V_n \to V$ in the topology of uniform convergence on the space of bounded functions, Where V is a fixed point of Λ. We claim that $V \in C(\Omega)$. Since each V_n is differentiable, we know that they are continuous, although the space of differentiable functions is not closed under the $||\cdot||_\infty$ norm, the space of continuous functions is. Therefore V must be continuous. We restrict our attention from $B(\Omega)$ to $C(\Omega)$, the space of continuous functions on Ω Let the correspondence $H : \Omega \times C(\Omega) \twoheadrightarrow \mathbb{R}$ be defined as follows:

$$(x, V) \mapsto \arg\max_{c\in\phi(x)} (u(c) + \beta\mathbb{E}(V(f(x, c) + Z(\omega)))$$

As a result of the continuity of u and f, it is straightforward to see that the map $(x, V, c) \mapsto u(c) + \beta\mathbb{E}V(f(x, c) + Z(\omega))$ is jointly continuous in $C(\Omega) \times \phi(x)$ for each x. Therefore, by the Berge maximum theorem, we see that the correspondence $V \mapsto H(x, V)$ must be upper hemicontinuous for each x [11].

□

As a result, we can begin with an arbitrary guess V_0, repeatedly apply the map Λ, and the resulting sequence of functions $(V_n)_{n\in\mathbb{N}}$ will converge with respect to the norm, $||\cdot||_\infty$, to some unique function V which is a fixed point of the map Λ. This value function solves the Bellman equation. Furthermore, if our sequence of policy functions $\pi_n(x)$ converges pointwise, we can ensure that the resulting function π is an optimal policy.

As a result of this procedure, solving the dynamic programming problem is straightforward as long as we can effectively evaluate the map Λ. This can prove to be difficult, as we are faced with finding the maximum of the right-hand side of the Bellman equation. We will approach this topic further in later sections.

3.1.2 First Order Conditions

We now aim to apply our theoretical discussion of policy function iteration to the solution of our optimization problem. Given the utility function for dividends of the bank, we say can derive the Bellman equation we aim to solve. We will denote the value function for this maximization problem starting at a given capital size M_t as $V(M_t)$.

$$V(M_t) = \max_{c_t\in\phi(x_t)} [u(d_t) + \beta\mathbb{E}_t V(M_{t+1})] \tag{3.1}$$

In order to solve the maximization problem for a given candidate value function, we must utilize the first order conditions of the above value function. There are four first order conditions, one for each variable. We begin with the first order condition for r_t.

$$-\beta r_b\mathbb{E}[Z_t^s V'(M_{t+1})] - \beta(r_t - r_b)\mathbb{E}[Z_t^s(\frac{\partial S_t}{\partial r_t} + \frac{\partial S_t}{\partial q_t}\frac{\partial q_t}{\partial r_t})V'(M_t)] = 0 \tag{3.2}$$

The first order condition for d_t

$$u'(d_t) - \beta r_b\mathbb{E}[Z_t^s V'(M_{t+1})] - \beta(r_t - r_b)\mathbb{E}[Z_t^s\frac{\partial S_t}{\partial q_t}\frac{\partial q_t}{\partial d_t}V'(M_{t+1})] = 0 \tag{3.3}$$

And finally, the first order conditions for a_t, and l_t, which are identical.

$$\beta r_b\mathbb{E}[V'(M_{t+1})] - \beta\mathbb{E}[Z_t^a V'(M_{t+1})] - \beta(r_t - r_b)\mathbb{E}[Z_t^s\frac{\partial S_t}{\partial q_t}\frac{\partial q_t}{\partial a_t}V'(M_{t+1})] = 0 \tag{3.4}$$

$$\beta r_b \mathbb{E}[V'(M_{t+1})] - \beta \mathbb{E}[Z_t^l V'(M_{t+1})] - \beta(r_t - r_b)\mathbb{E}[Z_t^s \frac{\partial S_t}{\partial q_t}\frac{\partial q_t}{\partial l_t} V'(M_{t+1})] = 0 \tag{3.5}$$

For simplicity we refer to the vector $c_t = (r_t, d_t, a_t, l_t)$, reducing the problem to finding value(s) $c^* \in \mathbb{R}^4$ That are optimal. Notice that these first order conditions depend only on the derivative of the value function rather than on the function itself. In order to use policy function iteration, we need the envelope condition. Simply differentiate the Bellman equation with respect to d_t on both sides, and it can easily be seen that $V'(M_t) = u'(d_t)$. Thus after we find the arguments which solve the optimization problem, we can use the optimal d_t to recover the value function at the next iteration as detailed in the previous section.

For any given candidate solution, c^*, we need conditions to test that are necessary to guarantee that c^* actually is a solution. In the case of a problem like the one we have detailed, we turn to the Karush-Kuhn-Tucker conditions. These conditions are standard in constrained optimization, so we refer the interested reader to [12].

Although in an ideal world we could solve these first order conditions at each level of capital in order to use the policy function iteration used in the last section, instead we only solve on a discrete grid of possible capital sizes $(M^{(n)})_{n=1}^N$. We then use a linear approximation to interpolate the value function beyond this limited set. In addition, notice that all of the first order conditions involve taking expectations of several random quantities, including endogenous ones. As a result, we must find ways to efficiently compute these quantities. In the next section we will cover how this is done in the context of our problem.

3.2 Numerical Solution

There are several intricacies which we will need to tackle in order to provide a numerical solution to this problem. One is how to compute the derivatives of q with respect to the control variables.

$$\frac{\partial q_t}{\partial r_t}, \frac{\partial q_t}{\partial d_t}, \frac{\partial q_t}{\partial a_t}$$

We have that $q = 1 - F(0; c_t, q)$, where $x \mapsto F(x; c_t, q)$ is the CDF of capital at a given control vector c_t and probability of failure q. Clearly we can compute the partial derivatives for q_t utilizing the implicit function theorem if we can compute the partial derivatives of F, using r_t as an example, we must find a way to compute the following.

$$\frac{\partial q_t}{\partial r_t} = \frac{\partial F / \partial r_t}{-1 - \partial F / \partial q_t}$$

Remember F was defined as follows.

$$F(0; c_t, q_t) = P(M_{t+1} - A(M_t) < 0) \tag{3.6}$$

Thus we need only compute a density for $M_{t+1} - A(M_t)$ and integrate to find the result. Thus we need to find integrals with respect to the distribution of:

$$(M_t + S_t(r_t, q_t, Z_{t+1}^s) - l_t - a_t - d_t)r_b + a_t Z_{t+1}^a + l_t Z_{t+1}^l - S_t(r_t, q_t, Z_{t+1}^s)r_t - A(M_t)$$

$$Z_{t+1}^s \sim \log N(\mu_s, \sigma_s^2),\ Z_{t+1}^a \sim \log N(\mu_a, \sigma_a^2),\ Z_{t+1}^l \sim \log N(\mu_l, \sigma_l^2)$$

Let F_s, F_a, F_l denote the cumulative distribution functions of Z_{t+1}^s, Z_{t+1}^a and Z_{t+1}^l respectively. We note that S_t is linear in it's random variable, and we define $s_t(r_t, q_t)Z_{t+1}^s = S_t(r_t, q_t, Z_{t+1}^s)$. Using the sums of random variables, we can see the following cumulative density function for $M_{t+1} - A(M_t)$ is appropriate.

$$F(0; c_t, q_t) = \int\int_{\mathbb{R}^2} F_l\left(\frac{-r_b(M_t - d_t - a_t - l_t) + s_t(r_t, q_t)z_s - az_a + A(M_t)}{l}\right)dF_s(z_s)dF_a(z_a) \tag{3.7}$$

Now, we can compute the partial derivatives of F, and using the implicit function theorem as shown above, compute the partial derivatives. There are many ways to compute integrals of the above form. For our purposes, we have chosen to utilize quasi-Monte Carlo. See [13] for a discussion of quasi-Monte Carlo methods and a comparison to traditional Monte Carlo. In our case we utilize Sobol quasi-random numbers and several transformations to manipulate them such that they are distributed in the manner that we wish. In our case, we would like out shocks (Z_t^s, Z_t^a, Z_t^l) to be log-normally distributed. Our procedure is as follows:

1. Generate a sequence of Sobol quasi-random numbers $Y_1, ..., Y_n$ such that $Y_i \in [0, 1]^3$.
2. Utilize a Box-Muller transform[14] to change these uniform quasi-random numbers into quasi-normally distributed numbers in $\mathbb{R}^3$
3. Shift and scale for the desired mean and variance.
4. Finally, Exponentiate.

Once we have generated quasi-random numbers $\{(z_i^s, z_i^a, z_i^l)\}_{i=1}^n$ that are distributed like the shocks (Z_t^s, Z_t^a, Z_t^l), we can use then as we would expect to calculate expectations. For example, suppose we would like to compute $\int h(z^s, z^a, z^l)dF_s(z^s)dF_a(z^a)dF_l(z^l)$.

$$\int h(z^s, z^a, z^l)dF_s(z^s)dF_a(z^a)dF_l(z^l) \approx \frac{1}{n}\sum_{i=1}^n h(z_i^s, z_i^a, z_i^l)$$

3.2.1 Algorithm

In this section we will summarize the algorithm used to solve the optimization problem. We are using policy function iteration, so to begin we discretize the capital into $\{M^i\}_{i=1}^N$. Our goal is to find an optimal policy c^i that solves the Bellman equation at each M^i. We now give our algorithm for policy function iteration.

In the following algorithm, we let $\phi(M, q)$ denote the set vectors $c \in \mathbb{R}^4$ that satisfy the constraints. $\Omega \subseteq \mathbb{R}^4$ is the entire set of possible controls. Therefore $\phi : \mathbb{R} \times [0, 1] \twoheadrightarrow \Omega$.

1. Make two guesses about the optimal c_t^i for $i = 1, ..., N$, these guesses must be in Ω. Call these $c^{i,0}$ and $\bar{c}^{i,0}$. Compute the resulting $q^{i,0} = Q(c^{i,0}, M^i)$ using Newton's method. Ensure $c^{i,0} \in \phi(M^i, q^{i,0})$

2. Interpolate $V'(M)$ using $\{c^{i,0}\}_{i=1}^{N}$ and the envelope condition: $V'(M^i) := u'(d_t^i)$

3. Use these two guesses and the maximization scheme to compute $\overline{c}^{i,1}$, $c^{i,1}$, and $q^{i,1} = Q(\bar{x}^{i,1}, M^i)$. Our method guarantees $\overline{c}^{i,1} \in \phi(q^{i,1}, M^i)$

4. $c^{i,0} \leftarrow \overline{c}^{i,1}$, $\overline{c}^{i,0} \leftarrow c^{i,1}$, and $q^{i,0} \leftarrow q^{i,1}$.

5. Return to Step 2

This type of routine is fairly standard. We will now consider the maximization procedure. We begin with two initial points $c^{i,0}$, $\overline{c}^{i,0}$ and $q^{i,0}$ such that $c^{i,0} \in \phi(M^i, q^{i,0})$ and $\overline{c}^{i,0} \in \Omega$. For each component $\overline{c}_j^{i,0}$ do the following.

1. Perturb $\overline{c}_j^{i,0}$, assigning it a value $\overline{c}_j^{i,0} \leftarrow (1+\xi)\overline{c}_j^{i,0}$ Where ξ is gaussian noise. Ensure it stays within Ω.

2. Use the secant method with the first order condition in that component to find a step size Δ

   (a) Assign $\overline{c}_j^{i,1} \leftarrow c_j^{i,0} - \Delta$.
   (b) Test if $\overline{c}^{i,} \in \Omega$, if not: $\Delta \leftarrow \alpha\Delta$ and go to (1)
   (c) Compute $q^{i,1} = Q(\overline{c}^{i,1}, M^i)$
   (d) Test if $\overline{c}^{i,1} \in \phi(q^{i,1}, M^i)$ if not: $\Delta \leftarrow \alpha\Delta$ and go to (1)

3. Update $c^{i,1} \leftarrow \overline{c}^{i,0}$

4. If repeating then update $\overline{c}^{i,0} \leftarrow c^{i,0}$ and $c^{i,0} \leftarrow \overline{c}^{i,1}$ return to Step 2.

This above maximization procedure can be considered a quasi-Newton method. These are common, however in principle this maximization procedure can be replaced by any given method for constrained maximization. Common methods in Python would be ”fsolve” in the Scipy package and ”root” in the numpy package along with a penalty term for being outside the feasible region.

Chapter 4

Results and Conclusions

4.1 Results

In this section we will estimate the model using the solution methods previously discussed. To begin we must specify a utility function for dividends, $u(d)$ and a reserve supply function S_t:

$$u(x) = \frac{\log(1+x)}{1+\log(1+x)}$$

$$S(r_t, q_t) = (r_r - r_b)^{.5} q_t^{1.1}$$

In addition, we let the risk-free interest rate, r_b equal 1.02. We let the reserve requirement be $\lambda = .1$. We allow the firm to falter from the reserve requirement with a probability of $p = .05$. After computation we end up with a number of functions that map a given capital level to an optimum for the control variable. We present these graphs below.

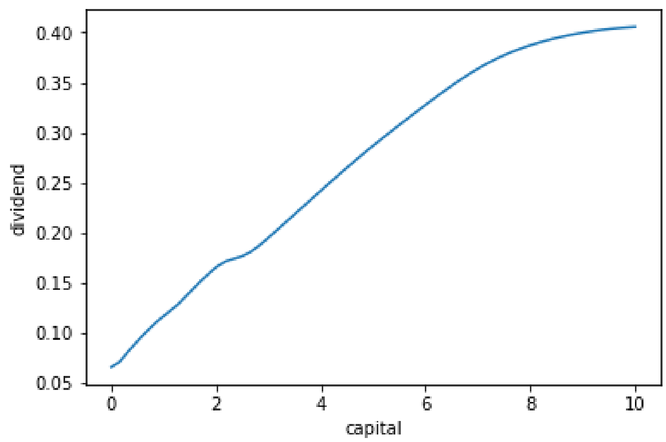

Figure 4.1: Dividend payout versus amount of capital available to the firm.

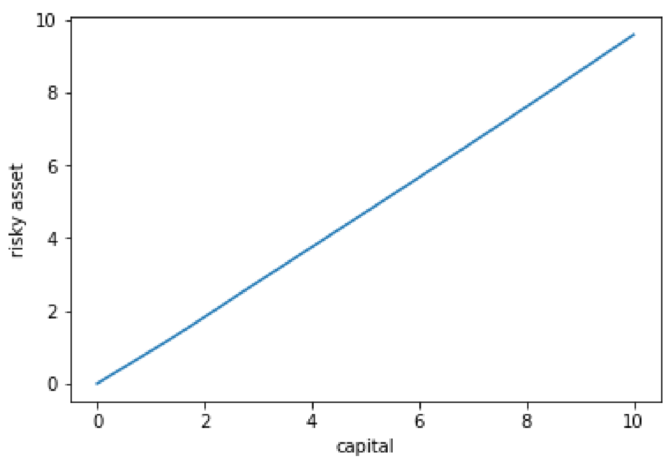

Figure 4.2: Amount of capital invested in risky assets versus amount of capital available to the firm.

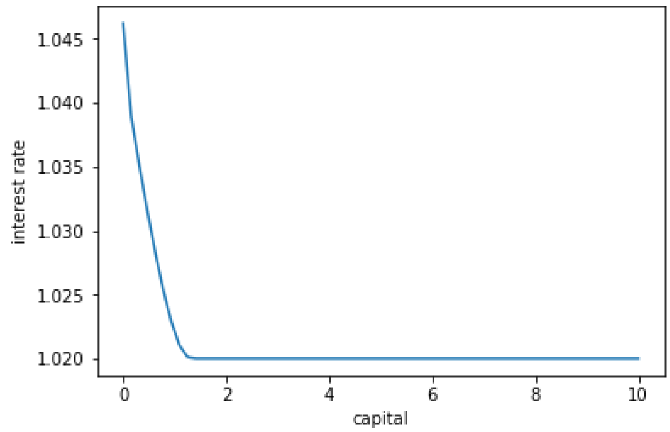

Figure 4.3: Internal interest rate offered versus amount of capital available to the firm.

As we can see, the responses are as we would expect, with the amount invested in risky assets and the amount paid out in dividends increasing with the amount of capital that the bank possesses. Note that the internal interest rate offered by the bank decreased with the amount of capital they possessed. Intuitively, this makes sense because smaller banks used these higher interest rates to draw reserves in order to invest in the capital and credit markets. We exclude the graph relating probability of failure with capital size because it is somewhat uninteresting. Namely, we find that under the restrictive reserve requirements that the bank has a probability 1 is survival for each capital level. As mentioned we allowed the bank to falter from the reserve requirement of $\lambda = .1$ with a probability $p = .05$. In the next graph we show how the variation of this parameter, as well as the reserve requirement λ effects the internal interest rate that the bank offers.

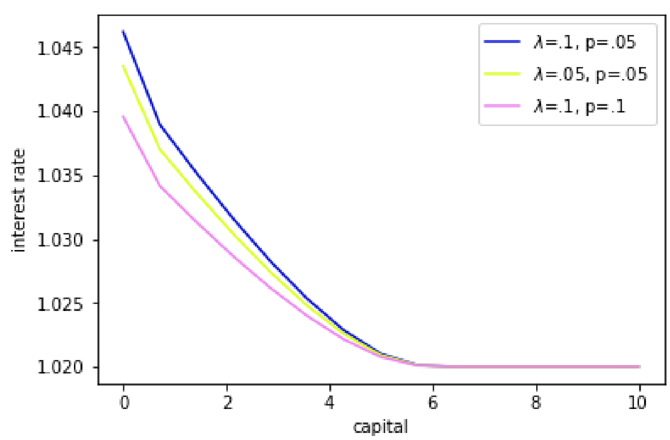

Figure 4.4: Variation of λ and p

As we can see, moving from the original model $\lambda = .1, p = .05$, the less stringent constraint to $p = .1$ allowed the smaller firms to offer a less competitive interest rate. In addition, we see that the move to $\lambda = .05, p = .05$ prompted a similar, but less severe decline.

4.2 Conclusion and Future Work

In future work, we aim to provide a more complete model of the interplay between banks and consumers by formally modeling the consumer's maximization problem, rather than simply providing a reserve supply function. Additionally, we find that the constraint that banks satisfy the reserve requirement almost surely to be very tight, in the future, we aim to incorporate an interbank market or Fed desk into our model, allowing the banks to borrow at some premium rate in the event they do not satisfy this requirement.

Our main contributions to the model of Gonzalez-Hermosillo Li is the addition of a Log-normal liquidity shock which is more representative of what banks face in reality. We also contribute a novel method of solving this problem, implementing quasi-Monte Carlo to solve the first order conditions. The solution method that we detail in chapter 3 is not specific to this problem, so it could be used to help other researchers solve similar stochastic dynamic programming problems.

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