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PROBLEM-SOLVING ABILITIES AMONG FIRST-YEAR UNDERGRADUATE STUDENTS WITH QUALIFYING AP CALCULUS EXAM SCORES

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ABSTRACT

During the past few decades, substantial reforms within the mathematics education community have radically redefined the notion of what it means for a student to be mathematically proficient to include, for example, adaptive reasoning and strategic competence (Kilpatrick, Swafford, & Findell, 2001). A review of relevant literature informs definitions of “problem” and “problem solving” as the terms appear in current interpretations of mathematical proficiency. Additionally, literature-based criteria for characterizing problem-solving performance arise in four areas: resources, heuristics, control, and belief systems. Guided by these criteria, this study investigates the problem-solving abilities of first-year undergraduate students who achieved qualifying scores on the AP Calculus AB Exam in order to determine to what extent and in what combination the problem-solving influences of adaptive reasoning, strategic competence, and established experience are employed in task solving.

In this exploratory study, first-year undergraduate students with qualifying AP Calculus Exam scores participated in a series of task-based interviews, which were videotaped and analyzed for evidence of the problem-solving influences. The interview tasks are modifications of common calculus tasks and were designed to be challenging problems for which students did not have preexisting solution strategies.

The results indicate that students did not exhibit adaptive reasoning to draw appropriately on their resources when forced experience—the recollection and forced unproductive use of ideas recalled from past work on mathematical tasks—inhibited the use of strategic competence to develop additional representations of the problems.
Displaying limited strategic competence, students who fixated on a single representation of the problem did not explore alternative strategies and draw on relevant resources. Students who did not initially employ rich adaptive reasoning did show evidence of the capacity for adaptive reasoning in response to scaffolding strategies that provided information about a missing piece of strategic competence. This combination of observations suggests that forced experience, with its unproductive use of ideas from earlier work with mathematical tasks, occurs in combination with an absence of strategic competence and subsequent lack of adaptive reasoning to inhibit students’ success in solving novel problems although students show the capacity for adaptive reasoning in these problem situations. Although more research must be done to generalize this result, the implications of the study for practice suggest the need to implement task modifications and scaffolding strategies in all mathematics classrooms in order to give students the opportunity to develop and exhibit strategic competence.
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Chapter 1: Background

We face an interesting paradox in the current state of mathematics education. Our mathematics instruction is being reformed in favor of developing students who have, as Kilpatrick and his coeditors (2001) term, “mathematical power” (p. 115) to solve problems, justify their reasoning, and communicate mathematics with others. However, the standardized assessments used to measure the achievement of these students reflect a belief that knowing how to find the correct answer as quickly as possible is a primary goal of mathematics instruction. Could it be that a student who is “qualified” according to a standardized assessment does not completely possess the mathematical power of which Kilpatrick and his colleagues speak so highly?

In this chapter, I discuss my motivation for wanting to study the problem-solving abilities of first year undergraduate students with qualifying AP Calculus Exam scores. I begin by providing a brief overview of the AP Program, followed by definitions and interpretations of the terms “problem” and “problem solving” as they appear in mathematics education literature. Finally, I discuss how I plan to use the relevant literature as a conceptual framework for this study.

Introduction

Each spring The College Board offers over thirty Advanced Placement (AP) Exams that give high school students the opportunity to earn college credit for or opt out of entry-level coursework (Mattern, Shaw, & Xiong, 2009). Specifically, the Calculus AB Exam assesses a student’s mastery of course content equivalent to the first semester of university calculus by assigning a score ranging from 1 to 5 based on the student’s
performance on multiple choice and free response items. According to The College Board (2010a), AP Exam scores of 3 (Qualified), 4 (Well Qualified), or 5 (Extremely Well Qualified) are equivalent to college course grades of B-/C+/C, B/B+/A-, and A, respectively. Typically students are awarded transfer credits or advanced placement by achieving a score of 3 or above on the AP Exam (The College Board, 2010a).

The College Board and ETS (the non-profit group that develops and administers the AP Exams) report that students who receive these qualifying scores have higher college GPAs than their peers who either did not take AP classes or received non-qualifying scores (Keng & Dodd, 2008; Morgan & Ramist, 1998). When comparing only college GPAs of the two student populations, this claim appears to be true. However, studies that controlled for a student’s non-AP math and science courses (Ferrini-Mundy & Gaudard, 1992; Klopfenstein & Thomas, 2009) and SAT Math scores (Dickey, 1986) find that enrollment in AP Calculus provides no statistically significant positive effect on a student’s first semester GPA or performance on calculus achievement tests. Furthermore, Klopfenstein and Thomas (2006) found that enrollment in AP Calculus has no effect on retention of introductory course content at the beginning of the second year. They believe this may be due, in part, to the amount of time necessary to cover the extensive amount of material on the AP Exam, which eliminates time available to focus on deep, conceptual understanding.

Despite the use of periodic reviews, studies, and course audits to ensure that a student’s AP scores truly reflect his or her ability compared to students enrolled in equivalent college courses (The College Board, 2010a), I am left with one major question: can a student who achieved a “qualifying” score on the Calculus AB Exam
actually solve a calculus problem? Before I can even begin to consider an answer to the aforementioned question, I must consider three additional prevailing questions: what exactly are mathematics problems, what is problem solving, and how does one measure a student’s problem-solving proficiency? I will refer to the mathematics education literature to gain insights upon which I will base the answers to these questions.

**Definition of a Problem**

The word “problem” is often used in both scholarly research and mathematical conversations, but at least two distinct interpretations of its meaning exist. In one sense, Webster’s defines a problem as “in mathematics, anything required to be done, or requiring the doing of something” (as cited in Schoenfeld, 1992, p. 337). For example, a teacher may ask students to do 20 homework problems, and the implication is that the students should come to class the following day with 20 tasks completed. However, if we take the solution strategy required for the task’s completion into account, the word “problem” no longer automatically applies. If the task can be accomplished using a procedure or algorithm that is readily known to the solver, it is generally referred to as an exercise (Kantowski, 1977; Lithner, 2000; Schoenfeld, 1992) or routine problem (Kilpatrick, Swafford, & Findell, 2001).

In contrast to conveying the notion of an exercise, the literature supports that a problem, sometimes called a non-routine problem, cannot be solved with a single procedural step or by directly applying a known algorithm (Erickson, 1999; Kantowski, 1977; Kilpatrick et al., 2001; Schoenfeld, 1992). In essence, a true problem requires a student to select, apply, and extend appropriate prior knowledge in a way that is not
immediately obvious to the student. For the purposes of this paper, the word “problem” will always be used as it is interpreted in the previous sentence, and the word “exercise” will be used to describe any other task that does not fit the criteria of a problem.

Perhaps more important than the definition of a problem, however, is the opportunity it provides for students to engage in meaningful mathematics. Hiebert and Wearne (2004) assert that a problem provides the context in which students begin to understand mathematics instead of simply performing procedures or calculations. With respect to the previous example of a teacher assigning 20 homework “problems” (which are, in fact, exercises), one might think completing those tasks correctly would be an indicator of understanding. According to Hiebert and Wearne, a more effective way to help students understand mathematics is to make it more problematic for them (2004); by allowing students to struggle a little on the way to a correct solution they will understand the underlying mathematics better than if they would mechanically apply a procedure 20 times.

**Definition of Problem Solving**

Even though the literature has established that problems are more than mere tasks that students do, I believe that the primary goal of a problem is ultimately to have students engaged in the task and searching for a solution. In my opinion, the operative word in the previous sentence is searching because it captures the necessity to look beyond the obvious procedure required in an exercise. The idea of searching for a solution also avoids the stipulation that a correct solution must be found and emphasizes the process leading up to a solution, be it correct or incorrect. Before I began my
research, it made intuitive sense to me to refer to the process mentioned in the preceding sentence as problem solving, but Schoenfeld (1992) detailed numerous other interpretations of problem solving as observed throughout the 1980s curricular reform. Clearly, the semantics of “problem solving” vary greatly, but through additional literature review I hoped to develop an operational definition of problem solving that would be useful in preparing my study. To do this, I looked not only at what problem solving could be, but also what problem solving definitely is not.

To provide an example of what problem solving definitely is not, Schoenfeld (1992) discusses a typical use of exercises from Milne’s *A Mental Arithmetic* (as cited in Stanic & Kilpatrick, 1988). Milne’s text models an appropriate algorithmic technique and then requires students to repeatedly practice the same isolated skill. Any extension of the algorithm introduced in the first example task is treated as a completely separate type of task accompanied by a “suggestion” that reduces what could potentially be a problem down to another exercise. Schoenfeld (1992) argues that the assumption is that students will “have a new technique in their mathematical tool kit” (p. 338) after completing a set of exercises in this manner, and that “the sum total of such techniques (the curriculum) reflects the corpus of mathematics the student is expected to master; the set of techniques the student has mastered comprises the student’s mathematical knowledge and understanding” (p. 338).

The previous non-example (at least according to my interpretation) of problem solving alludes to the fact that defining problem solving in a particular way also has implications for the nature of mathematics itself. Lampert (1990) believes that there are a set of “cultural assumptions” influenced heavily by years of being subject to the belief
that mathematics is “associated with certainty: knowing it, with being able to get the right answer, quickly” (p. 32). According to these assumptions, Lampert (1990) explains that a student does mathematics by following the “rules laid down by the teacher” (p. 32) and knows mathematics when he or she can apply the correct rule in response to a teacher’s question. This interpretation of doing and knowing seemed hollow and uninteresting to me, so I continued my search for a definition of problem solving that could underpin my study.

As I continued to read, my earlier suspicion that a semantic definition of problem solving would not be as useful as an operational definition that seemed to hold true. The most useful interpretations of problem solving I found described not so much what problem solving is but what it is intended to do: promote mathematical understanding by requiring students to think about, reason through, and construct a solution based on, but not limited to, their prior knowledge (Erickson, 1999; Webb, 1979). In this respect, Stanic and Kilpatrick (1988) reflect on three themes that have historically situated problem solving within mathematical curricula: problem solving as context, problem solving as skill, and problem solving as art. Of the three themes, I was most fascinated with and receptive to the notion of problem solving as art. Contrary to the previously mentioned interpretations of doing and knowing mathematics, problem solving as art reflects George Polya’s belief that mathematics consists of “information and know-how” (Stanic & Kilpatrick, 1988, p. 16). A student must know not only the procedures, algorithms, and theorems practiced using exercises but also how to select and apply each to problematic situations in which the selection is not immediately obvious. To extend one of my previous points, if a problem is the canvas upon which students can begin to
do and understand mathematics (Hiebert & Wearne, 2004), problem solving is set of brush strokes that leads to the desired understanding.

**Characteristics of Problem-Solving Performance**

Because of the imprecise nature of the previously discussed definition for problem solving, I had to carefully consider what would actually constitute evidence of a student’s problem-solving ability (or lack thereof). Schoenfeld (1985) gives some insight into the components necessary for “adequate characterization of mathematical problem-solving performance” (p. 15): resources, heuristics, control, and belief systems.

Schoenfeld (1985) describes resources as the “mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand” (p. 15). Simply put, resources are all of the skills, procedures, and understandings that a student has in his or her mathematical toolbox. An important distinction must be made that, although the correct use of procedures and algorithms is likely required in a successful problem-solving performance, a procedure or algorithm is a means to an end rather than the end itself. Webb (1979) studied the effects of student resources (which he called “conceptual knowledge”) in problem solving and found that they accounted for about 50% of the variance in student scores on problem-solving tasks. The second component of Webb’s (1979) study was the effect of student processes such as heuristics, which he found to account for 13% of variance in student scores.

As Webb observed, heuristics, or mathematical “rules of thumb” (Schoenfeld, 1985, p. 15) also play an important role in the problem-solving process. Their use in problem-solving tasks has been investigated through numerous additional studies.
(Carlson & Bloom, 2005; Kantowski, 1977; Lithner, 2000; Schoenfeld, 1982), most all of which cite Polya’s (1957) *How to Solve It: A New Aspect of Mathematical Method*. Two recurring heuristic strategies that students used in many of the above studies were drawing a diagram or considering a related problem, likely because their mathematical studies had introduced them to these strategies even in the absence of formal heuristic training.

A third component of problem-solving performance is control, which Schoenfeld (1985) characterizes as the “global decisions regarding the selection and implementation of resources and strategies” (p. 15). Control is necessary in problem solving because of the fact that the strategy necessary for obtaining a solution is not immediately obvious to the solver. Control can be observed in problem solving through a student’s decisions in planning the solution strategy, monitoring the progress, reflecting on new information, and adapting the strategy based on prior monitoring and reflection (Carlson & Bloom, 2005).

The final component of problem solving is a belief system that reflects the student’s “mathematical world view” and dictates how a student can use the first three components in a problem-solving setting (Schoenfeld, 1985, p. 15). Carlson and Bloom (2005) expand on Schoenfeld’s belief system to include attitudes, emotions, and values in what they call the “affective dimension” (p. 50). With respect to the affective domain, Carlson and Bloom found that students exhibited strong emotions of both frustration and joy while working on their solutions, and management of these emotions played a key role in each student’s persistence with difficult problems.
The four components described above—resources, heuristics, control, and belief systems—provide an overview of the indicators of student problem-solving performance. In the conceptual framework section, I discuss the extent to which I incorporate each of these components in the analysis of my data.

**Conceptual Framework**

Equipped with Schoenfeld’s problem-solving classification criteria as previously described, I now return to the preliminary question that sparked my interest in mathematical problem solving: can a student who achieved a qualifying score on the Calculus AB Exam actually solve a calculus problem? Any such problem, of course, would have a solution that is not immediately obvious to the student, but the solution would have to be accessible according to the student’s resources. Based on the student’s AP Calculus background, I could examine the Calculus AB course syllabus to determine, with relative certainty, a reasonable set of resources that a successful Calculus AB student likely would bring to the table. The AP Course Audit mandates that a student’s high school studies include all topics listed on the syllabus, and the standardized Calculus AB Exam measures the student’s proficiency of using and applying the procedures and concepts associated with the various content areas. By the above reasoning, it seems participants in this study likely would have adequate resources and non-negative views of mathematics. I did not choose resources or belief systems as a focus of my study but I keep them in mind in case there are major differences in these two areas among the participants that skew the results.
Instead, my hope is to gain some insight into how students access and apply their resources. I am drawn to the synergy between the heuristics and control aspects of problem-solving performance. Polya, historically known as the father of problem solving, gives a general summary of this synergy in his four-phase strategy of “How to Solve It” (1957): understanding the problem, devising a plan, carrying out the plan, and looking back. His devising a plan phase relies heavily on the heuristic strategy of looking for similarities between the problem at hand and problems from the past. Given that these AP Calculus students have likely had most of their mathematics experience in a classroom setting, my hypothesis is that they will draw heavily on their experience with tasks (be they problems or exercises) from past homework or assessments. In the understanding the problem phase, Polya (1957) suggests the use of heuristic strategies such as drawing a diagram, introducing notation, and sifting through given information. Control strategies play an important role in carrying out the plan and looking back, in which a problem solver must check and justify each step of the solution strategy as well as the final answer.

As I considered Polya’s problem-solving strategies, my thoughts returned to the literature by Kilpatrick and his coeditors (2001) and Lithner (2000). Kilpatrick and colleagues (2001) discuss the reform movement of the 1980s and 1990s that shifted the focus of mathematical learning from knowing basic procedures (proficiency with exercises) to “mathematical power” (p. 115). According to Kilpatrick and his coeditors, mathematical power consists of “reasoning, solving problems, connecting mathematical ideas, and communicating mathematics to others” (p. 115). Based on this interpretation of mathematical learning, Kilpatrick and colleagues list the strands of mathematical
proficiency as conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. In the ensuing paragraphs, I describe how the strands of strategic competence and adaptive reasoning are indicative of both heuristics and control in problem solving.

Kilpatrick and his coeditors (2001) describe strategic competence as the ability to “formulate, represent, and solve mathematical problems” (p. 116), which I believe ties in nicely with Polya’s heuristic strategies. In particular, the idea of student experiences comes to light in the authors’ distinction between novice and expert problem solvers. Kilpatrick and colleagues discuss how novice problem solvers see surface similarities between new and past problems, whereas more seasoned problem solvers can identify structural similarities that may make the potential use of a non-obvious procedure more apparent. Finally, the authors also explain the role of flexibility in strategic competence as a way for a student to develop several representations of the problem and choose the best one based on the particular problem.

According to Kilpatrick and his coeditors (2001), adaptive reasoning “refers to the capacity to think logically about the relationships among concepts and situations” (p. 129). Adaptive reasoning corresponds to the control aspects of Polya’s heuristics because it allows the student to continuously assess whether the strategies developed through strategic competence are indeed helping the student move productively to a solution. A student who exhibits adaptive reasoning does not indiscriminately perform procedures; instead, each procedure is chosen for a particular reason, and the student is able to mathematically justify why the procedure was selected and whether it was implemented correctly.
Lithner (2000) provides additional insight into the heuristics and control aspects of problem solving through his discussion of plausible reasoning and reasoning based on established experiences. A student’s reasoning is considered plausible if it is “founded on mathematical properties of the components involved in the reasoning, and is meant to guide towards what probably is the truth, without necessarily having to be complete or correct” (Lithner, 2000, p. 167). In contrast, the reasoning is based on established experiences if it is “founded on notions and procedures established as the basis of the individual’s previous experiences from the learning environment, and is meant to guide towards what probably is the truth, without necessarily having to be complete or correct” (Lithner, 2000, p. 167). In both plausible reasoning and reasoning based on established experiences, the intention to guide towards the probable truth provides evidence of the planning and decision-making aspects of the control component of problem solving.

My initial prediction that students would rely heavily on experience from their prior studies of mathematics is validated by Lithner’s description of reasoning based on established experiences, which can be included in the heuristic strategy of exploring related problems. In contrast to adaptive reasoning, reasoning based on established experience motivates the problem solver to choose a strategy based on the fact they have used it before in similar mathematics studies instead of providing a mathematical justification for why a particular strategy is chosen.

For the purposes of this study, I combine Lithner’s (2000) interpretations of plausible reasoning and reasoning based on established experience to define what I consider to be evidence of a student’s established experiences in problem solving. Students who make an explicit statement of something they did or learned in a
mathematics class will be considered to have exhibited evidence of their established experiences. Thus, it is possible for a student to exhibit evidence of established experience either in conjunction with adaptive reasoning (e.g., “I have done it before and I know why I did it”) or in the absence of adaptive reasoning (e.g., “I have done it before so I will do it again”).

In summary, my research interest lies in the synergy between the heuristics and control components of problem solving. I am particularly interested in the work of Kilpatrick and coeditors (2001) and Lithner (2000) and the roles that adaptive reasoning, strategic competence, and established experiences play in a student’s solution strategy. Thus, my study is designed to answer the following research question: when asked to solve introductory calculus problems that are structurally similar to but more problematic than typical textbook tasks, to what extent and in what combination do first year undergraduate students with qualifying AP Calculus Exam scores incorporate their established experiences, adaptive reasoning, and strategic competence?
Chapter 2: Methods

In this chapter I will outline the task development process, interview procedures, participant selection, and data analysis for the study. In the discussion of task development, I provide descriptions of typical textbook tasks and my efforts to problematize them for use in a series of task-based interviews. I also discuss the design and structure of the interview sessions to maximize the opportunities to observe evidence of the problem-solving influences found in the research question. For participant selection, I describe the recruitment process and provide background information from the students who volunteered their time for the study. Finally, I describe the data analysis procedures that give rise to the findings discussed in chapter 3.

Interview Tasks

The first step in designing the study was to develop the problem-solving tasks to use in the interviews based on the definition of a problem discussed in chapter 1. My goal, to draw on Hiebert and Wearne’s (2004) phrasing, was to make mathematics more problematic by choosing a task “just within the students’ reach, allowing them to struggle to find solutions and then examining the methods they have used” (p. 6). To illustrate this goal, Hiebert and Wearne (2004) discuss the example of students solving $|2x + 3| > 3$ after having learned how to solve $|x| > 3$. Rather than treating the former task as a new, separate example of solving inequalities, it can be made into a problem by asking students to apply what they know from the latter case to develop and reason through their own solution.
As a consequence of the definitions discussed in Chapter 1, the interview tasks could not be such that I asked a student to explicitly find a derivative or compute an integral because students learned numerous procedures (e.g., power rule, product rule, quotient rule, chain rule) in their AP studies to perform these computations. Application problems such as related rates, optimization, particle motion, and solids of revolution seemed like the logical place to begin my task selection. I considered application problems due to the necessity of translating a real-world situation into a constructive mathematical representation and selecting appropriate procedures to use in subsequent solution strategies. The aforementioned translation and selection processes provide students with opportunities to display their strategic competence and adaptive reasoning, respectively. I considered related rates, optimization, particle motion, and solids of revolution as the particular types of application problems due to the frequency with which they occur in the textbooks examined in the next paragraph. The fact that each type of task is found in the textbooks serves as another safeguard that students in the study will likely possess suitable resources to solve the interview tasks.

I hoped to avoid a situation in which students would merely reproduce solution strategies they developed in their previous calculus studies. To determine what the “typical” tasks were, I examined three calculus texts listed on the Calculus AB Example Textbook List (The College Board, 2010b)—Stewart (2003); Larson, Hostetler, and Edwards (1998); and Weir, Hass, and Giordano (2006)—to see the contexts in which the aforementioned topics were presented as examples in each book. The university at which I conducted this study uses Stewart’s textbook, and similarly sized research institutions considered in Ramist and Morgan (1997) use Weir and colleagues’ book. I chose to look
at the textbook examples because the university Calculus I course syllabus indicated a high dependence on the required textbook. Thus, a student in a classroom using one of those books would likely receive instruction that aligned closely with tasks similar to the examples. Therefore, what was presented explicitly in the textbook as a solution to the task would constitute (at least for a student who paid attention in class) a well-defined procedure that could be applied to similar tasks. Hence, in order for my tasks to problems I would have to ensure that their structure (context and content) was problematic enough that the solution strategy modeled in the example could not be directly applied. My summary is presented in Table 1.
Table 1: Summary of Contexts and Content for Example Problems Used in Calculus Texts

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<tbody>
<tr>
<td>Related Rates</td>
<td>Air into spherical balloon, ladder (using Pythagorean theorem), water into a tank, path of a searchlight (using right triangle trigonometry)</td>
<td>Air into spherical balloon, speed of an airplane (using Pythagorean theorem), angle of elevation (using right triangle trigonometry)</td>
<td>Water draining out of a cylindrical tank, angle of elevation (using right triangle trigonometry), speed of a car (using Pythagorean theorem)</td>
</tr>
<tr>
<td>Optimization</td>
<td>Maximizing fenced area, minimizing surface area of a can, closest point to a given point on a graph, boat traveling across a flowing river (using Pythagorean theorem), maximizing area of an inscribed rectangle</td>
<td>Maximizing volume of a box, closest point to a given point on a graph, minimizing area of a printed page</td>
<td>Maximizing volume of a box, minimizing surface area of a can, maximizing area of an inscribed rectangle</td>
</tr>
<tr>
<td>Particle Motion</td>
<td>Interpret distance/displacement as area under velocity curve (formulas with definite integrals)</td>
<td>Vertical motion (indefinite integrals)</td>
<td>Distance/displacement (estimate with finite sums)</td>
</tr>
<tr>
<td>Solids of Revolution</td>
<td>Rotate a region defined by given function or graph (formulas given, not called disc/washer method)</td>
<td>Rotate a region defined by given function or graph (“Disc Method” and “Washer Method”)</td>
<td>Rotate a region defined by given function or graph (“Disk Method” and “Washer Method”)</td>
</tr>
</tbody>
</table>

Note: Italic text indicates commonality among textbooks.

The commonalities listed in Table 1 give some indication as to what would constitute a reasonable set of resources to which a student would have access when solving a related rates, optimization, or solids of revolution problem (particle motion is discussed in detail in the next paragraph). The related rates examples in two of the three books model a growing sphere in the context of an inflating balloon. The optimization examples among the three textbooks, despite the differences in context, model strategies
for optimizing the area of a two dimensional figure. For both related rates and optimization tasks, the textbook examples indicate that applying the Pythagorean theorem is a strategy that is often employed in a solution process. Finally, all examples for the topic of solids of revolution involve interpretations of accumulating the areas of disks or washers; Larson and his colleagues (1998) and Weir and his coauthors (2006) even give names to the procedures corresponding to the aforementioned interpretation.

For the topic of particle motion, the textbook examples do not paint a picture of typical tasks. Additionally, in the textbook tasks labeled as “Exercises” there are very few particle motion tasks in any book, so I cannot look there for an interpretation of what “typical” might mean. In light of these discoveries from the textbooks, it seems more appropriate to classify particle motion tasks by the typical content of definite integrals and derivatives. Particle motion tasks can be problematic, in and of themselves, due to the context for which the use of derivatives or integrals is required. The context suggests no explicit strategy of using an integral or derivative, but the resources for integration or differentiation are the same familiar rules (see, for example, the inside front and back covers of Weir, et al., 2006) given the strategic competence to develop a constructive representation of the problem. If we apply a variant of Hiebert and Wearne’s (2004) suggestion and treat particle motion tasks as an extension of integration and differentiation exercises instead of explicitly modeling a particle motion solution process, we create a problematic situation in which students can apply their own strategic competence and adaptive reasoning.

In order to problematize the typical textbook tasks, I either created my own items or modified existing ones to reflect Thompson, Beckmann, and Senk’s (1997) eight
criteria for improving assessment questions, involving item format, reasoning, open-endedness, skill, level, representation, real context, and technology. Each specific task is discussed in detail later in this section; the review of the literature from Thompson and colleagues is meant to be an overview of the motivations behind my decisions. Furthermore, in the description of each task I discuss how the resources necessary to solve the problem fit within the reasonable set of resources I described using the commonalities among the textbooks.

The criteria identified by Thompson and her coauthors serve as my basis for problematizing a typical textbook task according to Hiebert and Wearne’s (2004) interpretation discussed in the first paragraph of the interview tasks section. My tasks require students to construct and justify their own responses (corresponding to the criteria of item format, reasoning, and open-ended), perform multiple nonobvious steps in the solution process (corresponding to the criteria of skill and level), and create representations from given data (corresponding to the criterion of representation). Additionally, the use of contexts outside of mathematics (corresponding to the criterion of real context) assesses a student’s ability to, as Thompson and colleagues describe (1997), translate across representations. Finally, each task is technology neutral (in the sense that it could be solved either with or without the use of technology) due to the stipulation in the university Calculus I course syllabus that calculators were prohibited on exams; I want to keep the tasks for the interviews as structurally similar to the typical course content as possible. However, I relaxed the constraint slightly to allow students to perform calculations more expediently with a four-function calculator in light of the very brief time allotted for each interview.
As an additional component of my task development, I thought it would be useful to create two orientation tasks that would familiarize the students with my interpretation of a problem and provide an opportunity for them to practice their justifications in a “semi-problematic” context. I wanted them to see the difference between a one-step exercise like those that they likely completed hundreds of times on their homework and a real problem that required interpretation, justification, and application of a variety of mathematical ideas, not all of which necessarily had to be from calculus. I hoped the orientation tasks, due to their comparative ease with respect to the ensuing problems, would set students up for success. Additionally, I wanted to acclimate the students to thinking aloud to give me information on the following issues: ideas from calculus they felt related to the problem, how and why they chose to apply any prior knowledge in their solution strategy, any strategies they considered but did not implement, and how confident they were that their solution strategy was correct.

After considering my interpretation of what a “problem” is, the idea of making the interview tasks problematic to provide an authentic problem-solving scenario, and the orientation goals I had for the first interview, I constructed a set of six interview tasks as described in Table 2. To parallel the details of the orientation tasks above, I discuss the choice, adaptation, and development of the four targeted problems in the description of each interview.
Table 2: Interview Tasks

<table>
<thead>
<tr>
<th>Task Name</th>
<th>Problem Text</th>
<th>Topic</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task 1.1: Derivative of a Polynomial Function exercise</td>
<td>If $f(x) = x^3 + 3x - 1$, what is $f'(x)$?</td>
<td>Typical Textbook Exercise</td>
<td>My own</td>
</tr>
<tr>
<td>Task 1.2: Derivative of an Absolute Value Function problem</td>
<td>Consider the function $f(x) =</td>
<td>x - 1</td>
<td>$. What is the value of the derivative of this function when $x = -1, 0, 1$?</td>
</tr>
<tr>
<td>Task 2.1: Radius of a Growing Sphere problem</td>
<td>At time $t$ ($t \geq 0$), the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is 2. Find the radius of the sphere as a function of $t$.</td>
<td>Related Rates</td>
<td>The College Board (2003)</td>
</tr>
<tr>
<td>Task 2.2: Strength of a Wooden Beam problem</td>
<td>A wooden beam has a rectangular cross section of height $h$ and width $w$. The strength of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest board that can be cut from a round log of diameter 24 inches?</td>
<td>Optimization</td>
<td>Modified from p212 #39 of Larson, Hostetler, &amp; Edwards (1998)</td>
</tr>
<tr>
<td>Task 3.1: Path of a Moving Particle problem</td>
<td>The velocity of a particle at time $t$ ($t \geq 0$, in seconds) is given by the function $v(t) = t^2 - 4t + 3$. What is the total distance traveled by the particle in its first 5 seconds of motion?</td>
<td>Particle Motion</td>
<td>My own</td>
</tr>
<tr>
<td>Task 3.2: Volume of a Metal Ring problem</td>
<td>A manufacturer drills a hole through the center of a metal sphere of radius 5 inches. The hole has a radius of 3 inches. What is the volume of the resulting metal ring?</td>
<td>Solids of Revolution</td>
<td>Modified from p421 Example 5 of Larson, Hostetler, &amp; Edwards (1998)</td>
</tr>
</tbody>
</table>

Note: The tasks are numbered to indicate the interview in which they were administered and the order in which they were presented within that interview. For example, the second task presented in the first interview is Task 1.2.
The tasks were distributed across three interviews (two problems each) that I intended to conduct during the 7th, 10th, and 13th weeks of the semester. I chose these weeks based on my study of the university Calculus I course syllabus and my hope that, for the most part, the topics covered in each interview would be discussed in class before I met with each student. Since I developed the tasks to be more problematic than those that the students would likely experience in class, I was not worried that students would be pigeonholed into reproducing a procedure or solution strategy they saw during those weeks. Instead, I hoped that reviewing the related topics in class would serve as a refresher for procedural skills and conceptual knowledge that would be among the students’ resources that could be used in the interviews.

The first interview, comprised of the Derivative of a Polynomial Function exercise and the Derivative of an Absolute Value Function problem, would serve as the aforementioned orientation to the study. The first task, which I characterized as being typical of a textbook exercise, had an immediate solution stemming from the application of the power rule to find a derivative. Additionally, the statement of the task indicated the need to find a derivative and served as a clue that the students would likely employ a strategy they had previously used in solving a similar task. To provide the “semi-problematic” context for the second task, I still alerted the student that solving the task would involve a derivative. However, since there is no formula for computing the derivative of an absolute value function, the student would have to determine an appropriate interpretation of the task (e.g., finding a slope on a graph, computing the derivative of two separate linear functions) before performing a computation according to
the familiar derivative rules that Weir and his coauthors (2006) list on the inside front cover of their textbook.

The major emphasis of the first interview was familiarizing the students with the types of problems they would see throughout the semester as well as the expectation that I would ask them to explain or justify their reasoning as they worked. I would ask the students to “think-aloud” (see, for example, Presmeg, 2008) for the remainder of the semester so I could gain insight into their thought processes that might not necessarily have been evident based only on written work. If the student remained silent for a period of 30 seconds or more, I would also ask open-ended questions as free as possible from the biases of my own solutions to the tasks in an attempt to identify strategies the student may have considered but did not choose to articulate by thinking aloud.

The second interview coincided with the portion of the semester that addressed related rates and optimization in the university Calculus I course. For this interview, I used the Radius of a Growing Sphere and Strength of a Wooden Beam problems. The Radius of a Growing Sphere problem required a set-up similar to that of the textbook examples of pumping air into a spherical balloon. I felt this was a problem because the rate of change for the volume was a function of the radius instead of a constant. Additionally, finding the derivative of the volume with respect to time only served to set up a separable differential equation as the first step to finding the radius of the sphere as a function of time. A task that was structurally similar to the related rates textbook examples I saw was, in fact, a problem because students had to apply and interpret given information, separate a differential equation, and solve an indefinite integral using initial conditions. Weir and his coauthors (2006) list the applicable integration rules on the
inside back cover of their textbook, so I consider them to be a reasonable part of a student’s resources.

Similarly, the Strength of a Wooden Beam problem is structurally similar to a typical optimization task in that a solution strategy to the problem would likely incorporate many of the “principles of problem solving” (Stewart, 2003, p. 80) that the author adapted from Polya’s *How to Solve It* (1957). Additionally, a solution would likely incorporate the use of a derivative set equal to zero to identify critical points of the function. However, the modifications I made before including it in the interviews make it problematic in a number of ways. The task itself, as it was stated in Larson, Hostetler, and Edwards (1998), would be problematic because the context is extremely unfamiliar. Whereas the examples from the textbook usually included signal words such as “maximum” or “minimum”, this task used the superlative word “strongest” as the primary clue to employ an optimization strategy. Additionally, the problem required students to make sense of some sort of proportionality, which was an unintended similarity to the Radius of a Growing Sphere problem but nevertheless added another contextual difference from typical textbook problems.

Larson, Hostetler, and Edwards (1998) included a diagram of the cross section of the beam next to the problem and also provided a hint that defined the strength of the beam. I removed both of these because I felt that the diagram made it too obvious that an application of the Pythagorean theorem, which I consider to be a reasonable part of a student’s resources based on my analysis of textbook examples, could be used to relate the width and height of the beam to each other. I also wanted the students to have the opportunity to show their strategic competence by defining an equation for the beam’s
strength on their own. In short, my intent was to eliminate the possibility that the problem statement would give an obvious indication of how to begin or solve the problem.

The third pair of tasks focused on the applications of integration and included the Path of a Moving Particle and Volume of a Metal Ring problems. The two problems are structurally similar to particle motion and volumes of revolution tasks, respectively. The context of the Path of a Moving Particle problem, however, asks for total distance traveled instead of displacement, and the students must consider three separate integrals to develop a correct final answer. The Volume of a Metal Ring problem differs from traditional volume of revolution problems because it deals with a real world context and does not explicitly mention the disk method or the equations that can be used to model the region to be rotated, both of which I consider to be reasonable inclusions in the student’s mathematical resources based on my analysis of the typical textbook tasks.

As with the other problem taken from Larson, Hostetler, and Edwards (1998), I removed their diagrams that showed both the three dimensional shape of the ring and the graphs of the intersecting functions \( y = 3 \) and \( y = \sqrt{25 - x^2} \). I thought these diagrams might have indicated that the intended solution strategy dealt with volumes of revolution, so I chose to omit them from my task to give the students an opportunity to exhibit their own strategic competence and adaptive reasoning.

**Interview Process**

During the actual interview sessions, students were given each task on a separate sheet of paper and asked to work through the problems while “talking aloud” about what
they were doing and why they chose to structure their solution in that particular way. I chose to video record each session to capture any nonverbal evidence (e.g., gestures, quizzical looks) as well as to give me the opportunity to review the verbal explanations at a later time. In the event that a student stated what they were doing but did not provide justification, I would ask clarifying questions about why they chose to use a particular procedure or idea from calculus or how confident they were that what they were doing was correct. If students ever reached a point in their work where they conceded that they did not know how to solve the problem, I would ask them what piece of information they felt they were missing and if they could outline the rest of a reasonable solution if they knew that information. The goal of those questions was to determine if the students still had the capacity for adaptive reasoning even if their strategic competence was not implemented perfectly.

When the students had exhausted all of their original ideas or had finished summarizing a reasonable solution, I tried to provide them with a hint about the information they identified as missing or a portion of another student’s work. I asked them to consider how the aforementioned hint might fit in the context of a solution to the problem to see if they could effectively display strategic competence and adaptive reasoning given a new established experience. These questions helped me to gather data from the students even if they were unable to develop a substantial solution (correct or otherwise) on their own.
Participants

With the design of the interview tasks taken care of, my focus shifted to finding volunteers to participate in the research study. I drafted an open recruiting letter asking for at least five first-year undergraduate volunteers who had scored either a 4 or 5 on the Calculus AB Exam, and the mathematics department at a large, mid-Atlantic research institution forwarded it to each instructor of Calculus I and II on my behalf. The prerequisite AP Exam score was an inclusion criterion for the study based on my own interest in seeing if an AP Calculus student could solve a real calculus problem, but it also allowed me to be fairly certain that among participants’ resources would be a high level of procedural fluency in calculus topics. Prospective participants were offered a $25 restaurant gift certificate to complete the series of three task-based interviews, each of which would last approximately 45 minutes.

Only two students, both of whom were engineering students enrolled in Calculus I, responded to my recruiting letter. Because I needed more volunteers for the study, my roommate’s sister asked some freshmen students with whom she went to high school to consider rereading my recruitment letter and possibly volunteering. Through that contact three additional students volunteered, and coincidentally all were engineering students as well but were enrolled in Calculus II. Since the majority of my study participants were in Calculus II and its course syllabus listed improper integrals and series as the topics coinciding with the interview weeks, the timing of the interviews coinciding with the topics in the Calculus I syllabus would inevitably have less of an impact on the established experiences of these students than I originally intended.
Although five students agreed to participate in the study at the beginning of the semester, recurring scheduling issues with one student prevented him from completing any of the interviews beyond the first. Although I recorded the data from his first interview in my original notes, he was considered dropped from the study and his data will not be considered in chapter 3. The four students who completed the entire series of interviews were coded with pseudonyms to ensure their privacy. Deb and Mary were the two original volunteers enrolled in Calculus I, and Dan and Nick comprised the second pair of volunteers enrolled in Calculus II. All four students received scores of 5 on the Calculus AB Exam during their junior year of high school, so there was over a year of time elapsed until they resumed their calculus studies in the first semester of college.

**Data Analysis**

After all the interviews were recorded, I reviewed the videotapes to find occurrences of strategic competence, adaptive reasoning, and established experiences as described in chapter 1. For the purposes of my analysis, I expanded upon the conceptual framework of the study to provide operationalized definitions of both strategic competence and adaptive reasoning as they appeared in student work. To reflect Kilpatrick and his coauthors’ (2001) interpretation of strategic competence as the ability to represent mathematics problems, I characterized student strategies in which they created graphical, pictorial, or symbolic representations of the problem as evidence of strategic competence. Additionally, I considered their interpretation of the problem text and the translation that occurred between the text and the aforementioned representations as evidence of strategic competence. Adaptive reasoning, as Kilpatrick and his colleges
(2001) describe, is the “glue that holds everything together” (p. 129). In my analysis, I classified an occurrence as adaptive reasoning if an interviewee provided a justification indicating why a particular procedure was performed or if an interviewee generated a valid result from a procedure.

I transcribed the dialogue from each session to include spontaneous student thoughts articulated as a result of my request to think-aloud, the questions I asked to clarify my understanding of student work, and the responses these questions elicited from students. Additionally, I cross-referenced the students’ verbal justifications from the videotape to the written work they performed on paper. This allowed me to link their evidence of strategic competence displayed when modeling and solving the problem to the adaptive reasoning exhibited in their justifications. I also identified any questions I asked or feedback I provided that might have altered the student’s original strategy; these occurrences are discussed as limitations of the study in chapter 5.

During this analysis, I recognized that I needed to qualify the different manifestations of established experiences that I saw the students use in their strategies. As discussed in chapter 1, my interpretation of established experience is slightly different than that of Lithner (2000). I proposed that students would always draw on their experiences from past calculus instruction, but the productivity with students’ experiences were applied to a strategy depended on the presence of adaptive reasoning to establish a link to the content of past problems beyond the context itself. Thus, established experience in Lithner’s (2000) study was always meant to guide students to a probable truth based on a contextual similarity, but I observed that established experience was not always employed productively in student strategies.
The instances of established experiences in student solutions seemed to fit within one of four categories that I developed within this study: recalled, forced, constructive, or induced. Recalled experiences were characterized by students saying that they remembered an idea or procedure from a previous mathematics class but could not work with it at all in their solution. For example, if a student saw a right triangle and said, “I remember there was something about the Pythagorean theorem that we used with the sides, but I don’t remember the formula,” this would be a recalled experience.

If the student remembered something (regardless of their confidence level in how correct it was) and tried to use it because it was their last resort, I classified it as a forced experience. For example, a student who computed the derivative of an equation simply because it appeared on a calculus test would be exhibiting a forced experience. In contrast, constructive experiences were carefully considered and meant to be useful; if the student had a justification for using a prior experience or tried to make sense of something they thought they remembered it was a constructive experience. If the same student on the calculus test decided to differentiate an equation because he needed to find a rate of change, he would be engaging in a constructive experience. Finally, I considered induced experiences that were brought about as a result of some sort of intervention on my part. If I would have told the student with the recalled experience about the Pythagorean theorem that \( a^2 + b^2 = c^2 \), then I would have provided him with an induced experience. The data I collected after an induced experience would assess the capacity for strategic competence and adaptive reasoning given the necessary prior knowledge.
In summary, my tasks were designed in order to provide extensive opportunities for students to engage with problematic mathematics and display evidence of the problem-solving influences in both their work and justifications during three interview sessions. Additionally, the tasks I created for this study should be accessible to AP Calculus students based on my review of the content and context of examples used in calculus textbooks; the commonalities among these textbooks serve as the basis for the reasonable set of resources a qualified AP Calculus student would likely possess. Using the written and verbal data I collected in the interview sessions and interpreted through my analysis, I will present the overall findings in chapter 3 as I highlight the evidence of strategic competence, adaptive reasoning, and established experience displayed by the qualified AP Calculus student participants.
Chapter 3: Findings

The findings presented in chapter 3 are my summaries of student solution strategies for each task. As I describe each student’s solution strategy, I highlight evidence I found indicative of strategic competence, adaptive reasoning, and established experience in both student work and corresponding justifications. This qualitative evidence will serve as the basis for the discussion in chapter 4 as I answer the research question: when asked to solve introductory calculus problems that are structurally similar to but more problematic than typical textbook tasks, to what extent and in what combination, do first year undergraduate students with qualifying AP Calculus Exam scores incorporate their established experiences, adaptive reasoning, and strategic competence.

I include evidence from each student’s interviews in my analysis. In some instances there were commonalities among strategies two or more students used that allow me to comment on those students as a group. However, the evidence from Nick’s interviews tends to be more extensive and detailed than the evidence from the other participants. Nick was the most articulate student of the group, so his genuine thought processes seemed clearer to me as I reviewed the interview tapes. Additionally, for a few of the tasks he employed multiple representations that provided additional opportunities for me to observe the influences of adaptive reasoning, strategic competence, and established experience that I considered for this study.
Task 1.1: Derivative of a Polynomial Exercise

As I expected, the Derivative of a Polynomial exercise elicited a nearly immediate one-line solution from each student. When I asked the students to justify their work, they mentioned “bringing the exponent down” and cited their wording of familiar rules from calculus like “the derivative of a constant is zero.” Two students referred to their manipulation of the exponents as “The Power Rule,” which gave me evidence that they were using a known procedure as the only step in their solution.

After the students were finished with the first task, I transitioned into the second task by explaining the difference between an exercise and a problem as discussed in chapter 1. The main point I shared with the students was my expectation that the subsequent tasks would require much more reasoning and justification than the first task. With that in mind, I told them I would appreciate if they could verbalize their thoughts about the problem by thinking aloud as they worked and answering any questions I had for them. During this explanation, students listened to my expectations and confirmed that they were willing to explain their work aloud while solving the remaining tasks.

Task 1.2: Derivative of an Absolute Value Function Problem

As I developed the tasks, I was worried that the Derivative of an Absolute Value Function problem (see Table 2) would be too easy to reveal any interesting insights into student problem solving. I was pleased for the purposes of this study to see, however, that all students were initially perplexed by the problem after the relative ease with which they completed the Derivative of a Polynomial exercise. Curiously, both Calculus II students represented the problem graphically at first by drawing the function graph as a
shift left of the graph of the basic function $y = |x|$. They executed these moves quickly and without hesitation, as if they were a learned procedure.

Nick, who admitted that he was “not very good at absolute value derivatives,” said that he was “pretty sure” the graph he drew was correct but insisted on double checking that the points on his graph were solutions to the original function. To check, he substituted the $x$ value of his graph’s vertex, -1, into the given function rule $y = |x – 1|$. When he realized that his graph was not correct (see Figure 1), he erased the original graph with vertex at (-1, 0) and redrew the function with the vertex at (1, 0). He said the function needed to be “shifted to the right because the -1 is inside.” Additionally, he commented, “I realize I did the thing that most people do when you see a minus one that you think it’s shifted to the left but it’s actually shifted to the right.” Nick’s justification for the shift based on the -1 being “inside” coupled with his generalization of how people interpret translated functions confirmed my belief that he was invoking the use of a procedure to graph the absolute value function in his solution. However, I interpret this as a constructive experience based on his use of adaptive reasoning skills to identify and correct the mistake almost immediately.

Figure 1. Nick’s graph of the translated absolute value function for Task 1.2

After Dan drew his graph (shown in Figure 2), he identified the slope on each “side” of the function graph noting -1 for $x < -1$ and 1 for $x > 1$. Without hesitation or
justification, he stated that the derivatives at \( x = 0 \) and \( x = 1 \) were both 1. When I asked why, he said it was because the slope was 1 at those points. He was puzzled by what the derivative would be at \( x = -1 \), which he called a “sharp turn” on the graph. As he pondered whether the slope at \( x = -1 \) would be undefined or 0, he decided to reexamine his graph. Dan identified his graphing error, saying, “Did I mess this up? Yeah, it’s supposed to be this way” before he drew a new absolute value function with vertex \((1, 0)\).

![Figure 2. Dan’s graph of the translated absolute value function for Task 1.2](image)

From his new graph, Dan used his interpretation of the derivative as a slope and said that at \( x = -1 \) and \( x = 0 \) the derivative was -1. However, without referring to that interpretation again he said that the derivative at \( x = 1 \) “does not exist.” When I asked him why he thought the derivative did not exist at \( x = 1 \) he said, “Because it’s not differentiable.” When I asked him what he meant by “it’s not differentiable,” he replied, “Because that’s what they told me,” which I characterized as a forced experience because Dan did not display the adaptive reasoning necessary to link his interpretation of the derivative as slope to a slope found at the vertex of his graph.

In contrast to the graphical representation that the Calculus II students used, the Calculus I students showed a preference for a symbolic representation of the problem as a piecewise function comprised of two linear functions. Both Calculus I students showed strong strategic competence by correctly defining the linear functions, but some questions
about their adaptive reasoning arose when they began to calculate derivatives. Initially, both Mary and Deb computed derivatives for the two separate linear functions and included \( x = 1 \) in the domain of the line with a positive slope. Mary’s and Deb’s work appear in Figures 3 and 4, respectively.

Mary specifically mentioned that the value of \( x \) had to fit within the domain of one of her linear functions. When I asked her how she chose which linear function rule to use to evaluate \( f(1) \), she explained that the absolute value function was always greater than or equal to zero. She then said she had to use the function rule for the line with a positive slope to obtain the inequality \( x - 1 \geq 0 \), which she solved as \( x \geq 1 \) for the domain. The absence of adaptive reasoning prevented her from considering the function’s differentiability at \( x = 1 \) based on the forced experience of defining the domain of an absolute value function.

![Figure 3. Mary’s piecewise function for Task 1.2](image)

Mary initially symbolically calculated answers of \(-1, 1, \) and \(1 \) for the derivatives at \( x = -1, 0, \) and \(1 \), respectively (see Figure 5). I asked her to reflect on whether or not her
answers seemed reasonable, and in doing so I feel that I did not allow her the chance to illustrate her adaptive reasoning skills by checking the result on her own. In any case, my question prompted an induced experience in which she said, “It looks wrong at zero. If you graph it, the slope would be -1 at zero.” Again illustrating strong strategic competence, she proceeded to graph the function as depicted in Figure 6.

![Figure 5. Mary’s symbolic work for Task 1.2](image)

![Figure 6. Mary’s graph of the translated absolute value function from Task 1.2](image)

When I asked Mary how the previous derivatives she computed related to the newly drawn graph, she stated that the derivative was a slope. As I tried to understand Mary’s work, I mistakenly voiced my interpretation of her work instead of allowing her
to justify it herself. In essence, my inexperience as an interviewer removed the opportunity for Mary to illustrate her adaptive reasoning. As I summarized my interpretation of her work, I mistakenly pointed at the graph’s vertex, \((1, 0)\), when I asked “So the slope at \(x = 0\) is?” In response to my error, she said, “\(x = 0\) is here” as she gestured to the point \((0, 1)\) on the graph and said the slope at that point was \(-1\).

As soon as she interpreted the slope from the graph, she noticed that she had previously symbolically calculated the derivative at \(x = 0\) as 1. By examining her symbolic work, she reasoned that she made an error and incorrectly included \(x = 0\) in the domain for the line with a positive slope. She exhibited her adaptive reasoning skills by rejecting her incorrect symbolic work in light of a new and correct graphical representation of the problem. Toward the end of the interview, I gave another summary of my interpretation of Mary’s work by saying, “at \(x = 1\) the slope is 1, and you are thinking of the derivative as a slope.” She confirmed that my interpretation of her work was accurate, and at that moment I could not think of a focusing question that could be used to prompt an induced experience of considering the differentiability of the function at \(x = 1\). I simply thanked her for volunteering and we ended the interview without giving her the opportunity to clarify her interpretation of the value of the derivative at \(x = 1\).

Deb spent by far the most time of any student completing the Derivative of an Absolute Value Function problem. Deb talked extensively about “conditions” that I interpreted as her way of trying to define the domain of the piecewise function, but I did not interject in her explanation. Deb attempted to symbolically compute the value of the derivative of the function at \(x = 1\), which she initially included in the “condition” for
where the expression inside the absolute value symbols evaluated to a positive number.

From her description, I assumed that Deb considered zero to be a positive number. Deb’s symbolic work and graph (discussed later) appear in Figure 7.

![Figure 7. Deb’s symbolic work (a) and graph (b) for Task 1.2](image)

Almost immediately after Deb told me the derivative at \( x = 1 \) was 1, she decided that \( x = 1 \) was “separate” because the “number in parenthesis” was zero for both linear functions. She admitted that she was confused because of the \( x \) values that were given in the problem statement and she thought she needed to fit \( x = 1 \) into one of the two functions she defined. In response to her admitted confusion, she exhibited her adaptive reasoning skills by explaining her belief that the derivative at \( x = 1 \) could be either -1 or 1 depending on which “condition” (which part of the domain of the piecewise function) applied. However, she said she was not sure what to do from there.

After about two minutes of silent thought, she stated, “The value of the derivative at one point shouldn’t be two different things”, and when I asked why she said, “because it’s at one point, and it’s the slope so there should only be one.” At this point in the interview she sketched an upside down absolute value function graph (as shown in Figure
7b) and said that at (1, 0), “There is a sharp point so there is no derivative. At that point the value of the function is zero coming from both sides but you can’t take the derivative of the function at that point which is kind of shown by taking the derivative of the two functions at that point and getting two different things.” I asked her if she could determine the slope of the line at the sharp point, and she said she could not because the derivatives coming from the left and right were not equal.

I was very curious about the upside down absolute value graph that she drew as potential evidence of strategic competence, so I asked her (mistakenly giving her an either/or choice) if it was a general graph or the given function. She stated that it was a general graph, and then restated that her final answer was that the value of the derivative at \( x = 1 \) does not exist because “there would be two derivatives at that point and it can’t happen.” Thus, the sketched graph was evidence of her adaptive reasoning instead of strategic competence of correctly graphing the given function.

Task 1.2 proved to be illustrative of the semi-problematic orientation task I described in the task selection section of chapter 2; the students practiced their verbal justifications while I practiced my questioning patterns and real-time analysis of student work. With the orientation to the study completed, Task 2.1 served as the students’ first experience with the targeted application problems discussed in chapter 2.

**Task 2.1: Radius of a Growing Sphere Problem**

The Radius of a Growing Sphere Problem (see Table 2) was one of two tasks (the other being The Volume of a Metal Ring Problem, discussed later) for which no student obtained a correct answer. Even more interesting, however, is that all solution processes
used in the task shared the common theme of interpreting the rate of change of the radius with respect to time as $\frac{1}{15}$. Each student obtained this result using the given information that the radius at $t = 0$ was 1 and the radius at $t = 15$ was 2. This lack of strategic competence necessary to correctly represent the problem led Deb, Nick, and Dan to a forced experience of finding a linear equation through the two points:

$$r = \left(\frac{t}{15} + 1\right).$$

Mary’s error is discussed in more detail later in this section.

Nick and Deb used the constructive experience of deriving the equation for volume of a sphere with respect to time. Nick’s and Deb’s constructive experiences were implemented as a result of identifying this problem as related rates from their past studies. Deb evaluated the derivative correctly (see Figure 8) and tried to substitute $\frac{dV}{dt} = \frac{k}{1}$ and $r = 1$ based on her interpretation that the rate of change of the volume was proportional to the reciprocal of the radius at time $t = 0$, which happened to be 1. Deb admitted that she was not sure if this strategy would lead to a solution, at which point she found the radius as a linear function of time (see Figure 9). She seemed satisfied that her solution gave the radius as a function of $t$ as stated in the problem, so I asked if her initial strategy would have also been a potential solution to the problem. She referred to it as “a different way to solve the problem” that didn’t end up working out, which indicates that her acceptance of the forced experience of finding the radius as a linear function of time inhibited her use of adaptive reasoning to consider the viability of her initial strategy.
In Mary’s first solution strategy, she interpreted $\frac{dr}{dt}$ as 1/15 and consequently interpreted “the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius” to mean that $\frac{dV}{dt}$ was 15, the reciprocal of the rate of change of the radius instead of the radius itself. As illustrated in Figure 10, she substituted her values for $\frac{dr}{dt}$ and $\frac{dV}{dt}$ into her correctly differentiated volume equation. However, she noticed immediately that she no longer had the variable $t$ in her equation, as she felt was required by the problem statement. She admitted that her reasoning to make the substitutions
described above was a guess, but it was the only option she had so she decided to see where it would take her. This statement indicated her use of a forced experience, and despite voicing her concern that she no longer had a $t$ variable, she said she would simply “solve for $r$ without having $t$.”

Figure 10. Mary’s substitution for Task 2.1 based on a forced experience

Mary considered her final answer for $r$ (as shown in Figure 10) and commented that it looked “weird” with the square root of $\pi$ included. Although this technically was evidence that she was reflecting on her result, I do not consider it to be evidence of adaptive reasoning because she attributed her unease to the appearance of her final answer instead of questioning the validity of her procedure. When she told me she had “no clue” where to go from that point, I provided the induced experience of helping her interpret the proportion correctly ($\frac{dV}{dt} = \frac{k}{r}$) to see if she would display adaptive reasoning leading to a reasonable solution. She asked if she needed to compute an integral, a potential constructive experience. When I declined to answer she abandoned that thought and performed symbolic work (see Figure 11) to obtain $r = \frac{\sqrt{15k}}{\sqrt{4\pi}}$. After she pondered whether the constant $k$ could actually be $t$ she simply switched the variables as
her final forced experience. She identified an algebraic mistake and modified her
previous solution so it became $r = \sqrt[3]{\frac{15t}{4\pi}}$, but she reported that she was still not more than
50% or 60% confident that her solution was correct.

Figure 11. Mary’s symbolic work for Task 2.1 after an induced experience

Nick began his work for the Radius of a Growing Sphere problem by indicating
that he felt this task was a related rates problem. He also wrote $f(0) = 1$ and
$f(15) = 2$ after reading “at $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is
2”. When I asked him why he felt this was a related rates problem, Nick said, “in these
problems they usually give you the radius increasing…well I guess it isn’t related rates.
A typical related rates problem would say, ‘as the radius increases, how does the surface
area increase’”. After Nick thought silently for a moment, he changed his mind and said
this task might actually be a related rates problem. Because he did not immediately give
a reason why the task might be a related rates problem, I classified this as a forced experience.

For the next step in his solution, he read the first line of the problem statement (see Table 2) and said, “the volume is equal to the reciprocal of the radius” as he wrote $V = t \left(\frac{1}{r}\right)$. Based on his incorrect interpretation, I gathered that he did not display the strategic competence necessary to model the problem, and at this point in his solution the mention of related rates was merely a recalled experience. Nick sat silently for a moment and then asked if he could move on to the second task and return to the first task later in the interview. I told him that would not be a problem, so he spent approximately 25 minutes on the second task before returning to the Radius of a Growing Sphere problem.

Nick’s work with the second task led him to believe that that task was an optimization problem (discussed later), and when he returned to his work on the Radius of a Growing Sphere problem he admitted that he used to confuse related rates and optimization. I asked him what types of things he would expect to see in the solution if the Radius of a Growing Sphere problem did end up being a related rates problem. The ensuing descriptions of his reasoning made me reconsider my interpretation that his earlier identification of the task as related rates was a forced experience; the general solution strategy he described indicated his use of a constructive experience.

He explained that he was fairly sure that he needed to “try to get down to two variables” and “take derivatives,” but he clarified that the derivative is usually left as $\frac{dV}{dt}$, change in volume over change in time, or $\frac{dr}{dt}$. I asked Nick to explain what the variables $V$, $r$, and $t$ stood for in his work, depicted in Figure 12. He interpreted the variables as volume, radius, and time, respectively.
Figure 12. Nick’s symbolic work for Task 2.1

Note: The top line previously read $V = t^{(1/r)}$ before he erased as described in the next paragraph.

After he clarified his interpretation of the variables, Nick looked at his original equation, $V = t^{(1/r)}$, and voiced his concern that it was no longer correct. He reread the first line of the problem and changed his symbols to $V = k^{(1/r)}$, citing his reasoning as, “it would have to be some number times one over $r$ to make it proportional.” His adaptive reasoning led to a partially correct equation but, because he still modeled the first line of the problem using the volume instead of the rate of change of the volume, he did not display effective strategic competence.

Based upon Nick’s discussion of the elements of a typical related rates solution and his correct interpretation of the variables in his work, I felt he was using the memories of his related rates studies as a constructive experience. However, he lacked the strategic competence necessary to correctly model the first line of the problem, “the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius.” I conjectured that if I provided an induced experience that allowed him to correctly model the problem he might be able to use his adaptive reasoning to develop a solution.

To help Nick interpret the first line of the problem, I asked him a series of questions based on his work (see Figure 12). I asked him to tell me how he interpreted the notation $\frac{dv}{dt}$ he had written on his paper. When he identified it as “the rate at which
the volume is increasing as \( r \) is increasing.” I referred him back to the first line of the problem to try to see if he would use adaptive reasoning to make a connection between his symbolic interpretation and the problem statement. In a similar manner, I asked him to tell me what his modified equation, \( V = \frac{k}{r} \), meant. He replied that it was the given volume in terms of \( r \). With both of his interpretations made explicit to me, I asked a final question of whether the \( V \) from the volume formula and the \( V \) from his modified equation were the same. He said, “They should be. Maybe I interpreted this [pointing to the first line of the problem statement] wrong and it [pointing to \( V = \frac{k}{r} \)] should be the rate at which the volume is increasing.” He then wrote \( \frac{dV}{dt} = \frac{k}{r} \), which indicated that his adaptive reasoning skills were strong given the necessary induced experience.

Perhaps acting too quickly, I next asked Nick if the two expressions he had written for \( \frac{dV}{dt}, 4\pi r^2 \left( \frac{dr}{dt} \right) \) and \( \frac{k}{r} \), were the same. Instead of responding immediately with a yes or no answer, Nick thought for a moment and said “They should be, but I’m just trying to be sure that in my mind that makes sense. It’s what the first line is trying to say, so I think I can set them equal to each other.” He did, and he solved the equation for \( \frac{dr}{dt} \) as illustrated in Figure 13.

![Figure 13. Nick’s symbolic work for Task 2.1 after my induced experience](image-url)
After Nick computed the result shown in Figure 13, he sat silently looking at his paper and admitted that he could not figure out how to proceed. I asked him if he could link his new work back to the first work he wrote down, \( f(0) = 1 \) and \( f(15) = 2 \), as an attempt to elicit an induced experience meant to gauge if he had the adaptive reasoning capacity to decide to use an integration strategy based on his work shown in Figure 13. Instead of integrating, Nick drew a graph of the radius at time \( t \) as shown in Figure 14. He assumed the linearity of the function and computed that the rate at which \( r \) increased was \( \frac{1}{15} \), which led him to find the same linear function for \( r \) as the other participants.

Referring back to the problem statement, he said, “they want me to find the radius of the sphere as a function of \( t \); isn’t that the radius as a function of \( r \)?” Once again, I too quickly reacted and asked if the previous manipulations with \( \frac{dV}{dt} \) were necessary to the problem, which prompted Nick to ask if this was a trick question.

![Figure 14. Nick’s graph of the radius as a linear function of time](image)

I asked Nick what his final answer was and whether it made sense. He said he would probably use the linear function as an answer, but it did not seem to work because he assumed that \( r \) increased at a constant rate. In the absence of a correct answer, he still exhibited some adaptive reasoning through his admission that his assumption, which I classified as a forced experience, likely was not correct. I asked if he could identify the spot in his work where there was something missing, and he identified the missing piece as “the equation that connects volume to the radius that’s increasing using these points
[radius at \( t = 0 \) and \( t = 15 \)]. I asked Nick to predict the next step of the solution, and he said he was having difficulty doing so because he did not remember the “steps” that he learned in calculus. His recalled experience of the “steps” indicated that he felt there was a procedure he once knew that would have allowed him to solve the problem correctly.

In contrast to the other three students, Dan did not consider using a derivative at all in his solution. When he read the task, he immediately discussed his interpretation of a reciprocal as meaning “one over the radius,” and within seconds he was developing a linear function rule, \( r = \left( \frac{1}{15} + 1 \right) \). Despite the lack of strategic competence, he still attempted to employ a constructive experience. As he was working, he kept saying, “the reciprocal of the radius is throwing me off. It makes it sound like I should be doing something else.” Despite this admission, he persisted with his computations (shown in Figure 15), so the absence of adaptive reasoning changed my interpretation of his initial constructive experience into a forced experience. When he obtained his linear function, he referred back to his interpretation of reciprocal as a fraction and was satisfied that his answer was plausible given that it contained a fraction.

![Figure 15. Dan’s symbolic work for the radius as a linear function of t](image)

When he told me that the linear function was his answer, I asked if he used any other information in the problem. He mentioned that he could have used the equation for
volume of a sphere from the formula sheet, and after I asked him if there was any
information in the problem statement he did not use he identified the first line and
rewrote the equation for volume by substituting \( r = \left( \frac{l}{15} + 1 \right) \) as shown in Figure 16.

After he wrote the new equation for volume I asked him what he felt about his original
linear function. He said he did not think it was correct now. When I asked why, he said it
was because I was asking a lot of questions about volume.

Figure 16. Dan’s substitution for Task 2.1

I decided to induce the experience of the equation \( \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \) used by
another student to see if Dan would exhibit adaptive reasoning by interpreting this work.
When I asked him to consider how the equation might be helpful in a solution to the
problem, he articulated his interpretation that the equation was the derivative of the
formula for volume with respect to time. Once again, he admitted that he felt his
previous work was incorrect, but his explanation this time involved a recalled experience
that usually (meaning usually in his past mathematics classes) the information given in
the problem is necessary for a solution. Dan thought for a few minutes and said he was
puzzled by the words “reciprocal of the radius” from the problem statement. He decided
he wanted to set this problem aside to work on Task 2.2.
When Dan returned to his work on Task 2.1, he wrote the equation for volume of a sphere and computed its derivative with respect to time, \( \frac{dV}{dt} = 4\pi r^2 \left( \frac{dr}{dt} \right) \). Suddenly and enthusiastically, he said, “I know! I’m supposed to get a value for \( \frac{dr}{dt} \) so I can relate radius and time.” He proceeded to explain his strategy of wanting to find a function relating radius and time, which he would differentiate to obtain \( \frac{dr}{dt} \). He thought he could substitute a new expression for \( \frac{dr}{dt} \) into his equation for \( \frac{dV}{dt} \) to “solve for \( \frac{dV}{dt} \).”

After describing that strategy, Dan reread the problem text “find the radius of the sphere as a function of \( t \).” After that statement, Dan displayed his adaptive reasoning by describing a new strategy of solving for \( \frac{dr}{dt} \) and integrating to find the radius as a function of \( t \). Despite his strong adaptive reasoning in that respect, he did not display the strategic competence necessary to formulate a representation for \( \frac{dV}{dt} \) that would have made his integration strategy possible. His strategic competence did not serve him well in that he applied the forced experiences of his prior work to develop multiple incorrect equations involving \( \frac{dV}{dt} \) and \( \frac{dr}{dt} \). His work appears in Figure 17.

![Figure 17. Dan’s symbolic work involving \( \frac{dV}{dt} \) and \( \frac{dr}{dt} \) for Task 2.1](image)

In general, the students struggled with Task 2.1 because they did not exhibit the strategic competence necessary to translate the problem text into a constructive representation of the growing sphere. In the next section, I describe how students...
exhibited more, though not complete, evidence of strategic competence for Task 2.2 than they did in Task 2.1.

**Task 2.2: Strength of a Wooden Beam Problem**

The Strength of a Wooden Beam problem was almost as difficult for students as the Radius of a Growing Sphere problem, though Nick and Dan were able to find the correct answer with assistance from one of my induced experiences. Immediately after reading the problem statement, each student drew a picture that served as the first incorporation of strategic competence for the task. The pictures appear in Figure 18.

![Initial sketches for Task 2.2](image)

*Figure 18. Initial sketches for Task 2.2*

*Note:* Clockwise from top left are sketches by Mary, Deb, Nick, and Dan.

Despite the differences in diagrams, all students modeled the equation for the strength of the beam as \( S = wh^2 \), and although they did not include a constant of proportionality I did not intervene. From solving the problems myself ahead of time, I knew that setting the derivative equal to zero and dividing through by \( k \) essentially made the constant of proportionality irrelevant.
Another commonality among many student solutions was the identification of the task as an optimization or maximization problem. All of the students except Mary identified this task by one of these names and said that they recognized it from their past calculus studies. Nick and Deb initially reasoned that the product $wh^2$ needed to be a maximum and they felt that having the width and height equal to each other as in a square would achieve the desired end. In contrast, Mary still felt the shape of the beam should be square but she chose the shape because she felt her calculations would be easier.

As all students worked through the problem, they illustrated strong adaptive reasoning skills by identifying that neither dimension could be equal to 24 inches because that would cut through the diameter of the log. As an additional illustration of adaptive reasoning, Mary, Dan, and Nick verbalized that the value of the squared variable in the equation would have to be larger to yield a maximum product because it would be multiplied twice instead of once.

Despite these commonalities, there were a number of unique solution strategies that illustrated different manifestations of adaptive reasoning, strategic competence, and established experience for each student. As illustrated in Figure 18, Deb’s initial representation differed from the others because she drew an upright rectangular prism instead of a rectangular cross section of the beam. She labeled the picture using three variables: $h$ for height, $w$ for width, and $t$ for thickness. She expressed her conjecture that she thought this was a “maximization problem” and stated that she needed to find an equation that would give her the “biggest width”. As previously mentioned, she used her adaptive reasoning skills to determine that the width of the beam had to be less than 24 inches or else the board would have no “thickness”.
Deb spent about a minute looking at her paper without saying or writing anything, so I asked her to discuss some things she expected would be part of a “maximization” task, as she called it. She said, “If I had a number that the width and height were equal to I could solve for one and plug it into the strength function, take the derivative, and set it equal to zero so I could find the point where it would be a max.” When I asked Deb what she felt she was missing in her work based on this description, she stated that she would need the equation to relate the width and height to each other.

Her general description of an optimization/maximization solution illustrated that Deb possessed the strategic competence to set up and solve the problem provided that she had an intermediate equation to use in substitution as described above. In an attempt to find such an equation, she employed the constructive experience of using the volume of a cylinder (the log) as illustrated in Figure 19. Using her adaptive reasoning to interpret the geometry modeled by her symbols, she stated that the volume of the beam, \( w h t \), could be no larger than \( 144\pi h \), the volume of the round log.

![Figure 19. Deb’s attempt to relate \( w \) and \( h \) in Task 2.2](image)

At this point in Deb’s work, I felt that she was misinterpreting the problem text “a wooden beam has a rectangular cross section of height \( h \) and width \( w \).” The \( h \) and \( w \) variables in her drawing and subsequent geometric interpretation were not the same \( h \) and
w variables used to measure the cross section, so I induced an explanation of interpreting the rectangular cross section using her variables of w and t, thickness. Deb accepted my explanation but asked me if she needed the equation for volume in her solution. I realized that my induced experience reflected what I wanted to see in a solution instead of what Deb was thinking, so I tried to encourage her to continue with her volume strategy if she could. However, she said that she did not know what to do with the inequality, \( wt < 144\pi \).

In response to Deb’s situation, I induced another experience of drawing a diagram of the rectangular cross section of the beam inscribed in the circular log as illustrated in Figure 20. Deb examined my sketch and said, “If you have a perfect circle and you cut a rectangle out of it, I feel like it has to be a square.” She reasoned that if the log was round it would have “the same diameter all the way around”, and the only possible inscribed rectangle would be a square. Although she did not elaborate on her explanation, I guessed that she believed that the diagonals of the rectangle were perpendicular diameters of the circle. In any case, she did not display the strategic competence to pictorially represent an inscribed rectangle.

![Figure 20: Sketch offered for induced experience for Deb’s work on Task 2.2](image)

Using her interpretation of the inscribed rectangle as a square, Deb used the Pythagorean theorem to immediately write \( 2x^2 = 24^2 \), with \( x \) representing the length of a
side of the square. She solved the equation and wrote her result as $x = 16.971$, which she referred to as the “maximum width.” Deb then described a plan to substitute the “maximum width” back into her strength function, which she would differentiate and set the derivative equal to zero to find the “max.” However, she did not describe what the “max” was that she wanted to find. According to her plan, Deb set up an equation $16.971h^2 = S$, but before she computed the derivative she identified that the $h$ variable in her strength equation actually referred to the $t$ variable from her original sketch. Instead of computing the derivative she said, “since the rectangle is a square the sides are both the same, so the dimensions [of the cross-section] are 16.971 by 16.971.”

I asked Deb to clarify one last time why the rectangular cross section had to be a square. She again referred to the “uniform circle” that had a diameter of 24, and she sketched Figure 21 in an attempt to show that the diagonals were perpendicular bisectors, so the figure had to be a square. This evidence validated my assumption that she thought the diagonals could only be perpendicular diagonals of the circle and indicated that Deb did not display the strategic competence or adaptive reasoning necessary to draw a non-square rectangle inscribed in the circle.

Figure 21: Deb’s illustration to justify why the inscribed rectangle was a square
As the interview drew to a close, I induced the experience of drawing a non-square inscribed rectangle and asked Deb if my construction was possible. She said, “I guess you could technically try that” and argued that the diagonals of the rectangle would have to be drawn in such a way that they were diameters of the circle. At this point, I induced the experience of the inscribed angle theorem (i.e., the measure of an inscribed angle is half the measure of the intercepted arc) to prove to Deb that the diagonals of the rectangle would always be diameters of the circle. I wanted to see, as I did with the other students, whether Deb would incorporate adaptive reasoning to modify her solution based on a new experience. However, Deb said she did not know anything else she could do to work toward a solution, and she concluded that she did not have a final answer.

Nick’s near immediate identification of the problem as a “maximization” task served as a constructive experience that gave him the idea to begin the problem by “figuring out how to put \( h \) in terms of \( w \) so that the equation will have two variables…\( S \), which is strength, and \( w \).” Curiously, Nick began by saying he wanted to find \( h \) in terms of \( w \) but after he paused to interpret the two variables as quoted in the previous sentence he switched his strategy to finding \( w \) in terms of \( h \). His interpretation of the variable in terms of which he wanted to define strength is discussed later in this chapter.

Nick made extensive use of geometric and trigonometric constructive experiences in his initial solution strategy as he tried to relate the height and width of the beam to each other. He examined the central angle created when drawing radii to the upper right corner of the rectangle and through the midpoint of the right side of the rectangle and he looked at the various trigonometric ratios of the angle based on his general drawing. His
symbolic trigonometric work and corresponding pictorial representation of the beam appear in Figure 22.

![Figure 22. Nick’s attempt to relate $w$ and $h$ using trigonometry in Task 2.2](image)

He rejected the possibility that the trigonometric function he needed was tangent because using the trigonometric function not only introduced a variable $\theta$ but also did not eliminate either $w$ or $h$. He tried to use sine and cosine, and even though he felt he was “overcomplicating” the problem he stated that he wanted to “see where it goes.” Nick’s final comment on the trigonometric function was a recalled experience that he felt he needed to use the inverse sine function from what he remembered from high school but could not apply that thought to any of his work.

As Nick silently pondered what he would do next, I asked him to describe the things he would expect to see in a solution to an optimization problem. He once again reiterated his goal to find $w$ in terms of $h$ and then described a graphical strategy to find the maximum value. He said he would look at the graph of the function (of a single variable) to find where the derivative was zero. Additionally, he would confirm that this
value was a maximum using a “double derivative” because he could test the concavity to make sure it was concave down. His sketch that he made while articulating this description appears in Figure 23.

![Image of sketch]

**Figure 23.** Nick’s graphical support for description of a typical optimization solution

In contrast to his peers, Nick illustrated additional strategic competence through his generation of multiple representations (e.g., graphical representation using height and width of the beam, graphical and symbolic representations using right triangle trigonometry, graphical and symbolic representations using the Pythagorean theorem) to attempt to solve the problem. Nick’s final idea was to investigate the area of the rectangular cross section to try to find some more information about the angles. He made one last sketch, which is illustrated in Figure 24. When he could not advance with this strategy, he returned once more to the trigonometric functions. Nick reached a point where he felt he could no longer work with any of his representations and he expressed his frustration that the only thing he needed to solve the problem was the intermediate equation relating $w$ to $h$. 
As I had done with other students when they ran out of their own ideas, I prompted an induced experience using the inscribed angle theorem to show that the angle in the corner of the rectangle, which I intentionally did not classify as a right angle, had to be half the measure of the intercepted arc. I did not want to make the application of the Pythagorean theorem immediately obvious by using the term “right angle”, and in doing so I hoped to observe Nick’s use of adaptive reasoning to make sense of the induced experience. He thought for a moment before saying, “Oh. I realize how stupid I have been this whole time” and wrote down an equation based on the Pythagorean theorem. Then he solved for \( w \) in terms of \( h \) as he had been describing throughout the interview. Nick began to compute the derivative as the product of the derivatives of the two terms, but in the interest of time I asked him to explain how he computed the derivative to see if he would correct the aforementioned procedural error. He said that he should have used the “product rule” and proceeded to perform a different differentiation procedure. The symbolic work described in this paragraph appears in Figure 25.
We were out of time for the second interview, but I asked Nick if he would be willing to return to his work on this task at our final interview. When we met again a month later, Nick had understandably forgotten the next step he was about to perform. He reviewed all of his work and I told him that I verified that the derivative he computed was correct if he would like to continue his solution. Nick remembered that he was trying to find the maximum strength of the beam, so he set the derivative equal to zero and commented, “that’s a lot of stuff!” Nick did not perform any additional symbolic work beyond what appears in Figure 25, but he displayed some excellent adaptive reasoning to identify that $h = 0$ would cause the derivative to be zero. However, he quickly rejected $h = 0$ as a potential solution based on the fact that it would not yield a maximum strength. Similarly, he thought that it might be possible to set $24^2 - h^2 = 0$, but he again rejected this possibility because it would yield a zero in the denominator.

Figure 25: Nick’s substitution and derivative of strength as a function of $h$
At this point, Nick illustrated additional adaptive reasoning skills by questioning his own strategy and my statement that his derivative was correct; he thought there had to be an easier way, and he wanted to find it. Nick had already surpassed the 45 minutes I originally anticipated for the interview, and although he wanted to keep working I needed to make a decision balancing my desire to see Nick’s evidence of the problem solving influences and the impending arrival of a scheduled university class to the interview room. I had reviewed Nick’s work before the interview, so in an attempt to expedite his familiarization with his previous work I pointed Nick in the direction of the equation based on the Pythagorean theorem that allowed him to make his first substitution. He immediately noticed that he could solve that equation for $h^2$ and make the substitution that would yield a polynomial function for strength. With 5 minutes remaining until the scheduled class was set to arrive, I asked him to summarize the rest of his solution instead of performing all of the arithmetic. He then performed the symbolic work illustrated in Figure 26. He concluded by saying that he would set the derivative equal to zero to find the value of $w$ at which strength of the beam would be a maximum, and he would confirm that it was a maximum by testing the second derivative to see if the function was concave down at that point.
Like Nick, Dan was able to find a correct answer to the problem after I prompted him to use an induced experience after I stated the inscribed angle theorem, but the path leading to his final answer contained fewer constructive experiences and many more forced experiences. Dan thought he recognized the problem as something he had done during his high school calculus studies, and he felt the solution involved a derivative somehow but could not articulate why. After he correctly identified the equation for strength of the beam, he used his forced experience and computed a “derivative” (note: the use of quotation marks indicates that it was not a valid derivative) using what looked like the product rule. His work appears in Figure 27.

Figure 27. Dan’s “derivative” for Task 2.2

He commented that he needed a way to relate \( h \) and \( w \), and he felt it would be of the form \( h + w \). As evidence of another forced experience, he defined the equation \( h + w = 24 \) and used substitution in his previously calculated “derivative” as illustrated in
Figure 28. Continuing despite his procedural mistakes, Dan found that the critical point of his “derivative” occurred at \( h = 0 \) and reasoned that his original assumption was not correct because the beam would not have maximum strength at \( h = 0 \). Although Dan exhibited some adaptive reasoning skills to determine that his procedure was not correct based on the result, he did not display the adaptive reasoning capacity necessary to recognize the invalidity of his original forced experience. I believe that if his “derivative” could have coincidentally been solved to yield a positive number for the height, he would have accepted that it was a maximum height.

![Figure 28. Dan’s “derivative” after substitution from a forced experience](image)

Note: Dan scribbled out this work after he stated that the beam could not have a maximum strength with a height of zero.

As we approached the end of the allotted interview time, I shared the same idea of the inscribed angle theorem as I provided to Nick. At first, Dan wondered what arcs had to do with the solution, and probably too quickly (as a result of the dwindling time) I reiterated that the inscribed angle theorem guaranteed that the diagonal of the rectangle was a diameter of the circle. Without saying another word, Dan put his pencil to the paper and wrote an equation that he said was based on the Pythagorean theorem:

\[ h^2 + w^2 = 24^2. \]

Then, he solved that equation for \( w \) in terms of \( h \), just as Nick did. He tried to use the substitution in his previously computed “derivative,” which confirmed my
suspicion that Dan did not display the adaptive reasoning necessary to disregard his incorrect derivative procedure. Figure 29 depicts his symbolic work.

Figure 29. Dan’s substitution of the equation based on the Pythagorean theorem into his “derivative”

*Note:* Dan crossed out this work as he articulated his reasoning as described in the ensuing paragraph.

I prompted an induced experience by asking Dan with respect to which variable he differentiated to gauge if he would display the adaptive reasoning necessary to realize that his procedure was invalid. He said that he did not differentiate with respect to either variable, and his statement led him to reexamine the original equation for strength. He said it made more sense to substitute $h$ (as a function of $w$) into his strength function, which was indicative of adaptive reasoning to simplify his ensuing calculations.

Dan ended the interview by correctly using a substitution for $h^2$ in the strength function, computing the derivative of the resulting polynomial function, and finding the zero of the derivative function. He then found the height of the beam from his Pythagorean theorem-based equation and confirmed that the width he found was a maximum using the first derivative test. His work appears in Figure 30.
In contrast to the optimization strategies employed by the other students, Mary employed geometric reasoning in her solution strategy without considering the use of a derivative. As mentioned earlier, she originally reasoned that the beam should be a square, but when I asked her why she felt this way she said that her choice was based on the fact that her calculations would be easier. Because there was no link to a physical interpretation of the beam or its strength, I classified this as a forced experience. Her diagram for the square beam appears in Figure 31.

After Mary determined the dimensions of her square beam using the Pythagorean theorem, I asked once again why she felt the beam had to be square. At this point, she
said another reason behind her choice was that, “it’s better to come up with something than not to put anything,” which validated my classification of her initial work with the square as a forced experience. She also admitted that she was thinking about how to model a general rectangular beam, but was unsure how to do so. This new admission was also evidence that Mary felt she did not possess the strategic competence to follow through with a constructive experience. Shortly after, however, Mary made two additional sketches (as shown in Figure 32) to try to model the rectangular beam using “similar triangles.” Despite her sketches, Mary admitted that could not identify a way to use them productively in her solution strategy.

Figure 32. Mary’s attempt to model the rectangular cross section of the beam

To end the interview, I asked Mary to describe what a correct solution would look like and how she would know it was the strongest beam. She said that she needed a “formula saying the big and little triangles [pointing to the triangle and altitude shown on the right side of Figure 32] are similar or some geometry formula relating angles to each other.” Although she employed her strategic competence to model the problem in various geometric ways, she did not display the adaptive reasoning necessary to use these constructive experiences to find a solution.
**Task 3.1: Path of a Moving Particle Problem**

The most common initial solution strategy used in the Path of a Moving Particle problem was to evaluate the integral of the velocity function, \( v(t) = t^2 - 4t + 3 \), from 0 to 5. Nick, Deb, and Dan all employed this strategy, but their decisions leading to the symbolic manipulation and consequential acceptance or discounting of the result illustrated different aspects of the three problem-solving influences.

Nick began his work by stating his belief that the distance traveled would be the integral of the velocity function. He said he wanted to graph the function, but he hesitated and did not show any work on his paper. Because of his hesitation, I asked Nick why he felt a graph would be a good first step in his solution strategy. Nick said, “when it comes to velocity functions and the distance traveled, I sometimes get mixed up…but I am pretty sure the area under a velocity graph shows the total distance traveled. I was going to draw a graph to help me think about it logically, but I looked at the velocity function and I don’t think that’s an easy graph to draw.” His explanation that he sometimes was “mixed up” made me believe that he had seen problems like this before, and based on his desire to “think about it logically” with a graph I conjectured that he was using a constructive experience.

Nick proceeded to compute an integral and established the limits of integration from 0 to 5 as illustrated in Figure 33. However, before he evaluated the integral he stopped and asked me, “wait, is it the total distance or the change in distance?” I asked him to describe how he interpreted the distinction between the two, and he explained that a squiggled line (see Figure 34) would represent “total distance traveled” from \( A \) to \( B \) but a straight line would represent “total change of distance” from \( A \) to \( B \).
Then, Nick described a scenario in which the graph of velocity would have area below the $x$-axis. He stated that if he were looking for the “total change in distance” he would have to subtract the area below the $x$-axis, but if he were looking for the “total distance traveled” he would have to add the three areas together. I considered Nick’s explanations to be indicative of strong adaptive reasoning, and since I felt he understood the distinction between the two potential interpretations of what the problem was asking for I decided to confirm that he was thinking about “total distance traveled” as I intended when I created the problem. His supporting graph, which coincidentally was surprisingly similar to the graph of the given quadratic function in terms of the area below the $x$-axis between $x = 1$ and $x = 3$ and the graph’s symmetry across the minimum point, appears in Figure 35.
After Nick explained his strategy to “add the three areas together” he found the zeros of the velocity function and set up three separate integrals as illustrated in Figure 36. In the interest of time and because he set up the integrals correctly, I provided him with the arithmetic values shown in his work. He then said, “so those [pointing to the first and second integrals in Figure 36] just cancel out…or wait…I’m trying to figure out whether they do cancel out or if it’s supposed to be the absolute value. I’m pretty sure it’s the absolute value because you can’t have a negative area and the original reason I found these points [pointing to the zeros of the quadratic function $t^2 - 4t + 3$] was so I could add this area and if I would have done the big integral it would have just been $4/3$.” In this explanation, Nick displayed adaptive reasoning to interpret the numeric values according to his previous constructive experience.
Mary’s solution strategy was almost exactly the same as Nick’s, but there was one instance of a forced experience that is worth noting. Mary added “+ c” to her first computed integral and set it equal to zero as illustrated in Figure 37. She then said that $S$, the “distance function” as she called it, was zero at $t = 0$, which she used to find $c = 0$ from the first step in her solution strategy. When I asked where the conditions came from, she answered, “Normally when you start you call distance at time zero equal to zero. You use it as a clue to sub in the distance equation to find $c = 0$ but I don’t think that’s right.” Mary’s articulation of her doubts indicated to me that she was originally using a forced experience but her use of adaptive reasoning allowed her to question the validity of her initial assumption.

Figure 37. Mary’s forced experience for Task 3.1

Note: Mary crossed out this symbolic work after she expressed her doubt that she could stipulate that the “distance” at time zero was zero.

Mary proceeded to set up and solve integrals in the same manner that Nick did, but she multiplied the second interval by $-1$ as shown in Figure 38. She illustrated both strategic competence and adaptive reasoning by setting up the integrals as she did and justifying her representation by saying, “The particle is traveling left so the distance is negative but I still need to count it as total distance.”
As mentioned earlier, Deb’s initial solution strategy involved computing the definite integral of the velocity function from $t = 0$ to $t = 5$. As soon as she obtained this result, she questioned whether she obtained the “change in position, not the total distance traveled.” As she thought about her result, she verbalized that a strategy that she thought about but did not use was finding the indefinite integral of the velocity function and choosing “numbers to do the distance formula.” She rejected this idea on the basis that she did not know what the constant of integration would be. This was evidence of a recalled experience that prompted Deb to illustrate some strategic competence by representing the problem as a definite integral, albeit with the incorrect limits of integration. However, the fact that she questioned her result was indicative of her adaptive reasoning skills.

When I asked Deb to describe the motion of the particle, she reasoned that the velocity was the first derivative of position. She found the zeros of the velocity function to determine the points at which the particle changed direction. Then, Deb illustrated strong strategic competence by generating a graphical representation of the velocity function, a choice she justified by interpreting the integral she thought she should compute as area under a curve. Deb plotted the points on the graph of $v(t)$ whose $x$-values were integers, and then evoked a forced experience of graphing the function with “straight lines because it’s easiest.” She elaborated, “Distance would be area under the curve, but when I took the integral it subtracted the area below the $x$-axis. Even though it
is negative area it [the particle] is still traveling that distance, so I have to count it.” In spite of her forced experience, Deb still displayed her adaptive reasoning to justify the choice of her strategy. Her graph appears in Figure 39.

Figure 39. Deb’s graph of $v(t)$ for Task 3.1

Deb geometrically calculated the area of the triangles found on her graph and wrote a numerical answer of 10.5 for total distance traveled. I asked her if this was an approximate or exact answer, and she stated that it was approximate. I asked her if it would be possible to find an exact answer, and she described her belief that she could if she knew “the original position [of the particle].” This knowledge would enable her to “take the integral, solve for the constant, and find the point where it starts and where it stops to plug in to the distance formula.” Deb said, “Because we don’t know the constant it’s just zero,” indicating a forced experience that led her to plot the points $(0, 0)$ and $(5, 6^{2/3})$ on a coordinate plane. Once again, in spite of the forced experience Deb illustrated her adaptive reasoning by explaining, “Since $c$ just shifts the function up or down, it would change the initial and final positions by the same amount. I don’t really need it.”

I asked Deb how she would answer the question now that she had her two plotted points, but before she responded she asked me if the question wanted “total distance as a
straight line, or how the graph actually moves.” As I had done with Nick’s interview, I declined to answer until after I asked how she interpreted the distinction between the two. She stated that she would use the distance formula strategy to find straight-line distance. She mentioned the term “total displacement,” and, when I asked her what she meant, she said, “every direction it goes, even if it goes below the axis.” Although I thought she understood total distance traveled but was referring to it as “total displacement,” when I asked her to illustrate with her pencil the “total displacement” between the points (0, 0) and (5, $6^{2/3}$) she once again described the straight-line distance.

To check her understanding in light of the seeming contractions in her terminology, I induced an experience of describing the “squiggle” that Nick drew between two points in his interview (as shown in Figure 34 from Nick’s description). I asked her what the “total distance traveled” would be between those two points, and she stated that it was “all the distance it moves along the graph”. However, when I asked her how she would find the total distance if she knew what the position function looked like between 0 and 5, she simply said, “I don’t know.” Despite her adaptive reasoning skills that allowed her to interpret the distinction between total distance and displacement, Deb did not employ sufficient strategic competence to make her strategy of using the graph of the position function a viable option for this problem.

Dan’s solution strategy to the Path of a Moving Particle problem was initially based on a forced experience. He could not decide whether the problem called for the use of a derivative or an integral, but he admitted that because he and I did not use integrals in the last interview he would probably need one now. Due to the lack of mathematical reasoning in his guess to integrate, I classified the motivation for his
strategy as a forced experience. However, Dan then employed a constructive experience by describing a strategy in which he would compute both an integral and a derivative to “see which answer makes more sense.”

Although Dan’s motivation for his strategy was based on a constructive experience, its implementation relied on the use of a number of forced experiences. He started by integrating $v(t)$ and substituting in $t = 5$ as shown in Figure 40, and his justification for the substitution was, “It is still a time.” When I questioned what his result meant, he said, “I don’t remember,” indicating the use of a forced experience in his computation. He was satisfied that his result for “total distance traveled” was reasonable on the basis that it was a positive number. To assess Dan’s capacity for adaptive reasoning, I questioned what he would have thought had the integral yielded a negative answer. He said he supposed it would be possible to have “negative distance traveled,” but he admitted that he “did not want to think about how it made sense,” again illustrating that his response was based on a forced experience.

![Figure 40. Dan’s initial indefinite integral for Task 3.1](image)

Dan computed the derivative of $v(t)$ as $2t - 4$, but before he interpreted this result he said, “wait, do I need a plus $c$?” In Dan’s solution strategy, the question of whether or not to include a constant of integration was one of his recurring concerns. Dan was only sidetracked by his first thought about the constant of integration for a moment before he
evaluated \( v'(5) \) as 6, which he interpreted as the acceleration. Dan disregarded this result based on his belief that acceleration would not help to find “total distance traveled,” so he did display some adaptive reasoning in the differentiation aspect of strategy.

Dan then returned to the question of needing a constant of integration. He said, “Supposing there is no plus \( c \), the answer is \( \frac{20}{3} \).” When I asked him if he thought he needed a constant of integration, he indicated another forced experience by saying, “When you take an integral you usually have a plus \( c \).” Dan thought for a minute, and then reasoned that he did not need a constant of integration by saying, “If this [the integral] were solving for distance traveled, if I had plus \( c \) I would already have a value for distance traveled at \( t = 0 \).” Through this explanation, I was able to understand Dan’s interpretation of the integral and recognized that he was still displaying adaptive reasoning in spite of a flawed conception of the position function.

Despite Dan’s admission that he did not think the constant of integration was necessary, he persisted in trying to find it. He thought he needed to evaluate the zeros of \( v(t) \), and during his ensuing explanation he described the movement of the particle as “moving forward from 0 to 1, moving back from where it came from 1 to 3, and then back the other direction from 3 to infinity.” His graphical work for representing the path of the particle appears in Figure 41a. I asked Dan to interpret his previous answer of \( \frac{20}{3} \) for “total distance traveled” relative to his interpretation of how the particle was moving. To do so, he drew lines as illustrated on the right side of Figure 41 and said, “It started at the origin and moved forward a little, then back a little, and then moved forward again. The total distance traveled would be from here to here [draws line segment shown in Figure 41b].”
Figure 41. Dan’s interpretations of the path of the particle (a) and “total distance traveled” (b)

Note: The line segment connecting \((1, f(1))\) and the point \((3, 0)\) did not appear on his original sketch—it was added to the figure as part of a later justification.

Whereas the other students offered two potential interpretations of “total distance traveled,” Dan indicated only one interpretation using his figure. I prompted an induced experience by indicating that I considered what he described to be displacement and asked if he could determine the total distance traveled if he were to consider the arrow he drew backwards as counting toward the total distance. Illustrating his strategic competence, Dan modeled my request as illustrated in Figure 42.

Figure 42. Dan’s integrals to model an induced definition of total distance traveled

Note: The operation in between the first and second integrals is subtraction.

I asked Dan why he chose to subtract the second integral, and he justified this choice because the integral from 1 to 3 would give a negative answer. However, as he hovered his pencil over the sketch shown in Figure 41 he felt that the area under the curve (though his graph depicted position, not velocity) from 1 to 3 would be positive unless there was a constant of integration. Thus, Dan confused himself by applying the forced experience of interpreting the integrals he set up as area under the graph of his
position function. He did not display the strategic competence or adaptive reasoning necessary to draw and consider the graph of the velocity function, which I believe would have led him to the correct answer in a similar manner as the other students.

Dan’s reasoning described in the paragraph above prompted him to abandon his idea to use definite integrals in his strategy. He returned to work on his indefinite integral shown in Figure 40 and substituted \( t = 1 \), which he computed as \( 1^{\frac{1}{3}} \). He interpreted the value \( 1^{\frac{1}{3}} \) as the “distance traveled” instead of the position of the particle. Similarly, he substituted \( t = 3 \) in the indefinite integral and computed a result of 0. He immediately said, “This isn’t right. The first value should be less than this value because it is moving backwards.” This puzzling result led Dan to display his adaptive reasoning by saying, “Oh wait, it [the particle] is. The difference between the point here [pointing at \( f(3) \)] and the point here [pointing at \( f(1) \)] is \( 1^{\frac{1}{3}} \).” Then, Dan modified his original graph (Figure 41) and created a new graph (Figure 43a) to reflect his new interpretation that the value of the “distance traveled function” was 0 at \( t = 3 \). He reasoned that the distance traveled between \( t = 1 \) and \( t = 3 \) was \( 1^{\frac{1}{3}} \) because the 0 he previously computed meant that the particle traveled backward the same distance as it moved from \( t = 0 \) to \( t = 1 \). Finally, Dan substituted \( t = 5 \) into the indefinite integral and obtained a value of \( 20^{\frac{1}{3}} \). He added the distances (i.e., \( 1^{\frac{1}{3}}, 1^{\frac{1}{3}}, \) and \( 20^{\frac{1}{3}} \)) together to obtain his final answer of \( 28^{\frac{2}{3}} \).

![Figure 43](image-url)

(a) (b)  

Figure 43. Dan’s modified “distance traveled function” (a) and pictorial justification (b)
The students’ work with Task 3.1 resulted in three correct final answers and one nearly correct approximation of the final answer. For this task, students identified appropriate representations for the problem based on constructive experiences and persisted with these representations for the duration of their solution processes. However, relative success on Task 3.1 did not guarantee similarly successful results for Task 3.2.

**Task 3.2: Volume of a Metal Ring Problem**

For the Volume of a Metal Ring problem, every student except Nick began the task by discussing a geometric approach of computing the volume of the entire sphere and then subtracting the volume of the piece that was drilled out. When asked what the shape of the drilled out section would look like, Deb and Dan described it as a cylinder with a rounded piece on the top and bottom. In contrast, Mary felt that the shape cut out of the sphere would be a smaller sphere. Each student’s sketch of the cutout appears in Figure 44.

![Figure 44. Sketches of piece cut out from metal sphere by Dan (a), Deb (b), and Mary (c)](image)

Dan and Deb exhibited the strategic competence necessary to correctly model the physical interpretation of the metal ring, but both students said that they could not geometrically determine the volumes of the rounded portions of the cutout piece. I asked
if they could think of another way to solve the problem without using explicit geometric formulas, and neither student displayed additional strategic competence to develop another representation of the problem. In both interviews, we were nearing the end of the allotted time so I induced an experience of a modified version of Nick’s sketch (as shown in Figure 45; Nick’s work will be discussed later) to see if students could use the new experience of strategic competence as a springboard for displaying additional adaptive reasoning. I asked them to describe any way they felt the sketch related to the problem and how it might be useful as a potential strategy to solve the task.

![Figure 45](image)

Figure 45. My sketch given to provoke an induced experience for Deb and Dan

Deb responded to my drawing shown in Figure 45 by admitting a recalled experience that she remembered something involving the use of an integral but she could not remember how to work with it. She went on to say, “it’s something with a figure and if you rotate it you get the area of a figure like this.” When I asked her what portion would be rotated according to her interpretation, she identified her belief that both the top and bottom shaded regions would be rotated around the $x$-axis. When I asked if she could remember anything else about the idea of rotation, she said, “I just remember it has to do with integrals. It was one of the last pieces of calculus and we haven’t done it yet in [her college Calculus I] class.”
Dan responded to the induced experience by explaining his belief that he could find the areas of the shaded region and a rectangle around it (sketched in Figure 46) to help him determine information about the rounded top of the cutout piece. There were only 5 minutes remaining in the interview, so in the interest of time I induced the experience that the white space in the graph was the hole in the ring created after the drilling was done. Dan thought silently for approximately 1 minute and said, “I still don’t know.” To try to gain additional evidence of his capacity for adaptive reasoning, I induced one final experience as I showed him the sketch shown in Figure 45, told him that the student who drew that sketch computed an integral in his solution, and asked him to consider how an integral would relate to either the picture or a solution strategy. Dan thought silently for at least 2 minutes before conceding that he had no additional ideas, so I ended the interview at that time. In contrast to his work with the Strength of a Wooden Beam problem, Dan did not illustrate additional evidence of adaptive reasoning after an induced experience in the Volume of a Metal Ring problem.

Figure 46. Dan’s interpretation of the induced experience

As previously mentioned, Mary interpreted the physical representation of the problem as a small sphere being cut out of the larger sphere, so this aspect of her strategic competence was not as strong as that displayed by the other students in the study. She
proceeded to find the volume of the two spheres and subtracted the volume of the smaller sphere from the volume of the larger sphere as shown in Figure 47.

When Mary reached for the calculator to find a decimal approximation for the volume, I assumed she was nearly finished with her solution. I induced an experience of illustrating the drilling motion as boring straight down through the sphere so the circular hole as viewed from above would have a radius of 3 inches. I then asked her if she thought the cutout shape would still be a sphere, to which she responded, “Are you saying it’s going to be a different one?” Before I had a chance to answer she said, “I think you need to subtract out the sphere because all you have left is the ring. I could have done a formula with big $R$ and little $r$, but I know I would make a mistake.”
interpreted the previous sentence to be her invoking an established experience involving
the difference of the volumes of two cylinders that she might be associating with the
volume of two spheres, so I asked her to continue discussing that idea if she could. She
went on to explain her idea that she needed to subtract out the volume of the smaller
sphere from the larger sphere. Incidentally, the first line would yield the same result she
previously computed (as shown in Figure 47), but Mary did not display the adaptive
reasoning necessary to link her two symbolic representations. Thus, I classified her work
(including the writing shown in Figure 48) as a forced experience.

![Figure 48. Mary’s symbolic work with an “alternate” strategy](image)

*Note:* Mary crossed out these equations when she said she would make a mistake with the
“big R and little r” strategy.

After hearing Mary’s second interpretation of the cutout piece as a sphere, I
decided to induce a more explicit description of the cutout piece as a cylinder. I said, “If
you drill down through the sphere using a circular motion, the shape that is cut out is a
cylinder and not a sphere.” My description did not mention the rounded top and bottom
pieces of the cutout piece that Deb and Dan mentioned, and Mary did not exhibit any sort
of adaptive reasoning to question my admittedly imprecise word choice. Thus, she
employed a forced experience based on my description and calculated the volume of a
right circular cylinder of height 10 to subtract from the volume of the sphere. Her work
appears in Figure 49.
Figure 49. Mary’s work to find the volume of a cylindrical cutout piece

Note: Mary crossed her out her drawing of the sphere after hearing my explanation as described in the next paragraph.

Shortly after Mary computed this result, she exhibited her adaptive reasoning by explaining her belief that she did not draw the cylinder correctly. I mistakenly provided her with the reason why her drawing was wrong (not accounting for the rounded piece of which Dan and Deb spoke) instead of providing her with the opportunity to exhibit her own evidence of adaptive reasoning and strategic competence. After my description, Mary conceded that she did not know how to compute the volume of that piece using a formula as she did for the spheres and cylinders. In an attempt to salvage some opportunity for Mary to exhibit adaptive reasoning and strategic competence, I asked her if there were a way that she could represent the problem without using a geometric formula.

As Mary thought, she said, “Well, you used an integral in the first one [Task 3.1], but I don’t think I have to use an integral, do I?” In response to her rhetorical question, I asked her if she could use an integral in her solution, to which she said, “No, I can’t… wait….” Because she could not decide whether or not an integral would fit within a reasonable strategy for the problem, I asked her what purpose she felt computing an integral would serve within a potential strategy. She said that an integral would be used
for area explained that she thought she could rotate an area to find volume but could not remember how, which was indicative of a recalled experience. Her graphical interpretation of an integral as an area appears in Figure 50.

Figure 50. Mary’s interpretation of an integral as area under a curve

As Mary considered her recalled experience, she exhibited adaptive reasoning and strategic competence that helped her recast the recalled experience as a constructive experience. She drew several sketches (one of which appears in Figure 50) and described finding an area between a curve and a line that she could rotate around the $x$-axis to find a volume. When she admitted that she did not know what this area would be, I attempted to induce the same experience as I did for Deb and Dan using the sketch shown in Figure 45 to see if she would identify the area she needed given a new piece of strategic competence. Mary used hand gestures to illustrate rotation around the $y$-axis, and then subsequently around the $x$-axis.

I asked her which interpretation made more sense to her, and she said that the rotation around the $x$-axis made more sense. She then outlined a strategy of rotating the semicircle to find the volume of a sphere and then using the horizontal line to rotate and find the volume of the cylinder. Although she did not mention the limits of integration or the intersection point of the two graphs, I was fairly confident that she understood the
general strategy for finding a volume of revolution but was missing the strategic competence to set up the integral. To try to provide a hint to evoke strategic competence, I induced one final experience of accumulating the areas of small disks to create a volume. Mary mentioned the idea of taking a limit because she thought there would be a large number of these disks, but ultimately she decided she could no longer make any additional progress with the task and we ended the interview.

Of all the students in the study, Nick exhibited by far the most evidence of the problem-solving influences in his work on Task 3.2. His strategic competence was evident immediately through his sketch and corresponding interpretation of what the metal ring would look like after the hole was drilled (see Figure 51). Nick considered finding the volume of the ring with “a geometric approach,” but in contrast to his peers he rejected this idea in favor of using a strategy with integrals. When I asked him why he made this choice, he said, “I would be losing the top part of the sphere and I don’t know how to calculate that. I definitely remember doing these things where you had two different lines and you would find the difference of integrals and that shows you the area between that you would rotate around the y-axis. I don’t have a very good memory so I need to brainstorm to figure out how you go about these problems.”

Figure 51. Nick’s interpretation of the metal ring for Task 3.2
Nick’s explanation illustrated the consideration of a constructive experience, but unlike his peers Nick exhibited the adaptive reasoning necessary to reject the geometric idea quickly so it did not turn into a forced experience. Additionally, Nick illustrated strategic competence combined with another constructive experience to consider a new representation of the problem involving an integral. Furthermore, his desire to “brainstorm” illustrated his adaptive reasoning to help incorporate the constructive experience into a reasonable strategy.

Nick’s brainstorm for the integral strategy led him to draw a graphical representation of his interpretations of the physical description of the metal ring (see Figure 52a). His original sketch and description reflected the strategy he originally proposed of rotating an area around the \( y \)-axis, but exhibited additional adaptive reasoning and strategic competence by saying, “Come to think of it, I want to graph it this way [pointing to Figure 52b] because I am better at thinking of them rotating this way [around the \( x \)-axis].” When I asked him what the difference was between the two pictures, he again exhibited strong adaptive reasoning based on a constructive experience by saying, “The resulting shape will be no different, but it will be rotated 90 degrees. More often than not, we find integrals with \( y \, dx \) instead of \( x \, dy \), so this [Figure 52b] will let me find the area and wrap it around the \( x \)-axis, which will be half of the whole shape.”
Figure 52. Nick’s initial graphical representations

I asked Nick to clarify what he meant by “half of the whole shape”, and he responded by saying that he could use the symmetry of the graph to evaluate the integral using only half of the graph and then multiply his result by 2 to find the area that needed to be rotated, which was once again indicative of strong adaptive reasoning. Nick’s next step was to define equations for both the horizontal line and the upper part of the circle. He found the equation for the horizontal line easily, but he admitted, “There is no way I would remember the equation for a circle, but right now in [the Calculus II course] we are doing parametric equations and we are doing a lot with circle graphs. This [writes $x = \cos$ and $y = \sin$] would be parametric equation style, but I’m going to try to get it into a Cartesian equation. I’m not sure if sine and cosine are matched up correctly so I need to check.”

Nick checked that his parametric definition was correct by saying, “Yes $y$ should be sine because it needs to be opposite over adjacent.” Based on his strong adaptive reasoning up to this point I was inclined to believe that he misspoke and said “adjacent” instead of hypotenuse, especially given his correct interpretation of the trigonometric functions in his work for Task 2.2. Then, he defined an equation for the circle as $x^2 + y^2 = 1$, a choice he justified using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$. He then
solved the equation as $y = \sqrt{1-x^2}$. I realized that he failed to account for the radius of the sphere being equal to 5, but I waited to intervene to see if he would exhibit similar adaptive reasoning to realize his relatively minor error.

He began his next sentence with, “So now that I have the formula for the sphere…,” which I interpreted as acceptance of the previous result and a desire to continue with the solution strategy. I stopped him to induce the experience that the equation of a circle centered at the origin was $x^2 + y^2 = r^2$. Given this induced experience, Nick reasoned that his graph showed the radius of the circle as 5 and his equation should actually have been $y = \sqrt{25-x^2}$.

The next step in Nick’s strategy was to find the intersection point between the graphs of the horizontal line and the upper half of the circle. Nick found the intersection point as (4, 3) by solving $3 = \sqrt{25-x^2}$ and used $x = 4$ as an upper limit of integration (see first line of Figure 53) to find to find the area underneath each curve. Next, he said that the area between the two curves could be found by subtracting the integrals from each other (see last line of Figure 53).

Figure 53. Nick’s integrals to find area under each curve (first line) and area between the curves (second line)
I asked Nick how setting up the integral as he did in Figure 53 would help him find the volume of the metal ring. He said that the area under the curve is the area of a “cross-section,” and he would need to use the “equation of a circle, either area or volume, to rotate it [the cross-section] around the x-axis.” I asked him to elaborate on his idea of the equation of a circle, to which he responded that he thought he needed to incorporate π into his integral somehow with the “volume” of a circle. I thought he misspoke and meant to say area, so I asked him to clarify whether he meant area or volume. Nick said, “I meant volume. Did I say area earlier? I didn’t mean to say that.” Although Nick was not correct (a circle does not have volume), he was still exhibiting adaptive reasoning to try to make sense of his representation. Given the presence of adaptive reasoning, I still considered Nick to be working with a constructive experience, albeit with a missing piece of strategic competence to set up his integrals correctly for a solid of revolution.

Nick proceeded to explain that he was confident that the integrals appearing in Figure 53 would give him the “area of the cross-section,” but he could not remember the equation he needed to use that area and a rotation of the region around the x-axis to find a volume. This was Nick’s first indication of a recalled experience in his work with Task 3.2, and in an attempt to see if he would display additional adaptive reasoning in a slightly different situation I asked him if there was a way he could build up the formula using the ideas he mentioned about π and “volume” of a circle (I was careful to use his interpretation). Nick referred to another recalled experience that he felt the problems he did in Calculus I involved $2\pi$ inside the integral, but he was not positive that he was remembering correctly. Given the lack of adaptive reasoning, Nick evoked the forced experience of setting up the integral as shown in Figure 54.
Despite Nick’s forced experience of writing the integral shown in Figure 54, he described some potential pieces of strategic competence as he continued to narrate his recalled experiences from Calculus I. He said he remembered “finding an outer radius and then the inner radius, and then you subtract the volume you’re getting with the inner radius from the outer radius.” Once again, I felt that Nick was almost on the verge of a fully productive strategy, so I continued to ask him how the idea of integrals would allow him to get a volume to see if he would show use his adaptive reasoning skills to develop the correct integral. He said, “An integral allows you to take the cross-section and rotate it around the axis. The space it passes through when it’s rotated creates a volume. A volume is just an area with thickness to it, so that rotating gives it the thickness.” Nick’s explanation illustrated very strong adaptive reasoning of the concept behind volumes of rotation, but his strategic competence was still not fully evident because he kept referring to the area as a cross section instead of the difference in measures of the areas of the two disks with the outer and inner radii he mentioned before.

I asked Nick to refer back to his integral (see Figure 54) and tell me which part of it modeled the thickness he spoke about. This question was meant to see if any ensuing work would be based on a constructive experience given his new evidence of adaptive reasoning or if it was still evidence of a forced experience. He said that the difference of the integrals was the area, and the $2\pi$ represented the thickness. He said, “$2\pi$ does make sense with the thickness I am thinking about because if you were to watch a point trace
around the $x$-axis it would trace out the circumference of the circle, which is $2\pi r$.’’ This explanation indicated that Nick was trying to use the integral as part of a constructive experience even though his reasoning was not completely correct. He then said, “If this were a math test, I would definitely write down the answer that this [the integral shown in Figure 54] gave me.”

When I asked him if he could compute the integral, he first said that he could, but then paused because he could not identify a way to integrate the square root function. He integrated the constant part of the integral with ease, but after approximately 2 minutes he did not identify a way to integrate the square root function. With approximately 10 minutes left in the interview, I wanted to see if I could prompt an induced experience of strategic competence that would allow him to combine the various pieces of his strategy into a correct strategy and subsequent solution. I described the process of accumulating the areas of disks with thickness of $dx$ along the $x$-axis as opposed to rotating the area of the cross-section that he had been describing. Nick said that he understood the process I described and that he remembered looking at pictures of “tiny disks stacked on each other,” but he said that he could not incorporate that understanding into his integral.

Despite Nick’s extensive use of adaptive reasoning and attempts at employing constructive experiences, he ultimately did not exhibit the strategic competence necessary to find a correct solution to the task. In any case, the extensive inclusion of the problem-solving influences in his strategies was valuable evidence related to the answer to my research question, which I discuss in detail in chapter 4.
Chapter 4: Discussion

In this chapter, I discuss the answer to my research question in light of the findings presented in chapter 3 and the background literature discussed in chapter 1. First, I will consider whether the students’ experiences with the tasks I developed reflected the nature of a problem and problem solving as those phenomena were discussed in chapter 1. Then, I respond to the research question—when asked to solve introductory calculus problems that are structurally similar to but more problematic than typical textbook tasks, to what extent and in what combination do first year undergraduate students with qualifying AP Calculus Exam scores incorporate their established experiences, adaptive reasoning, and strategic competence. I first highlight the extent to which established experience, strategic competence, and adaptive reasoning were incorporated in student solution strategies. Finally, I discuss combinations of influences in terms of how various interactions among the three problem-solving influences manifested themselves in student solution strategies.

Student Reception to Tasks as Problems

According to the definition of a problem from chapter 1, each task used in this study (with the intended exception of Task 1.1) was a problem because no student was able to identify an immediate procedure to use as a strategy that led to a direct answer. This is not to say, however, that students were unable to outline a reasonable solution strategy shortly after reading the problem text. Specifically, for Tasks 1.2, 2.1, and 3.1 the study participants were able to use constructive experiences to determine plausible
initial strategies, though not a single student was able to implement that strategy as originally stated to obtain a correct answer.

Throughout each interview, students exhibited strategic competence to generate representations of the problem and adaptive reasoning to monitor their results and progress. Therefore, the modifications I implemented to make the tasks more problematic than typical textbook tasks met the intended purpose of providing students with a context in which they could demonstrate their understanding of mathematics instead of merely performing procedures and calculations. As discussed in chapter 1, the evidence of strategic competence and adaptive reasoning accounts for the heuristics and control categories of “knowledge and behavior necessary for an adequate characterization of mathematical problem-solving performance” (Schoenfeld, 1985, p. 15).

The two remaining categories, resources and belief systems (Schoenfeld, 1985), were present in the students’ work and explanations even though I did not purposefully design the interviews to measure the extent and combination of their inclusion as I did with the problem-solving influences. Each task was designed to be more problematic than a typical textbook problem in several of the ways articulated by Thompson and colleagues (1997), but the resources necessary to implement a solution strategy were all included in the set of reasonable resources I identified as a result of the textbook analysis. The student work and justifications I heard validated that, for the most part, each participant was familiar with the basic procedures and concepts of differentiation and integration found in the textbooks (i.e., Larson, Hostetler, & Edwards, 1998; Stewart, 2003; Weir, Hass, & Giordano, 2006).
My observations included a few cases in which students did not display evidence of correct applications of their resources when working with forced experiences. However, I hypothesize that the students did not apply the resources correctly because their work with forced experiences inhibited the development of additional representations of the problem; I do not believe that the students could not apply their resources correctly in different contexts. I support the previous conjecture with evidence that Nick and Dan were able to identify and correct procedural derivative errors after I asked them to explain their work on Task 2.2. In both instances I noticed that their work was incorrect, but rather than waste what little time we had in the interview by allowing them to continue their strategy with incorrect procedural results I decided to intervene. My questions induced constructive experiences in which students had the opportunity to display additional occurrences of adaptive reasoning in a new context, and each of the two students ultimately displayed the correct application of his resources.

I obtained evidence of students’ individual belief systems as they explained their work and voiced frustrations or excitement related to the implementation of their solution strategies. Students were disappointed at times that they could not develop correct solutions, and often commented that they felt they would have done more tasks correctly if the interviews took place immediately after studying the appropriate textbook section. Nick, in particular, exhibited his motivation by volunteering to stay longer than the anticipated 45 minutes for the second and third interviews. He also made a comment that he was going to go home and review related rates problem after he failed to develop a correct solution to Task 2.1. In short, these students took the tasks seriously and showed a strong desire to know the answer to each task after the interview ended. These brief
glimpses into the student belief systems once again validate that the students were working with genuine mathematics problems as they completed the interview tasks.

**Established Experience**

As I expected, students drew heavily upon their established experience from classroom settings; they continually spoke about things their teachers did or problems that they had seen on assessments or homework assignments in ways similar to those Lithner (2000) described as reasoning based on established experiences. The students, for the most part, were able to employ constructive experiences at the beginning of their solution strategies. Although these strategies were not necessarily correct due to a lack of strategic competence, they were nevertheless meant to be productive in the pursuit of a solution. The students who were able to develop correct solutions made extensive use of constructive experiences in their problem-solving processes. For the students who were not able to develop correct solutions, the absence of adaptive reasoning and additional strategic competence caused what few constructive experiences they did employ to quickly transition to forced experiences.

As students persisted with forced experiences that involved a single representation of the problem, they engaged in a “wild goose chase” (Schoenfeld, 1985, p. 32) of trying to satisfy a particular goal without identifying its importance in the overall strategy. Student comments (according to their belief systems) such as “If this were a test, it would be my answer” (Nick) or “It’s better to come up with something than to not put anything” (Mary) provided insight into why the forced experiences were implemented as often as they were. The comments, particularly with respect to symbolic manipulation, reflected
the students’ belief that “doing” mathematics in the procedural sense is a desirable way to
develop a reasonable solution.

The extent to which recalled experiences were implemented in student solution
strategies was much less than I expected. In most instances, evidence of recalled
experiences usually occurred shortly before a student gave up on a strategy implemented
as a result of a forced experience. Invoking a recalled experience usually prompted the
student to give up on a particular solution strategy in favor of investigating a different
representation for the problem or admitting they had no other ideas upon which to
develop additional strategies. In the particular case of Task 2.2, the recalled experience
of needing to find an equation relating the width and height of the beam together
motivated me to provide students with the final type of experience considered in the
study: induced experience.

My attempts to provoke induced experiences served as the intended intervention
in a student’s solution process to gauge adaptive reasoning capability in the absence of
sufficient strategic competence. Students’ work with the Strength of a Wooden Beam
and Volume of a Metal Ring problems involved the most instances of induced
experience, and in each case the students illustrated additional evidence of adaptive
reasoning in response to the new context of the induced experience. The attempts to
provoke induced experiences were not explicit instructions to complete the task; instead,
I presented information that the student could implement in a proposed strategy using
adaptive reasoning. Nick’s and Dan’s successes in response to the induced experience
provoked by the description of inscribed angle theorem in Task 2.2 showed that capacity
for logical thought was present given a new context. In chapter 5, I discuss the
implication of the illustrated capacity for logical thought with respect to scaffolding student questioning in practice.

**Strategic Competence**

Strategic competence was most evident in the tasks for which most students outlined a fairly correct overall solution strategy shortly after reading the problem. As previously discussed, the students were able to identify a general strategy as a result of incorporating constructive experiences, which helped to generate an appropriate representation of the problem. In contrast, the absence of strategic competence necessitated the incorporation of forced experiences in the solution strategy. In a similar manner, the absence of adaptive reasoning while working with forced experiences prevented the student from displaying additional evidence of strategic competence. My findings from this study indicated an “all-or-nothing” aspect of strategic competence for these particular students; they generally developed one strategy or representation on their own and did not deviate from it unless I provided an induced experience. The students seems to invest “all” in one strategy, regardless of its ineffectiveness, with “nothing” for other strategies until provoked.

The lack of strategic competence that I observed beyond the initial representation is somewhat troubling, particularly among a group of engineering students who will be expected to interpret problems and develop solutions to them throughout their college studies. The task modifications I implemented gave students the opportunity to illustrate their strategic competence, but in many cases they did not.
Adaptive Reasoning

As previously mentioned, the absence of adaptive reasoning led to extensive, unproductive work with forced experiences. Adaptive reasoning was, however, commonly included in a student’s solution strategy in the form of checking the accuracy of procedures or making sense of physical conditions (e.g., determining the constraint for the maximum width of the beam). Additionally, students illustrated multiple instances of adaptive reasoning after an induced experience, which suggests they have the capacity for logical thought given an alternative representation or a missing piece of strategic competence.

Of the problem-solving influences considered in this study, I gathered the least qualitative data on the spontaneous inclusion of adaptive reasoning in student solution strategies. One cause may be that the lack of strategic competence to generate multiple representations of the problems led to fewer opportunities in which to exhibit adaptive reasoning. My questioning may have prompted students to illustrate adaptive reasoning that they may not have evoked themselves if I had remained silent. In chapter 5, I discuss these and other limitations of the study in more detail.

Interaction Among the Problem-Solving Influences

As noted in the previous sections, the discussion of the extent to which the problem-solving influences were included in a student’s solution strategies was not complete without identifying, at least briefly, the absence or presence of one influence that likely affected the extent to which a separate influence was implemented. In this section I synthesize observations about the influences to answer to the overall research
question regarding the prevalence of and interactions among strategic competence, adaptive reasoning, and established experience in student solution strategies as I distinguish between successful and unsuccessful problem-solving attempts. This answer is not meant to provide a definitive statement of the problem-solving proficiency of qualified AP Calculus students; instead, it reflects the results of an exploratory study that brings to light ideas to consider both in practice and in further research. Relevant considerations based on the exploratory nature of the study are discussed in more detail in chapter 5.

Most correct solutions given in the interview sections began with a representation selected as a result of a constructive experience, as is evidenced in the students’ work with several tasks. For Task 1.2, Dan and Nick selected a graphical representation of the absolute value function, whereas Deb selected a symbolic representation. In each case, the students persisted with the initial representation throughout the problem, so they illustrated their strategic competence by choosing an appropriate type of representation at the beginning of the solution process and applying their mathematical resources in a correct manner to obtain a solution.

Guiding the application of mathematical resources was adaptive reasoning, which allowed Nick and Dan to correct their graphical errors and Deb to make sense of her “conditions.” In Dan’s case, his statement that the derivative was undefined at the vertex of the graph was indicative of a forced experience based on a rule his teacher told him; although he obtained a correct answer he did not display his adaptive reasoning to justify why it was so, and thus was not confident in his answer. In contrast, Nick and Deb were confident of their answers because they used adaptive reasoning to explain their answers
related to an interpretation of derivative as slope. Since Deb began with a symbolic interpretation, the new graphical interpretation was also indicative of additional strategic competence and flexibility to translate across representations.

For Task 2.2, Nick’s and Dan’s correct solutions began with valid pictorial representations of the rectangular cross section of the wooden beam and appropriate symbolic representations of the strength function. As was the case with Task 1.2, the strategies branched off in two separate directions based on the presence of adaptive reasoning (Nick) and the absence of adaptive reasoning (Dan). Nick’s extensive adaptive reasoning, combined with the constructive experience of identifying the problem as an optimization task, allowed him to monitor his progress with his initial trigonometric representation of the problem. He knew that the tangent function was not going to help him achieve his goal of finding \( w \) in terms of \( h \), so he exhibited flexibility in his strategic competence by trying to use sine and cosine. What he failed to realize, however, is that he defined both \( w \) and \( h \) in terms of \( \theta \). He could have modeled strength as a function of only \( \theta \) and derived that equation according to his optimization strategy.

In contrast to Nick’s approach, Dan’s strategy employed numerous forced experiences in which he symbolically manipulated any and all work he had on the page. The forced experiences, combined with a lack of adaptive reasoning, prevented Dan from exhibiting the same flexibility that I saw from Nick. Both students, however, responded to the induced experience of the inscribed angle theorem by generating a correct representation using the Pythagorean theorem. For both students, there was translation across an initial pictorial representation and a new symbolic representation of the problem. Additionally, the induced experience provided the missing piece of strategic
competence, and the students’ subsequent use of adaptive reasoning led to correct final answers.

The final task that yielded correct answers was Task 3.1. Once again, students who obtained correct final answers began with the strategy of computing integrals and generally persisted with this representation throughout their solution process. However, both Nick and Dan exhibited additional strategic competence by employing graphical representations of the movement of the particle, yet again illustrating the ability to translate across representations. As with his work on Task 2.2, Dan did not exhibit adaptive reasoning and consequently persisted with numerous forced experiences throughout his solution. I still am not sure how he generated his last representation of the problem, but once his strategic competence allowed him to model the problem in a way that he could analyze, his adaptive reasoning shone through in what I feel is a very unorthodox, though correct, solution.

In general, when successful in solving the problem, students showed strong, though not perfect, initial strategic competence coupled with recurring evidence of adaptive reasoning. This allowed them to develop a solution with only minor adjustments to the initial strategy, but their reasoning was guided in most cases by the flexibility to translate across symbolic and pictorial representations of the problem. Further, for every student except Dan the strong initial strategic competence was based on a constructive experience. Dan’s solutions almost always differed from other students’ approaches in that he persisted with forced experiences in the absence of adaptive reasoning, but once he obtained the correct piece of strategic competence
(induced or otherwise) he did display the adaptive reasoning necessary to develop a correct solution.

One final aspect to note is that the induced experiences I provided gave a hint about a missing piece of strategic competence. Once the students understood what the problem text was saying, they were consistently able to employ their adaptive reasoning to make sense of the new information. These results beg the question of whether we are preparing students well enough to interpret and model problematic situations. It may well be that the component missing in current mathematics classrooms is simply the opportunity to display strategic competence. I will discuss this possibility in more detail in chapter 5.

The unsuccessful problem-solving efforts generally began with poor strategic competence to model the problem. For Task 2.1, no student could correctly interpret and model the rate of change of the volume using the notion of proportionality to the reciprocal of the radius. Even after identifying the task as a related rates problem, the students employed forced experiences of manipulating work on their paper that they generated by the initial absence of strategic competence. Furthermore, the absence of strategic competence and adaptive reasoning in the presence of forced experiences precluded the students from showing the same flexibility in translating across representations that made them successful in their other problem-solving efforts.

Similarly, only Nick considered using a solid of revolution to model Task 3.2. The others began with constructive experiences of trying to use geometric volumes, but once again the lack of adaptive reasoning and additional strategic competence caused students to persist with unproductive forced experiences. In short, the considerable
extent to which I classified forced experiences as being used in unsuccessful problem solving efforts was motivated by the fact that students did not display strategic competence and adaptive reasoning in their strategies. When students evoked what I classified as forced experiences in their work, they also showed a seeming inability to apply new instances of adaptive reasoning and strategic competence that could generate a constructive experience.

Despite the large number of unsuccessful problem-solving efforts, there was positive evidence of the capacity for adaptive reasoning given a missing piece of strategic competence. For Task 2.1, Nick responded to the induced experience of the correct interpretation of \( \frac{dV}{dt} \) by setting up a separable differentiable equation. If he would have considered computing an integral, I am confident he could have arrived at a correct answer because he used his integration resources correctly given a constructive symbolic representation of Task 3.1. In Task 3.2, Nick was only missing the interpretation of the volume of rotation as an accumulation of areas of disks. In contrast to the other students’ insistence that they could not make additional progress because they forgot the formulas they learned, Nick exhibited strong adaptive reasoning by trying to recreate the formulas on his own. Although I did not consider student belief systems as a major influence in this study, I did note that Nick exhibited the desire to recreate the formulas whereas the other students in the study did not.

To summarize, when students were able to employ strong initial strategic competence along with adaptive reasoning throughout their solution process, they generally were successful in their problem-solving efforts. Furthermore, the use of constructive experiences and their ability to translate across representations allowed them
to exhibit more instances of adaptive reasoning, which in turn gave them more confidence that their answers were correct. In contrast, unsuccessful problem-solving efforts were characterized by extensive use of forced experiences and the absence of both strategic competence and adaptive reasoning. However, students given a missing piece of strategic competence still showed the capacity for adaptive reasoning. The implications of these results, both in practice and further research, will be discussed in chapter 5.
Chapter 5: Conclusions

In this chapter I first discuss the implications of this study for practice, particularly with respect to opportunities for scaffolding in mathematics classrooms. Additionally, I suggest implications of this study for high school AP Calculus courses. Finally, I discuss the limitations of my research and suggestions for further research.

Implications for Practice and AP Calculus

As was mentioned in chapter 4, the students in this study showed promising evidence of the capacity for adaptive reasoning when they understood something that might be have been absent from their work due to a piece of strategic competence that was previously missing. Based on the tasks I observed in my review of the calculus texts, I believe that part of the problem is that the statement of tasks and the inclusion of hints or diagrams for any problem that is slightly different from the textbook example take away valuable opportunities for students to exhibit strategic competence on their own. In the particular case of AP Calculus instruction, depending on McMullin’s (2010) classification of the free-response “problems” on the AP Exam according to their content and context discourages students from exhibiting strategic competence. By giving explicit suggestions about how each task should be solved, we reduce a potential problem, which is consequently an invitation to exhibit strategic competence, to a mere exercise involving reproductive thinking.

An implication for practice, then, is that teachers can and should employ Thompson and colleagues’ (1997) task modifications to provide students with the opportunities to display and develop strategic competence that are currently missing.
Specifically, the criteria of real context, representation, and reasoning reflect increased emphasis on the solution process of developing a representation, translating across representations, and justifying why particular procedures are being implemented instead of on the product of a final answer.

In conjunction with task modifications, Hiebert and Wearne (2004) argue that teachers can make extensions of exercises (see Schoenfeld’s [1992] discussion of Milne, as cited in Stancic & Kilpatrick, 1988) more problematic simply by avoiding “stepping in and doing too much of the mathematical work too quickly” (p. 7). If we can stimulate a student’s curiosity and foster an inherent desire to solve the problem, then students will be meaningfully engaging in the task in the manner discussed in chapter 1. Silver (1988) echoes this sentiment by saying that students in a classroom setting employing true problem solving would “engage in doing mathematics rather than having it done to them” (p. 276).

The means to the end of having students engaged with real problems could be implementing scaffolding strategies in all mathematics classrooms (Anghileri, 2006; Erickson, 1999). Since the 1990s, scaffolding strategies have been researched in response to the theory that traditional mathematics instruction may not be helping students develop what Kilpatrick and his coeditors (2001) termed “mathematical power” (p. 115). In particular for AP Calculus, if we attempt to make mathematics accessible to the student by using McMullin’s (2010) method of traditional instruction for the short-term goal of passing the AP Exam, we may actually hinder their ability to succeed in solving novel problems in postsecondary education. The hindered ability to succeed in solving novel problems was especially evident in my study with respect to solids of
revolution, which are included in McMullin’s (2010) characterization of free response questions. I saw first hand that students who were deemed qualified according to their AP Exam struggled to implement strategic competence and adaptive reasoning away from the familiar textbook task context for solids of revolution.

In contrast to traditional instruction in which students are passive observers to mathematics that teachers tell them should be done (Baxter & Williams, 2009), scaffolding strategies allow the teacher to support learning by encouraging the students to develop representations and strategies on their own (Anghileri, 2006). With respect to strategic competence, students should not be shown and told how the teacher would represent, for example, the Strength of a Wooden Beam problem. Instead, students should be engaged with finding their own representation of the task through the discussion of transferrable representational tools that could be applied to a variety of problematic situations (Anghileri, 2006). In the context of the Strength of a Wooden Beam problem, this might manifest itself through the teacher asking students to develop three distinct strategies to relate the variables \( w \) and \( h \) together. The students could then discuss and evaluate the relative merits of each strategy instead of simply finding the solution using the Pythagorean theorem-based strategy that Larson and colleagues (1998) suggest through their hints and diagrams.

In summary, I am encouraged by the results of my study indicating that students do exhibit the capacity for adaptive reasoning and additional strategic competence in spite of imperfect initial strategic competence. Through the student interviews, I saw how provocative induced experiences encouraged students to display more adaptive reasoning and generate additional representations of the problematic situations different
from those that they otherwise evidenced. My work with Nick in Task 2.1, in particular, shows the benefits of using focusing questioning to help students connect isolated pieces of their strategic competence together into a fully synthesized solution strategy.

As previously described in chapter 3, Nick began his work on the Radius of a Growing Sphere problem with an incorrect interpretation that the volume of the sphere, as opposed to the rate of change of the volume of the sphere, was proportional to the reciprocal of the radius. I intentionally focused my questions to scaffold him toward applying his isolated pieces of strategic competence (e.g., interpretation of the variables $V$, $r$, and $t$; notion of proportionality; differentiation with respect to time) into a fully synthesized integration strategy for the problem. Although I was ultimately unsuccessful in helping Nick obtain the correct answer the problem, he was able to implement constructive experiences and adaptive reasoning in his solution process to develop an alternative symbolic representation of the phrase “the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius” from that with which he started.

Nick was successful in finding a correct answer to the Strength of a Wooden Beam problem after I provoked an induced experience by using the inscribed angle theorem to prove that the diagonal of the beam had to be a diameter of the round log. The application of a Pythagorean theorem-based equation to relate the variables $w$ and $h$ together proved to be the piece missing from Nick’s initial evidence of strategic competence. Perhaps more important than the correct answer, though, was the additional evidence of adaptive reasoning and strategic competence he displayed as he worked with his trigonometric representation of the relationship between $w$ and $h$. If Nick would have immediately developed a Pythagorean theorem-based strategy as I anticipated students
would have done without my task modifications, I doubt I would have seen these instances of adaptive reasoning and strategic competence that illustrated a non-obvious solution strategy for the Strength of a Wooden Beam problem.

From these two representative results, I believe that scaffolding strategies and task modifications provide students with more opportunities to develop and exhibit strategic competence and adaptive reasoning than they would have in traditional show-and-tell instruction. The additional representations developed using strategic competence combined with thoughtful use of scaffolding strategies that retain the problem status of classroom tasks will allow students to take charge of their own understanding and engage in meaningful mathematics, thus paving the way for continued success throughout their educational careers.

**Limitations of the Study and Suggestions for Further Research**

As I mentioned previously, I consider this to have been an exploratory study based on my own curiosity of the problem-solving abilities of a population of students I hope to teach in the very near future. The most basic limitation of this research was the time allowed for each interview; I had only 45 minutes to understand how a student showed evidence of three distinct indicators of problem-solving performance. Additionally, the limited time precluded me from persisting with questioning after students evoked induced experiences near the end of an interview (e.g., all students in Task 2.1, Deb in Task 3.1, Deb and Mary in Task 3.2). With respect to constructive experiences, the time limitation also precluded Nick from investigating his trigonometric representation of the Strength of a Wooden Beam problem, which likely would have
resulted in a correct final answer if he would have used the Pythagorean identity 
\[ \sin^2 x + \cos^2 x = 1. \]

In noted cases my questioning and inexperience as an interviewer also were 
limitations to the study as I funneled students toward what I expected to see instead of 
trying to focus them on their own thoughts. For example, in the Radius of a Growing 
Sphere problem Dan stated that he thought he needed to incorporate the volume of the 
sphere in his strategy, and when I asked why he said, “Because you keep asking me 
questions about volume.” The effects of these funneling questions were fewer 
opportunities in which the students could demonstrate evidence of spontaneous strategic 
competence and adaptive reasoning, which counteracted my goal of designing 
problematic tasks for which students could apply extensive adaptive reasoning and 
strategic competence in their solution processes.

In addition to noting my inexperience, I also attribute these funneling questions to 
my bias of wanting to see the students succeed by making suggestions that would lead to 
correct answers. My mistake, however, was that these suggestions were based on my 
interpretations of the problems and not the interpretations the students were trying to 
explain to me. Additionally, I now recognize that students could implement components 
of successful problem solving even without obtaining a correct final answer. Given more 
time in my own classroom, I hope to build upon my experiences from this study and 
improve my focusing questioning strategies.

Another limitation of the study was the AP background, and consequently the 
mathematical resources, of each student. All students received qualifying scores on the 
Calculus AB Exam during their junior year of high school. There was more than a year
time in which they were not engaged with calculus on a daily basis before they participated in my study. This gap in their calculus experience manifested itself in a deterioration of some of the resources I expected students would possess for each interview, the most noticeable of which was an understanding of solids of revolution.

My assumption in designing the interviews was that each volunteer would possess a similar set of resources that could be applied to each task, but in practice the similarities among the student resources when comparing the Calculus I and Calculus II students were not always apparent. For example, both Calculus II students evoked an induced experience and applied a resource of a Pythagorean theorem-based strategy that led to a correct final answer for Task 2.2. In contrast, neither Calculus I applied the resources necessary to obtain a correct final answer.

I thought the likelihood of a common set of resources among study participants would afford each student similar opportunities to display evidence of strategic competence and adaptive reasoning. In retrospect, it may be that differences in resources that I suspected but could not measure within the interviews accounted for some of the differences in number of additional representations beyond the first generated for the problems. For future research, I recommend that quantitative data of performance on a set of tasks like Task 1.1 be included to control for the extent to which differences in student resources influence the implementation of strategic competence and adaptive reasoning in task solving.

One final limitation of my study deals with student belief systems and the reasonable expectation that a student who scored a 5 on the Calculus AB Exam could be enrolled in university Calculus II. For the Calculus I students participating in my study, I
did not account for their own belief systems of why they chose to repeat the Calculus I course instead of accepting advanced placement into university Calculus II. Could it be that they felt their resources were not on par with those of students who had completed Calculus I in college, and thus they chose to repeat the course rather than accept credit? Conversely, could the Calculus II students’ choice to accept credit and advanced placement be indicative of their positive self-perceptions of ability to use and apply knowledge from Calculus I? The answers to these questions might provide more insight into the limitation of differences among student resources, so in future studies I would consider an introductory interview to gain insight into each student’s belief systems before proceeding with the problem-solving tasks.

In light of these limitations, I offer some final suggestions for my own future research or for directions to be considered by the mathematics education community in general. First, more careful attention must be focused on the heuristic and control strategies modeled and taught during AP Calculus instruction. Based on the results of this exploratory study, I am inclined to believe that there is room for improvement in both of these areas. To this end, I believe there should be a study of how students develop and display strategic competence and adaptive reasoning in an AP Calculus course and what types of instruction are most conducive to the development of these two problem-solving influences.

Next, I propose an investigation of the extent to which problems rather than exercises might be incorporated and valued in college and AP Calculus courses. From the sample exams provided by the large, mid-Atlantic research institution in which I conducted my study, I could not identify a novel problem among them. It may be that the
AP Calculus Exam is adequately preparing students to succeed in a post-secondary calculus environment devoid of problems, in which case we should seriously consider how effective traditional instruction in college mathematics courses is in preparing the next generation of mathematicians, scientists, and engineers.

Finally, I recommend that teachers conduct their own informal exploratory research to determine the extent to which their students exhibit evidence of strategic competence, adaptive reasoning, and established experience as problem-solving influences in their work on classroom tasks. As a future teacher myself, my eyes were opened throughout this experience and I truly believe that providing students with the opportunities to engage in meaningful mathematics will empower them to develop the adaptive reasoning and strategic competence necessary for continued success throughout their lives.
Appendix: Sample Solution Elements for Interview Tasks

The following pages contain the task statements and portions of written work for at least one strategy to solve each problem.

First Interview

Task 1.1: If \( f(x) = x^3 + 3x - 1 \), what is \( f'(x) \)?

\[
f'(x) = 3x^2 + 3
\]

(application of a term-by-term derivative procedure sometimes referred to as "The Power Rule")

Task 1.2: Consider the function \( f(x) = |x - 1| \). What is the value of the derivative of this function when \( x = -1, 0, \) and \( 1 \)?

Interpret derivative as slope

\[
f'(-1) = -1 \quad \text{slope of linear function graph}
f'(0) = -1 \quad y = -(x-1) \text{ is } -1
\]

\( f'(1) \) does not exist because multiple tangent lines can be drawn through \((1,0)\)
Second Interview

Task 2.1: At time $t$ ($t \geq 0$), the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is 2. Find the radius of the sphere as a function of $t$.

$$V_{\text{spher}} = \frac{4}{3} \pi r^3$$

"volume of a sphere increasing at a rate proportional..."$ \Rightarrow \frac{dV}{dt} = \frac{k}{r}$

$$\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt} \quad \text{use equivalent expression} \quad \frac{dV}{dt} = \frac{k}{r}$$

$$4 \pi r^2 \frac{dr}{dt} = \frac{k}{r}$$

Separable differential equation

$$\int 4 \pi r^2 \, dr = \int k \, dt$$

Integrate

$$\pi r^4 = \frac{k t}{4} + C$$

Apply initial conditions

$$\pi (1)^4 = k(0) + C$$

$$\pi = C$$

$$\pi (2)^4 = k(15) + \pi$$

radius at $t = 15$ is 2

$$16\pi = 15k + \pi$$

$$15\pi = 15k$$

$$k = \pi$$

$$\pi r^4 = \pi t + C$$

Substitute values for $k$ and $C$

$$r^4 = t + 1$$

radius as a function of time

$$r = \sqrt[4]{t+1}$$
Task 2.2: A wooden beam has a rectangular cross section of height $h$ and width $w$. The strength of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest board that can be cut from a round log of diameter 24 inches?

$$S = kw h^2$$

$$24^2 = h^2 + w^2$$  
Because measure of inscribed angle ($90^\circ$) is half the measure of the intercepted arc ($180^\circ$ = semicircle), the diagonal of any inscribed rectangle is a diameter of the log.

$$h^2 = 24^2 - w^2$$  
Solve for $h^2$ (simplifies derivative)

$$S = kw (576 - w^2)$$  
Substitute to find $S$ as function of $w$

$$S = 576kw - kw^3$$  
$$S' = 576k - 3kw^2 = 0$$  
Substitute to find critical points that could be maximum value

$$3k(192 - w^2) = 0$$  
$$192 = w^2$$  
$$8\sqrt{3} = w$$  
Must be positive because $w$ is a length

$$h^2 = 576 - 192$$  
$$h^2 = 384$$  
$$h = 8\sqrt{6}$$

$$S'' = -6kw < 0 \Rightarrow \text{concave down} \Rightarrow w \text{ is max}$$

Dimensions of cross-section of beam $8\sqrt{3}$ in $\times 8\sqrt{6}$ in.
Task 3.1: The velocity of a particle at time $t$ ($t \geq 0$, in seconds) is given by the function $v(t) = t^2 - 4t + 3$. What is the total distance traveled by the particle in its first 5 seconds of motion?

Integral of velocity is position

Integrating velocity from $[0, 5]$ yields displacement, not total distance.

Need to account for particle moving backwards from $(1, 3)$ → take absolute value of integral

\[
\int_0^1 (t^2 - 4t + 3) \, dt = \left. \frac{t^3}{3} - 2t^2 + 3t \right|_0^1 = \frac{1}{3} - 2 + 3 = \frac{4}{3}
\]

\[
\left| \int_1^3 (t^2 - 4t + 3) \, dt \right| = \left| \left. \frac{t^3}{3} - 2t^2 + 3t \right|_1^3 \right| = \left| 9 - 18 + 9 - \left( \frac{1}{3} - 2 + 3 \right) \right| = \left| -\frac{4}{3} \right| = \frac{4}{3}
\]

\[
\int_3^5 (t^2 - 4t + 3) \, dt = \left. \frac{t^3}{3} - 2t^2 + 3t \right|_3^5 = \frac{125}{3} - 50 + 15 - \left( \frac{125}{3} - \frac{105}{3} \right) = \frac{20}{3}
\]

Total Distance = \[\frac{4}{3} + \frac{4}{3} + \frac{20}{3} = \frac{28}{3}\] units

Note: Displacement is \[\frac{4}{3} - \frac{4}{8} + \frac{20}{3} = \frac{20}{3}\] units
Task 3.2: A manufacturer drills a hole through the center of a metal sphere of radius 5 inches. The hole has a radius of 3 inches. What is the volume of the resulting metal ring?

Piece cut out difficult to find geometrically.

\[ x^2 + y^2 = 25 \]
\[ y = \frac{25 - x^2}{2} \]

Rotate shaded region using an integral.

Idea: accumulate areas of disks to find volume: \((\pi r^2)\)

Need limits of integration:

\[ 3 = \sqrt{25 - x^2} \]
\[ g = 25 - x^2 \]
\[ x = 5 \]

\[ y = \frac{25 - x^2}{2} \]

Outer radius \( R = \sqrt{25 - x^2} \)

Inner radius \( r = 3 \)

Radius of disk in shaded region to be accumulated.

\[ \pi \int_{-4}^{4} \left( \sqrt{25 - x^2} \right)^2 - 3^2 \, dx = \pi \int_{-4}^{4} 25 - x^2 - 9 \, dx \]

\[ = \pi \int_{-4}^{4} 16 - x^2 \, dx \]

\[ = \pi \left[ 16x - \frac{x^3}{3} \right]_{-4}^{4} \]

\[ = \pi \left[ 64 - \frac{64}{3} - (-64 + \frac{64}{3}) \right] \]

\[ = \pi \left[ \frac{128}{3} - \frac{128}{3} \right] \]

\[ = \frac{256\pi}{3} \cdot \frac{3}{3} \]

\[ = \frac{256\pi}{3} \]
References


http://fcweb.sd36.bc.ca/~joyce_c/Courses/FOV1-001F3C76/FOV1-00206561/


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