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Theory, Methods and Applications of the Classical Hypergeometric Orthogonal Polynomial Sequences of Sheffer and Jacobi

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Abstract

The purpose of this Schreyer Honors Thesis is to explore theories, methods and applications of orthogonal polynomial sequences. This thesis is designed to be approachable and understandable by any undergraduate mathematics or physics major at The Pennsylvania State University in order to serve as a learning resource. This paper is to be comprehensive and self-consistent; that is, we hope to cover all prerequisite information here that is required for the understanding of this paper, hence the thorough introduction. Throughout this paper, several definitions, terminologies and notations are used, as listed in Chapter 1. The presentation of polynomial sequences here closely follow that in [1] and [2]. Limit, series and integral relations involving orthogonal polynomials are presented in Chapter 2. Chapter 3 covers the inverse method and Schrödinger form for various orthogonal polynomial sequences. Chapter 4 introduces applications of orthogonal polynomials in physics and numerical analysis.

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Chapter 1

Introduction to Functional Analysis

We need to introduce three concepts: metric spaces, normed vector spaces and inner-product spaces. We follow the presentation of these concepts by [3].

1.1 Metric Spaces

A **metric** (property of distance) on a set X is a function, where $X^2 = X \times X$,

$$\begin{aligned} d : X^2 &\to \mathbb{R}^+ \\ (x, y) &\to d(x, y) \end{aligned} \tag{1.1.1}$$

such that, for every $x, y, z \in X$,

1. Positive-definiteness: $d(x, y) \geq 0, \quad d(x, y) = 0$ if and only if $x = y$
2. Symmetry: $d(x, y) = d(y, x)$
3. The Triangle Inequality: $d(x, y) \leq d(x, z) + d(z, y)$

If X has a metric d, we say it is a metric space, denoted by (X, d).
Example: If $x, y \in \mathbb{R}^3$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, we have the Euclidean metric

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

If X is a normed vector space, then the norm of the space, denoted $||\cdot||$, induces a metric on X defined in a natural way: if $\mathbf{a}, \mathbf{b} \in X$, then $d(\mathbf{a}, \mathbf{b}) = ||\mathbf{b} - \mathbf{a}||$. So, what is a norm?

1.2 Normed Vector Spaces

A normed space X is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with the **norm** (property of length) $||\cdot|| : X \to \mathbb{R}$ such that, for every $\mathbf{x}, \mathbf{y} \in X, \ \lambda \in \mathbb{R}$,

1. Positive-definiteness: $||\mathbf{x}|| \geq 0, \quad ||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
2. Scaling-homogeneity: $||\lambda\mathbf{x}|| = |\lambda|||\mathbf{x}||$
3. The Triangle Inequality: $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$

If X has a norm $||\cdot||$, we say it is a normed space, denoted by $(X, ||\cdot||)$.
Example: If $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{x} = \langle x_1, x_2, x_3 \rangle$, we have the Euclidean norm

$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

If a space has an inner-product, usually denoted $\langle , \rangle$, the space will induce a norm in a natural way: $||\mathbf{x}||^2 = \langle \mathbf{x}, \mathbf{x} \rangle$. So, what is an inner-product?

1.3 Inner-Product Spaces

An inner-product space X is a space with the **inner-product** (property of angles) $\langle\ ,\ \rangle : X^2 \to \mathbb{F},\ \mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, such that, for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X,\ \lambda \in \mathbb{F}$,

1. Positive-definiteness: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0, \quad \langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$
2. Scaling-homogeneity: $\langle \mathbf{x}, \lambda \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$
3. Complex-Scaling: $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \overline{\lambda} \langle \mathbf{x}, \mathbf{y} \rangle$
4. Complex-Symmetry (remember, the complex conjugate of a real number is itself): $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
5. Distribution: $\langle \mathbf{x}, \mathbf{z} + \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$ and $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$
6. Angles: $\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||\,||\mathbf{y}||}$

We also define the projection of two vectors. If $\mathbf{x}, \mathbf{y} \in X$ then the scalar projection of $\mathbf{x}$ onto $\mathbf{y}$ is

$$\text{proj}_{\mathbf{y}}\mathbf{x} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \tag{1.3.1}$$

Example: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, where $\mathbf{x} = \langle x_1, x_2, x_3 \rangle, \mathbf{y} = \langle y_1, y_2, y_3 \rangle$, we have the dot-product

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

An inner-product always induces a norm. Given the norm, we can test if the norm was induced by an inner-product using the Parallelogram Law, which is obeyed by all norms induced by an inner-product:

$$||\mathbf{a} + \mathbf{b}||^2 + ||\mathbf{a} - \mathbf{b}||^2 = 2||\mathbf{a}||^2 + 2||\mathbf{b}||^2 \tag{1.3.2}$$

We also the the Cauchy-Schwarz inequality

$$\langle \mathbf{x}, \mathbf{y} \rangle^2 \leq ||\mathbf{x}||^2 ||\mathbf{y}||^2 \tag{1.3.3}$$

We now ask ourselves, what was the purpose introducing these concepts? So we could discuss orthogonality.

1.4 Orthogonality

Two vectors $\mathbf{x}, \mathbf{y}$ are said to be **orthogonal** (generalization of the quality of perpendicularity from $\mathbb{R}^n$) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Before we continue, we must introduce a new concept: the Kronecker Delta, defined as

$$\delta_{nm} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \tag{1.4.1}$$

A collection of orthogonal vectors $\{e_k\}_{k\in\mathbf{J}}$, where $\mathbf{J}$ is some countable index, has the property due to the definition of orthogonality, $e_i \in X, \langle e_n, e_m\rangle = ||e_n||^2\delta_{nm}$. If they span X, then they can form a basis for X. That is, if $\mathbf{x} \in X$ then

$$\mathbf{x} = \sum_{k\in\mathbf{W}} c_k e_k, \quad c_k = \text{proj}_{e_k}\mathbf{x}$$

The set of all polynomials is a vector space, and an orthogonal polynomial sequence can form a basis of said space over the support interval, denoted Ω of the orthogonal polynomial. So, let us introduce concepts of polynomial sequences.

1.5 Foundations of Polynomial Sequences

We start with introducing some foundational terms.

Definition 1.5.1. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a set of polynomials such that the degree of $P_n(x)$ is exactly n, written as* $\deg(P_n(x)) = n$. *We call this set a* ***polynomial sequence***

Definition 1.5.2. *A set of polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ is said to be a monic polynomial sequence, or simply* ***monic****, if* $\deg(Q_n(x) - x^n) \leq n - 1$. *That is, the leading coefficient is 1.*

Definition 1.5.3. *The* ***weight*** *function, $w(x)$, is a non-negative and integrable function over some interval $\Omega \subset \mathbb{R}$ such that $\int_\Omega w(x)dx > 0$ and $w(x) > 0 \forall x \in \Omega$.*

A polynomial sequence can be made monic by simply dividing by the leading coefficient in the polynomial sequence. The bridge between polynomial sequences and orthogonal polynomials sequences is the moment-functional. We start by requiring that the nth moment μ_n, defined as

$$\mu_n = \int_\Omega x^n w(x)\mathrm{d}x \quad n = 0, 1, 2... \tag{1.5.1}$$

be finite. We can thus construct the sequence $\{\mu_n\}_{n=0}^{\infty}$. From here, we can define the operator

$$\mathcal{L}[f(x)] = \int_\Omega f(x)w(x)\mathrm{d}x \tag{1.5.2}$$

So, $\mathcal{L}[x^n] = \mu_n$. We say that $\mathcal{L}$ is the **moment-functional** determined by the moment sequence $\{\mu_n\}_{n=0}^{\infty}$. $\mathcal{L}$ is also an inner-product over the vector space of polynomials, a subspace of $\mathbf{L}^2\Omega$. That is, for f, g polynomials,

$$\langle f, g\rangle = \mathcal{L}[f, g] = \int_\Omega f^*(x)g(x)w(x)\mathrm{d}x$$

Where $f^*(x)$ is the complex conjugate of $f(x)$. For our purposes, we will only consider polynomials with real coefficients being integrated on some interval of the reals, in which case $f^*(x) = f(x)$. Since polynomials are dense, we can extend this inner-product to all of $\mathbf{L}^2\Omega$. We also require that $\mathcal{L}$ to be positive-definite. That is,

$$\mathcal{L}[f^2(x)] \geq 0, \ \mathcal{L}[f^2(x)] = 0 \text{ if and only if } f(x) = 0$$

Definition 1.5.4. *Two polynomials related by some index, say f_n, f_m, are said to be* ***orthogonal*** *with respect to moment-functional $\mathcal{L}$ (or, equivalently, with respect to weight $w(x)$) if*

$$\mathcal{L}[f_n, f_m] = K_n \delta_{nm} \tag{1.5.3}$$

Where $K_n = \mathcal{L}[f_n, f_n]$ will be referred to as the ***squared norm****. If $K_n = 1$, the polynomials are said to be* ***orthonormal****.*

Example: Let $\Omega = [-\pi, \pi]$ and $w(x) = 1$. Then, for $n \neq 0, m \neq 0$.

$$\mathcal{L}[\cos(nx), \cos(mx)] = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\mathrm{d}x = \pi\delta_{nm} \tag{1.5.4}$$

where $\cos(nx)$ can be thought of as a polynomial via its Taylor expansion.

Definition 1.5.5. *A set $\{P_n(x)\}_{n=0}^{\infty}$ is called an* ***orthogonal polynomial sequence*** *(OPS for short) with respect to some moment functional $\mathcal{L}$ (or, again, with respect to weight $w(x)$) if*

1. *$\{P_n(x)\}_{n=0}^{\infty}$ is a polynomial sequence, as defined above*
2. *$\mathcal{L}[P_n(x), P_m(x)] = K_n\delta_{nm}$, that is, any two terms in the sequence are orthogonal. $K_n = \mathcal{L}[P_n(x), P_n(x)]$ is the squared norm of an OPS, and, if $K_n = 1$, the OPS is an orthonormal polynomial sequence (ONPS).*

Theorem 1.5.6. *If $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS, then it forms a basis on the vector space of polynomials. That is, if $\pi(x)$ is a polynomial such that $\deg(\pi(x)) = n$, then,*

$$\pi(x) = \sum_{k=0}^{n} c_k P_k(x) \tag{1.5.1}$$

where

$$c_k = \frac{\mathcal{L}[\pi(x), P_k(x)]}{\mathcal{L}[P_k(x), P_k(x)]}$$

That is, c_k is the projection of $\pi(x)$ onto $P_k(x)$ via the moment-functional associated with $\{P_n(x)\}_{n=0}^{\infty}$.

Corollary 1.5.6.1. *If $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS with respect to some $\mathcal{L}$, then it is unique up to some arbitrary coefficient. That is, if $\{R_n(x)\}_{n=0}^{\infty}$ is an OPS with respect to the same $\mathcal{L}$, then*

$$R_k = q_k P_k, \quad q_k \neq 0, \quad \forall\ k \in \mathbf{W}$$

Corollary 1.5.6.2. *If $\pi(x)$ is a polynomial such that $\deg(\pi(x)) = j,\ j < n$, then, $\mathcal{L}[\pi(x), P_n(x)] = 0$*

Proof. From Eq. (1.5.1), we can write $\pi(x) = \sum_{k=0}^{j} c_k P_k(x)$. So,

$$\begin{aligned}\mathcal{L}[\pi(x), P_n(x)] &= \int_{\Omega} \pi(x)P_n(x)w(x)\mathrm{d}x = \int_{\Omega}\left(\sum_{k=0}^{j} c_k P_k(x)\right) P_n(x)w(x)\mathrm{d}x \\ &= \sum_{k=0}^{j} c_k \int_{\Omega} P_k(x)P_n(x)w(x)\mathrm{d}x = \sum_{k=0}^{j} 0 = 0\end{aligned}$$

where the orthogonality of the OPS made each integral zero. An important special case is

$$\pi(x) = x^j$$

. □

Theorem 1.5.7. *If $\{P_n(x)\}_{n=0}^{\infty}$ an OPS with respect to $\mathcal{L}$ and with squared norm K_n, then the set $\{p_n(x)\}_{n=0}^{\infty}$, where*

$$p_n(x) = \frac{P_n(x)}{\sqrt{K_n}}$$

has the property

$$\mathcal{L}[p_m(x), p_n(x)] = \int_{\Omega} p_m(x)p_n(x)w(x)dx = \delta_{nm}$$

*and is called the associated ONPS. We say that $p_n(x)$ is the **normal** version of $P_n(x)$, and that $P_n(x)$ has been **normalized**.*

It would be careless if the following distinction was not brought to light: $\mathcal{L}$ is a *continuous* inner-product. There exists a *discrete* inner-product that is obeyed by certain orthogonal polynomial sequences (e.g. the Charlier, Meixner and Krawtchouk polynomials, etc.):

$$\sum_{x\in\Omega_2} P_n(x)P_m(x)w(x) = K_n\delta_{nm} \tag{1.5.2}$$

where $\Omega_2 \subset \mathbf{W}$.

Orthogonal polynomial sequences are a type of *special function*. An example of a special function is the Gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1}\mathrm{e}^{-t}\mathrm{d}t, \quad \mathrm{Re}\, z > 0 \tag{1.5.3}$$

The gamma function is the generalization of the factorial, and has the following three properties:

1. If n is a positive integer: $\Gamma(n) = (n-1)!$
2. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
3. $\Gamma(z) = \frac{\Gamma(z+1)}{z}$

The gamma function appears in the definition of some orthogonal polynomials, hence the brief digression to cover its properties here.

1.6 Properties of Orthogonal Polynomials

Now that we have an idea of what an orthogonal polynomial sequence *is*, we will introduce the associated concept of the recursion relation.

We first begin by introducing new concepts, starting with what is hopefully familiar. Consider the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{1.6.1}$$

Next, we introduce the Pochhammer symbol $(a)_k$:

$$(a)_k := a(a+1)(a+2)\cdots(a+k-1), \quad (a)_0 := 1 \tag{1.6.2}$$

and

$$(a_1, ..., a_j)_k := (a_1)_k \cdots (a_j)_k \tag{1.6.3}$$

We can now generalize the binomial theorem.

Definition 1.6.1. *The Classical Hypergeometric Function ${}_rF_s$ is a power series of the form*

$${}_rF_s\left(\begin{matrix} a_1,\ldots a_r \\ b_1,\ldots,b_s \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots a_r)_k}{(b_1, \ldots, b_s)_k} \frac{x^k}{k!} \tag{1.6.4}$$

Corollary 1.6.1.1. *To see that the Hypergeometric function is a generalization of the binomial theorem, observe that*

$${}_1F_0\left(\begin{matrix} a \\ - \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!} = (1-z)^{-a} \tag{1.6.5}$$

From here, we introduce a theorem.

Theorem 1.6.2. *Every OPS has a hypergeometric form, and the definition of the OPS is its hypergeometric form.*

Now, we must introduce the concept of generating functions, following [4].

Definition 1.6.3. *An infinite sequence $\{c_n\}_{n=0}^{\infty}$ has the ordinary generating function (OGF)*

$$G(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots \tag{1.6.6}$$

where the series on the right hand side of Eq. (1.6.6) *is called the explicit form. There is also an implicit form if the series converges.*

As an example, consider the sequence of all 1's:

$$G(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

where we used the properties of geometric series to obtain $\frac{1}{1-x}$, the implicit form. Another example, returning to Eq. (1.6.1), where $y = 1$, we get

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{1.6.7}$$

So, for the finite sequence $\{\binom{n}{k}\}_{k=0}^{n}$, the left hand side of Eq. (1.6.7) is the implicit generating function, and the right hand side is the explicit generating function. Now, we introduce the concept of exponential generating functions (EGF).

Definition 1.6.4. *A sequence $\{c_n\}_{n=0}^{\infty}$ has the exponential generating function:*

$$G(x) = \frac{c_0}{0!} + \frac{c_1 x}{1!} + \frac{c_2 x^2}{2!} + \frac{c_3 x^3}{3!} + \cdots + \frac{c_n x^n}{n!} + \cdots \tag{1.6.8}$$

Example, again using the sequence of all 1's:

$$\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \mathrm{e}^x \tag{1.6.9}$$

Where the left hand side of Eq. (1.6.9) is the explicit form, and the right hand side is the implicit form, obtained using the Taylor expansion of e^x. A sequence has both an OGF and an EGF, but a general rule-of-thumb is: if the elements of the sequence are not constants, use an EGF. And now we state a theorem.

Theorem 1.6.5. *Every OPS has a generating function, either exponential or ordinary. We say that function **generates** the OPS.*

From here, we introduce a key concept of orthogonal polynomial sequences: the recursion relation. For example, consider the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

The Fibonacci sequence obeys the following **recursion relation**:

$$F_n = F_{n-1} + F_{n-2}, \quad n > 1, \quad F_0 = 0, F_1 = 1$$

That is, we can determine the nth Fibonacci number from adding the previous two. We can now introduce another theorem.

Theorem 1.6.6. *It is a necessary and sufficient condition that an OPS $\{P_n(x)\}_{n=0}^{\infty}$ satisfies an **unrestricted three-term recurrence relation**:*

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad A_n A_{n-1} C_n > 0 \tag{1.6.10}$$

where $P_{-1}(x) = 0, P_0(x) = 1$.

Corollary 1.6.6.1. *If $Q_n(x)$ is the monic form of $P_n(x)$, then it is a necessary and sufficient condition that $\{Q_n(x)\}_{n=0}^{\infty}$ satisfies the following **monic three-term recurrence relation**:*

$$Q_{n+1}(x) = (x - b_n)Q_n(x) - c_n Q_{n-1}(x), \quad c_n > 0 \tag{1.6.11}$$

where $Q_{-1}(x) = 0, Q_0(x) = 1$.

Corollary 1.6.6.2. *If $\{p_n(x)\}_{n=0}^{\infty}$ is an ONPS (and, remember, every OPS has an associated ONPS), then $p_n(x)$ satisfies following* ***orthonormal three-term recurrence relation***

$$a_{n+1}p_{n+1}(x) = (x - b_n)p_n(x) - a_n p_{n-1}(x) \tag{1.6.12}$$

Now, it is trivial to show[1] that, by comparing Eq. (1.6.12) to Eq. (1.6.11), we get $a_n = \sqrt{c_n}$. So, we can also write Eq. (1.6.12) as

$$\sqrt{c_{n+1}}p_{n+1}(x) = (x - b_n)p_n(x) - \sqrt{c_n}p_{n-1}(x)$$

Next, we define three functions which will become readily important. For a given weight $w(x)$, domain Ω and associated ONPS $\{p_n(x)\}_{n=0}^{\infty}$, we have

$$\nu(x) = -\ln(w(x)), \tag{1.6.13}$$

$$A_n(x) = a_n\left(\left.\frac{p_n^2(y)w(y)}{y-x}\right|_{\partial\Omega} + \int_{\Omega}\frac{\nu'(x)-\nu'(y)}{x-y}p_n^2(y)w(y)\mathrm{d}y\right), \tag{1.6.14}$$

$$B_n(x) = a_n\left(\left.\frac{p_n(y)p_{n-1}(y)w(y)}{y-x}\right|_{\partial\Omega} + \int_{\Omega}\frac{\nu'(x)-\nu'(y)}{x-y}p_n(y)p_{n-1}(y)w(y)\mathrm{d}y\right) \tag{1.6.15}$$

where a_n is from Eq. (1.6.12).

Theorem 1.6.7. *The following differential equation and relations are true*

$$p_n'(x) = -B_n(x)p_n(x) + A_n(x)p_{n-1}(x), \tag{1.6.16}$$

$$B_n(x) + B_{n+1}(x) = \frac{x-b_n}{a_n}A_n(x) - \nu'(x), \tag{1.6.17}$$

$$B_{n+1}(x) - B_n(x) = \frac{a_{n+1}A_{n+1}(x)}{x-b_n} - \frac{a_n^2 A_{n-1}(x)}{a_{n-1}(x-b_n)} - \frac{1}{x-b_n} \tag{1.6.18}$$

[1]Proof, courtesy of Dr. Chuck Yeung, Professor of Physics at The Pennsylvania State, Behrend.
Begin by Normalizing Eq. (1.6.11)

$$p_{n+1} = \frac{Q_{n+1}}{||Q_{n+1}||} = \frac{||Q_n||}{||Q_{n+1}||}(x-b_n)\frac{Q_n}{||Q_n||} - \frac{||Q_{n-1}||}{||Q_{n+1}||}c_n\frac{Q_{n-1}}{||Q_{n-1}||}$$

So

$$p_{n+1} = \frac{||Q_n||}{||Q_{n+1}||}(x-b\_n)p_n - \frac{||Q_{n-1}||}{||Q_{n+1}||}c_n p_{n-1}$$

Putting in form Eq. (1.6.12)

$$\frac{||Q_{n+1}||}{||Q_n||}p_{n+1} = (x-b\_n)p_n - \frac{||Q_{n-1}||}{||Q_n||}c_n p_{n-1}$$

So comparing this to Eq. (1.6.12) gives

$$a_{n+1} = \frac{||Q_{n+1}||}{||Q_n||}$$

While

$$a_n = \frac{||Q_n||}{||Q_{n-1}||} = \frac{||Q_{n-1}||}{||Q_n||}c_n = \frac{1}{a_n}c_n$$

So

$$c_n = a_n^2 \to a_n = \sqrt{c_n}$$

where Eq. (1.6.18) is called the string equation. These equations become the foundation of Section 3.2.

1.7 Apell and Sheffer Polynomials

We now begin discussing the Apell and Sheffer classifications of polynomials sequences, which are a focus in this paper.

Definition 1.7.1. *A polynomial set* $\{P_n(x)\}_{n=0}^{\infty}$ *is said to be* ***Apell*** *if there exists*

$$A(t) = \sum_{n=0}^{\infty} a_n t^n, \quad a_0 = 1$$

such that

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{1.7.1}$$

Example: We get the well-known Taylor Expansion:

$$\mathrm{e}^{xt} = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n$$

when

$$a_0 = 1, a_n = 0 \quad \forall \quad n \geq 1$$

Thus, the polynomial set $\{\frac{x^n}{n!}\}_{n=0}^{\infty}$ is Apell.

Corollary 1.7.1.1. *An alternative definition of Apell is that*

$$P_n'(x) = P_{n-1}(x) \tag{1.7.2}$$

.

Proof. Observe that differentiating Eq. (1.7.1) reveals, from the left-hand-side:

$$\frac{\mathrm{d}}{\mathrm{d}x} A(t)\mathrm{e}^{xt} = tA(t)\mathrm{e}^{xt} = \sum_{n=0}^{\infty} P_n(x)t^{n+1} = \sum_{n=1}^{\infty} P_{n-1}(x)t^n \tag{1.7.3}$$

and, from the right-hand-side:

$$\sum_{n=1}^{\infty} P_n'(x)t^n \tag{1.7.4}$$

Thus, comparing coefficients reveals that

$$P_n'(x) = P_{n-1}(x) \tag{1.7.5}$$

. □

From here, we note that the derivative is really just an operator acting on P_n, and we can generalize to state the following theorem:

Theorem 1.7.2. *For a given polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ there exists a unique operator, called the* ***backward shift operator*** *$\mathcal{J}$ such that*

$$\mathcal{J}[P_n(x)] = P_{n-1}(x) \tag{1.7.6}$$

Definition 1.7.3. *We say that the polynomial set is of* ***Sheffer Type 0*** *if $\mathcal{J}$ has the following form:*

$$\mathcal{J}[y(x)] = \sum_{n=1}^{\infty} c_n y^{(n)}(x) \quad c_1 \neq 0 \tag{1.7.7}$$

The generating function of $\mathcal{J}$ is the power set

$$J(t) := \sum_{n=1}^{\infty} c_n t^n, \quad c_1 \neq 0 \tag{1.7.8}$$

with inverse

$$H(t) := \sum_{n=1}^{\infty} s_n t^n, \quad s_1 = c_1^{-1} \neq 0 \tag{1.7.9}$$

That is, $H(J(t)) = J(H(t)) = t$.

For each $\mathcal{J}$ there exists infinitely many associated polynomial sets. However, there exists only one set, $\{B_n(x)\}_{n=0}^{\infty}$, called the **basic set**, with the property that

$$B_0 = 1, B_n(0) = 0 \quad n = 1, 2, 3... \tag{1.7.10}$$

The set has the generating function

$$\mathrm{e}^{xH(t)} = \sum_{n=0}^{\infty} B_n(x) t^n \tag{1.7.11}$$

where $H(t)$ is as it is defined in Eq. (1.7.9). We are now beginning to see a further connection between being Apell and Type-0 - this leads to the following theorem, written exactly as stated in [2]:

Theorem 1.7.4. *The set $\{P_n(x)\}_{n=0}^{\infty}$ corresponds to the operator $\mathcal{J}$ and is of Sheffer Type 0 if and only if the sequence $\{a_n\}_{n=0}^{\infty}$ exists such that*

$$A(t)e^{xH(t)} = \sum_{n=0}^{\infty} P_n(x) t^n \tag{1.7.12}$$

where

$$A(t) := \sum_{n=0}^{\infty} a_n(x) t^n \quad a_0 = 1 \quad \textit{and} \quad H(t) := \sum_{n=1}^{\infty} s_n t^n, \quad s_1 = 1 \tag{1.7.13}$$

1.8 Orthogonal Sheffer Type-0 Polynomials

Now, determining which Sheffer Type-0 Polynomials are also orthogonal is a very laborious task, not covered here (again, see [2]). We will however show the results, which were proven by J. Meixner to be unique [5]. Here listed are the definitions (in hyper-geometric form), orthogonality, recursion relation, monic recursion relation and generating function for the six polynomials, all obtained from [6]. The orthogonality may be discrete or continuous, hence we write them out explicitly instead of using the $\mathcal{L}$ notation.

1. Laguerre (note: the Laguerre polynomials are actually a family of orthogonal polynomials, hence the α)

   (a) Definition:
   $$L_n^{(\alpha)}(x) := \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}\middle| x\right) \tag{1.8.1.1}$$

   (b) Orthogonality:
   $$\int_0^\infty L_n^{(\alpha)}(x)L_m^{\alpha}(x)\mathrm{e}^{-x}x^{\alpha}\mathrm{d}x = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{nm}, \quad \alpha > -1 \tag{1.8.1.2}$$

   (c) Recurrence Relation
   $$(n+1)L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \tag{1.8.1.3}$$

   (d) Monic Recurrence Relation
   If we set
   $$Q_n(x) = \frac{n!}{(-1)^n}L_n^{(\alpha)}(x)$$
   we get
   $$Q_{n+1}(x) = (x-2n-\alpha-1)Q_n(x) - n(n+\alpha)Q_{n-1}(x) \tag{1.8.1.4}$$

   (e) Generating Function
   $$(1-t)^{-\alpha-1}\mathrm{e}^{\frac{xt}{t-1}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n \tag{1.8.1.5}$$

2. Charlier (note the Charlier polynomials are a function of two variables, hence the a in the function arguments)

   (a) Definition:
   $$C_n(x;a) := {}_2F_0\left(\begin{matrix} -n,-x \\ - \end{matrix}\middle| -\frac{1}{a}\right) \tag{1.8.2.1}$$

   (b) Orthogonality:
   $$\sum_{x=0}^{\infty} C_n(x;a)C_m(x;a)\frac{a^x}{x!} = a^{-n}\mathrm{e}^a n!\delta_{nm}, \quad a > 0 \tag{1.8.2.2}$$

(c) Recurrence Relation

$$aC_{n+1}(x;a) = (n+a-x)C_n(x;a) - nC_{n-1}(x;a) \tag{1.8.2.3}$$

(d) Monic Recurrence Relation
If we set

$$Q_n(x) = (-a)^n C_n(x;a)$$

we get

$$Q_{n+1}(x) = (x-n-a)Q_n(x) - naQ_{n-1}(x) \tag{1.8.2.4}$$

(e) Generating Function

$$\mathrm{e}^t\left(1-\frac{t}{a}\right)^x = \sum_{n=0}^{\infty} \frac{C_n(x;a)}{n!} t^n \tag{1.8.2.5}$$

3. Hermite

(a) Definition:

$$H_n(x) := (2x)^n {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix} \middle| -\frac{1}{x^2} \right) \tag{1.8.3.1}$$

(b) Orthogonality:

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\mathrm{e}^{-x^2} = \sqrt{\pi}2^n n!\delta_{nm} \tag{1.8.3.2}$$

(c) Recurrence Relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{1.8.3.3}$$

(d) Monic Recurrence Relation
If we set

$$Q_n(x) = 2^{-n}H_n(x)$$

we get

$$Q_{n+1}(x) = xQ_n(x) - \frac{n}{2}Q_{n-1}(x) \tag{1.8.3.4}$$

(e) Generating Function

$$\mathrm{e}^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \tag{1.8.3.5}$$

4. Meixner (note: the Meixner polynomials are functions are three variables)

(a) Definition:

$$M_n(x;\beta,c) := {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix} \middle| 1-\frac{1}{c} \right) \tag{1.8.4.1}$$

(b) Orthogonality:

$$\sum_{x=0}^{\infty} M_n(x;\beta,c)M_m(x;\beta,c)c^x\frac{(\beta)_x}{x!} = \frac{c^{-n}n!}{(\beta)_n(1-c)^\beta}\delta_{nm}, \qquad \beta > 0 \quad \text{and} \quad 0 < c < 1 \tag{1.8.4.2}$$

(c) Recurrence Relation

$$c(n+\beta)M_{n+1}(x;\beta,c) = [n - x + (n+\beta+x)c]M_n(x;\beta,c) - nM_{n-1}(x;\beta,c) \tag{1.8.4.3}$$

(d) Monic Recurrence Relation
If we set

$$Q_n(x) = (\beta)_n\left(\frac{c}{c-1}\right)^n M_n(x;\beta,c)$$

we get

$$Q_{n+1}(x) = \left(x - \frac{n+(n+\beta)c}{c-1}\right)Q_n(x) - \frac{n(n+\beta-1)c}{(1-c)^2}Q_{n-1}(x) \tag{1.8.4.4}$$

(e) Generating Function

$$\left(1-\frac{t}{c}\right)^x(1-t)^{-x-\beta} = \sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}M_n(x;\beta,c)t^n \tag{1.8.4.5}$$

5. Meixner-Pollaczek (note: the Meixner-Pollaczek polynomials are a function of two variables)

(a) Definition

$$P_n^\lambda(x;\phi) := \frac{(2\lambda)_n e^{in\phi}}{n!}{}_2F_1\left(\begin{matrix}-n,\lambda+ix\\ 2\lambda\end{matrix}\middle|1-e^{-2i\phi}\right) \tag{1.8.5.1}$$

(b) Orthogonality

$$\int_{-\infty}^{\infty} P_n^\lambda(x;\phi)P_m^\lambda(x;\phi)e^{(2\phi-\pi)x}|\Gamma(\lambda+ix)|^2 dx = \frac{2\pi\Gamma(n+2\lambda)}{(2\sin(\phi))^{2\lambda}n!}\delta_{nm}, \quad \lambda > 0 \quad \text{and} \quad 0 < \phi < \pi \tag{1.8.5.2}$$

(c) Recurrence Relation

$$(n+1)P_{n+1}^\lambda(x;\phi) = 2[x\sin\phi + (n+\lambda)\cos\phi]P_n^\lambda(x;\phi) - (n+2\lambda-1)P_{n-1}(x;\phi) \tag{1.8.5.3}$$

(d) Monic Recurrence Relation
If we set

$$Q_n(x) = \frac{n!}{(2\sin\phi)^n}P_n^\lambda(x;\phi)$$

we get

$$Q_{n+1}(x) = \left(x + \frac{n+\lambda}{\tan\phi}\right)Q_n(x) - \frac{n(n+2\lambda-1)}{(2\sin\phi)^2}Q_{n-1}(x) \tag{1.8.5.4}$$

(e) Generating Function

$$\left(1-\mathrm{e}^{i\phi}t\right)^{-\lambda+ix}\left(1-\mathrm{e}^{-i\phi}t\right)^{-\lambda-ix}=\sum_{n=0}^{\infty}P_n^{\lambda}(x;\phi)t^n \tag{1.8.5.5}$$

6. Krawtchouk (note: the Krawtchouk polynomials are functions of three variables)

(a) Definition

$$K_n(x;p,N):={}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix}\middle|\frac{1}{p}\right),\quad n=0,1,2,\ldots,N \tag{1.8.6.1}$$

The Krawtchouk polynomials are self-dual, and thus obey

$$K_n(x;p,N)=K_x(n;p,N),\quad x,n\in 1,2,\ldots,N \tag{1.8.6.2}$$

(b) Orthogonality

$$\begin{aligned}\sum_{x=0}^{N}K_n(x;p,N)K_m(x;p,N)\binom{N}{x}p^x(1-p)^{N-x}\\ =\frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n\delta_{nm},\quad 0<p<1\end{aligned} \tag{1.8.6.3}$$

and, due to being self-dual:

$$\begin{aligned}\sum_{n=0}^{N}K_n(x;p,N)K_n(y;p,N)\binom{N}{n}p^n(1-p)^{N-n}\\ =\frac{\left(\frac{1-p}{p}\right)^x}{\binom{N}{x}}\delta_{xy},\quad 0<p<1 \text{ and } x,y\in 0,1,2,\ldots,N\end{aligned} \tag{1.8.6.4}$$

(c) Recurrence Relation

$$\begin{aligned}p(N-n)K_{n+1}(x;p,N)=[p(N-n)+n(1-p)-x]K_n(x;p,N)\\ -n(1-p)K_{n-1}(x;p,N)\end{aligned} \tag{1.8.6.5}$$

(d) Monic Recurrence Relation

If we set

$$Q_n(x)=(-N)_n p^n K_n(x;p,N)$$

we get

$$\begin{aligned}Q_{n+1}(x)=-[p(N-n)+n(1-p)-x]Q_n(x)\\ -np(1-p)(N+1-n)Q_{n-1}(x)\end{aligned} \tag{1.8.6.6}$$

(e) Generating Function

$$\left(1-\frac{1-p}{p}t\right)^x(1+t)^{N-x}=\sum_{n=0}^{N}\binom{N}{n}K_n(x;p,N)t^n \tag{1.8.6.7}$$

1.9 Jacobi Polynomials

Section 2.1 details properties concerning the Jacobi polynomials. Hence, we cover them here, following [7].

1. The Jacobi polynomials have an intimate relationship with the beta function, as it is through the beta function the moment-functional associated with the Jacobi polynomial is often evaluated. Thus, we define it here

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad \operatorname{Re} x > 0,\ \operatorname{Re} y > 0 \\ &\text{with the property} \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \end{aligned} \tag{1.9.1.1}$$

2. Jacobi polynomials have the weight function

$$w(x;\alpha,\beta) := (1-x)^{\alpha}(1+x)^{\beta} \tag{1.9.2.1}$$

and thus we have

$$\mu_0 = \mathcal{L}[x^0] = \int_{-1}^{1} w(x;\alpha,\beta)\mathrm{d}x = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \tag{1.9.2.2}$$

which was evaluated using Eq. (1.9.1.1).

Now, we can list the definition of the Jacobi polynomials, followed by special cases.

3. The Jacobi polynomials

   (a) Definition

$$P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!}{}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-x}{2}\right) \tag{1.9.3.1}$$

   and they have the property that, if $x \to -x$, then $\alpha \to \beta$. That is,

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= (-1)^n P_n^{(\beta,\alpha)}(-x) \\ &= \frac{(-1)^n(\beta+1)_n}{n!}{}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix} \middle| \frac{1+x}{2}\right) \end{aligned} \tag{1.9.3.2}$$

   (b) Orthogonality

   First, we define

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\alpha+\beta+n+1)(\alpha+\beta+2n+1)} \quad \alpha,\beta > -1 \tag{1.9.3.3}$$

   So, using $h_n^{(\alpha,\beta)}$ as defined in Eq. (1.9.3.3) and $w(x;\alpha,\beta)$ as defined in Eq. (1.9.2.1), we have

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)w(x;\alpha,\beta)\mathrm{d}x = h_n^{(\alpha,\beta)}\delta_{nm}, \tag{1.9.3.4}$$

(c) Recursion Relation

$$\begin{aligned}\frac{2(n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2(n+1))}P_{n+1}^{(\alpha,\beta)}(x)\\ = \left[x+\frac{\alpha^2-\beta^2}{(\alpha+\beta+2n)(\alpha+\beta+2(n+1))}\right]P_n^{(\alpha,\beta)}(x)\\ -\frac{2(n+\alpha)(n+\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}P_{n-1}^{(\alpha,\beta)}(x)\end{aligned} \tag{1.9.3.5}$$

(d) Monic Recursion Relation

If we set

$$Q_n(x) = \frac{2^n n!}{(n+\alpha+\beta+1)_n}P_n^{(\alpha,\beta)}(x) \tag{1.9.3.6}$$

we get

$$\begin{aligned}Q_{n+1}(x) =&\left[x+\frac{\alpha^2-\beta^2}{(\alpha+\beta+2n)(\alpha+\beta+2(n+1))}\right]Q_n(x)\\ &-\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}Q_{n-1}(x)\end{aligned} \tag{1.9.3.7}$$

(e) Generating Function

If we set

$$R = \sqrt{1-2xt+t^2}$$

we get

$$\frac{2^{\alpha+\beta}}{R(1+R-t)^\alpha(1+R+t)^\beta} = \sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n \tag{1.9.3.8}$$

4. Ultraspherical or Gegenbauer Polynomials. These polynomials are a scaled Jacobi polynomial, with $\alpha = \beta = \nu - 1/2$

(a) Definition

$$\begin{aligned}C_n^\nu(x) :=& \frac{(2\nu)_n}{(\nu+1/2)_n}P_n^{(\nu-1/2,\nu-1/2)}(x)\\ =& \frac{(2\nu)_n}{n!}{}_2F_1\left(\begin{matrix}-n,n+2\nu\\ \nu+1/2\end{matrix}\middle|\frac{1-x}{2}\right)\end{aligned} \tag{1.9.4.1}$$

(b) Orthogonality

$$\int_{-1}^{1}C_n^{(\nu)}(x)C_m^{(\nu)}(x)(1-x^2)^{\nu-1/2}\mathrm{d}x = \frac{(2\nu)_n\sqrt{\pi}\Gamma(\nu+1/2)}{n!(n+\nu)\Gamma(\nu)}\delta_{nm} \tag{1.9.4.2}$$

(c) Recursion Relation

$$(n+1)C_{n+1}^{(\nu)}(x) = 2(n+\nu)xC_n^{(\nu)}(x) - (n+2\nu-1)C_{n-1}^{(\nu)}(x) \tag{1.9.4.3}$$

(d) Monic Recursion Relation

If we set

$$Q_n(x) = \frac{n!}{2^n(\nu)_n} C_n^{(\nu)}(x) \tag{1.9.4.4}$$

we get

$$Q_{n+1}(x) = xQ_n(x) - \frac{n(n+2\nu-1)}{4(n+\nu-1)(n+\nu)} Q_{n-1}(x) \tag{1.9.4.5}$$

(e) Generating Function

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{(\nu)}(x) t^n \tag{1.9.4.6}$$

5. Chebyshev. These can be derived from both the Jacobi and the Ultraspherical polynomials. There are two types: the First Kind, denoted $T_n(x)$, and the Second Kind, denoted $U_n(x)$.

(a) Definition of the First Kind

i. Ultraspherical

$$T_n(x) = \lim_{\nu\to 0} \frac{n+2\nu}{2\nu} C_n^{(\nu)}(x) \tag{1.9.5.1}$$

ii. Jacobi

$$T_n(x) := \frac{P_n^{(-1/2,-1/2)}(x)}{P_n^{(-1/2,-1/2)}(1)} = {}_2F_1\left(\begin{matrix} -n,n \\ 1/2 \end{matrix} \middle| \frac{1-x}{2} \right) \tag{1.9.5.2}$$

iii. Trigonometric

$$T_n(\cos\theta) = \cos(n\theta) \tag{1.9.5.3}$$

(b) Definition of the Second Kind

i. Ultraspherical

$$U_n(x) = C_n^{(1)}(x) \tag{1.9.5.4}$$

ii. Jacobi

$$U_n(x) := \frac{P_n^{(1/2,1/2)}(x)}{P_n^{(1/2,1/2)}(1)} = (n+1){}_2F_1\left(\begin{matrix} -n,n+2 \\ 3/2 \end{matrix} \middle| \frac{1-x}{2} \right) \tag{1.9.5.5}$$

iii. Trigonometric

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{1.9.5.6}$$

(c) Orthogonality

For the First Kind:

$$\int_{-1}^{1} T_n(x)T_m(x)(1-x^2)^{-1/2}\mathrm{d}x = \begin{cases} \frac{\pi}{2}\delta_{nm}, & \text{if } n \neq 0 \\ \pi\delta_{nm}, & \text{if } n = 0 \end{cases} \tag{1.9.5.7}$$

For the Second Kind:

$$\int_{-1}^{1} U_n(x)U_m(x)(1-x^2)^{1/2}\mathrm{d}x = \frac{\pi}{2}\delta_{nm} \tag{1.9.5.8}$$

(d) Recursion Relations

$$\begin{aligned} &T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \\ &n \geq 1,\ T_{-1}(x) = 0,\ T_0(x) = 1,\ T_1(x) = x \end{aligned} \tag{1.9.5.9}$$

$$\begin{aligned} &U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \\ &n \geq 0,\ U_{-1}(x) = 0,\ U_0(x) = 1 \end{aligned} \tag{1.9.5.10}$$

(e) Monic Recurrence Relations
If we set

$$\begin{aligned} t_n(x) &= 2^{-n}T_n(x) \\ u_n(x) &= 2^{-n}U_n(x) \end{aligned} \tag{1.9.5.11}$$

we have

$$\begin{aligned} &t_{n+1}(x) = xt_n(x) - \frac{1}{4}t_{n-1}(x) \\ &n \geq 1,\ t_{-1}(x) = 0,\ t_0(x) = 1,\ t_1(x) = \frac{x}{2} \end{aligned} \tag{1.9.5.12}$$

$$\begin{aligned} &u_{n+1}(x) = xu_n(x) - \frac{1}{4}u_{n-1}(x) \\ &n \geq 0,\ u_{-1}(x) = 0,\ u_0(x) = 1 \end{aligned} \tag{1.9.5.13}$$

(f) Generating Functions

$$\begin{aligned} \frac{1 - xt}{1 - 2xt + t^2} &= \sum_{n=0}^{\infty} T_n(x)t^n \\ \frac{1}{1 - 2xt + t^2} &= \sum_{n=0}^{\infty} U_n(x)t^n \end{aligned} \tag{1.9.5.14}$$

6. Legendre, or Spherical Polynomials: Jacobi polynomials with $\alpha = \beta = 0$.

(a) Definition

$$P_n(x) := P_n^{(0,0)}(x) = {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix} \middle| \frac{1-x}{2} \right) \tag{1.9.6.1}$$

(b) Orthogonality

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = \frac{2}{2n+1}\delta_{nm} \tag{1.9.6.2}$$

Notice that the weight function is $w(x) = 1$.

(c) Recurrence Relation

$$(n+1)P_{n+1}(x) = x(2n+1)P_n(x) - nP_{n-1}(x) \tag{1.9.6.3}$$

(d) Monic Recurrence Relation

If we set

$$Q_n(x) = \frac{2^n}{\binom{2n}{n}} P_n(x) \tag{1.9.6.4}$$

we have

$$Q_{n+1}(x) = xQ_n(x) - \frac{n^2}{(2n-1)(2n+1)} Q_{n-1}(x) \tag{1.9.6.5}$$

(e) Generating Function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \tag{1.9.6.6}$$

We now begin looking into properties, methods and applications of orthogonal polynomial sequences.

Chapter 2

Theorems

2.1 Theory

This section will lay some foundational groundwork for various orthogonal polynomials and then lead into proofs, Section 2.2. Before we begin, we must also list important properties of the pochhammer symbol Eq. (1.6.2) that were used throughout the proofs.

The most straightforward yet useful is the following:

$$(1)_n = n! \tag{2.1.1}$$

Written as a combination, we have that

$$\frac{(n)_k}{k!} = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!} \tag{2.1.2}$$

The sign of the argument of the pochhammer symbol can be altered through the following relation, where the second line is obtained via Eq. (2.1.2)

$$\begin{aligned}(-n)_k &= (-1)^k(n-k+1)_k \\ &= \frac{(-1)^k n!}{(n-k)!}\end{aligned} \tag{2.1.3}$$

The pochhammer symbol has a strong connection to the Gamma function, through the following relation:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{2.1.4}$$

This then leads to the duplication formula, in both pochhammer and Gamma function form:

$$\begin{aligned}\Gamma(2z) &= \frac{2^{2z-1}\Gamma(z)\Gamma(z+1/2)}{\sqrt{\pi}} \\ (2a)_{2n} &= 2^{2n}(a)_n(a+1/2)_n\end{aligned} \tag{2.1.5}$$

Combining Eq. (2.1.1) and Eq. (2.1.5) reveals that

$$(2n)! = (1)_{2n} = 2^{2n}(1/2)_n n! \tag{2.1.6}$$

For the ratio of two pochhammer symbols with the same argument but different index, we have

$$\frac{(a)_n}{(a)_m} = \begin{cases}(a+m)_{n-m} & n > m \\ \frac{1}{(a+n)_{m-n}} & m < n\end{cases} \tag{2.1.7}$$

Finally, we have the Chu-Vandermande Sum

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-n)_k(b)_k}{(c)_k}\frac{1}{k!} &= \sum_{k=0}^{n} \frac{(-n)_k(b)_k}{(c)_k}\frac{1}{k!} \\ &= \frac{(c-b)_n}{(c)_n}\end{aligned} \tag{2.1.8}$$

And now we are ready to start covering rudimentary orthogonal polynomial theory. As a reminder, $L_n^{(\alpha)}(x)$ refer to the Laguerre polynomials, $H_n(x)$ refer to the Hermite polynomials, $P_n(x)$ refer to the Legendre polynomials and $C^\lambda(x)$ refer to the ultraspherical polynomials. Please feel free to review Section 1.8 to obtain a refresher of their properties, but we will list the pertinent properties for each proof as we need them. We shall start by focusing on the Legendre Polynomials as a case study of the development of the theory behind orthogonal polynomials.

2.1.1 The Legendre Polynomials

In this section, we will focus on the common development of a theory behind an orthogonal polynomial. Usually, some equation, called the Rodrigues formula, is derived for a polynomial. From the formula, a hypergeometric series representation is discovered, and the orthogonality of the polynomial is proven over some interval, called the support interval. Finally, it is then shown that the orthogonal polynomial forms a basis for measurable functions on the support interval. We will use the Legendre polynomials as our case study, and shall start with deriving the hypergeometric series representation.

We start by defining the Legendre polynomials as

$$P_n(x) = \frac{(-1)^n}{n!2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x^2)^n\right] = \frac{1}{n!2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(x^2-1)^n\right] \tag{2.1.1.1}$$

Many use this equation, the Rodrigues formula for the Legenred polynomials, as the definition of the Legendre polynomials, and derive all other properties from this definition. That is, an orthogonal polynomial's Rodrigues formula is historically the starting point in the development of an orthogonal polynomial. Now, we shall use the binomial theorem, Eq. (1.6.1), on Eq. (2.1.1.1) to get the following series relation.

$$P_n(x) = \frac{1}{n!2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\sum_{k=0}^{n}\binom{n}{k}x^{2n-2k}(-1)^k\right] \tag{2.1.1.2}$$

Now, observe that the derivative will be nonzero when $2n-2k \geq n$, which means $\lfloor n/2 \rfloor \geq k$. And, for $n \neq m$, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^m = \frac{m!}{(m-n)!}x^{m-n}$. So now we get

$$P_n(x) = \frac{1}{n!2^n}\sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n}{k}\frac{(2n-2k)!}{(n-2k)!}x^{n-2k}(-1)^k = \sum_{k=0}^{\lfloor n/2 \rfloor}\frac{(2n-2k)!}{(n-2k)!2^n}\frac{x^{n-2k}(-1)^k}{k!(n-k)!} \tag{2.1.1.3}$$

Now, the duplication formula, Eq. (2.1.5), tells us that

$$\Gamma(2(n-k)) = \frac{2^{2n-2k-1}\Gamma(n-k)\Gamma(n-k+1/2)}{\Gamma(1/2)}$$

or, by multiplying through by $(2n-2k)$

$$(2n-2k)! = \frac{2^{2n-2k}(n-k)!\Gamma(n-k+1/2)}{\Gamma(1/2)} = 2^{2n-2k}(n-k)!(1/2)_{n-k}$$

where the Gamma relation for the pochhammer symbols, Eq. (2.1.4), was used in the final step. Inserting, we find that

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(2x)^{n-2k}(-1)^k(1/2)_{n-k}}{k!(n-2k)!} \tag{2.1.1.4}$$

This is a valid series representation of the Legendre polynomials. However, it is not the standard often used. In Section 2.1.2, we will discuss that the Legendre polynomials are defined in terms of the Jacobi-polynomials $P_n^{(\alpha,\beta)}(x)$, defined by Eq. (1.9.3.1), where $\alpha = \beta = 0$. In Problem 2 of Section 2.2, we also show that Eq. (2.1.1.4) is equivalent to $C_n^{(1/2)}(x)$ where $C_n^{(\lambda)}(x)$ are the Ultraspherical polynomials defined in Eq. (2.1.2.1). So we thus have

$$P_n(x) = P^{(0,0)}(x) = C_n^{(1/2)}(x).$$

Which means we have

$$P_n(x) = C_n^{(1/2)}(x) = {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix} \middle| \frac{1-x}{2} \right) \tag{2.1.1.5}$$

We next set out to prove that the Legendre polynomials are orthogonal over the support interval $\Omega = [-1, 1]$. For f, g polynomials continuous on $[-1, 1]$, define the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x \tag{2.1.1.6}$$

Thus, in in terms of our inner-product, we have that if $\{P_n(x)\}_n^\infty$ denotes the Legendre orthogonal polynomial sequence and $n \neq m$, then $\langle P_n(x), P_m(x) \rangle = 0$.

Proof. Observe that the Legendre polynomials have the form $P_m(x) = \sum_{j=0}^m a_j x^j$ which means that using the inner-product defined in Eq. (2.1.1.6) yields $\langle P_n(x), P_m(x) \rangle = \sum_{j=0}^m a_j \langle x^j, P_n(x) \rangle$. So, let's focus on $\langle x^j, P_n(x) \rangle$. We shall use the Rodrigues formula for the Legendre polynomials. Using the Legendre polynomial and x^j within the inner-product yields the integral

$$\langle x^j, P_n(x) \rangle = \frac{1}{2^n n!} \int_{-1}^{1} x^j \frac{\mathrm{d}^n (x^2-1)^n}{\mathrm{d}x^n} \mathrm{d}x \tag{2.1.1.7}$$

Now, do integration by parts of $u = x^j$ to get

$$\langle x^j, P_n(x) \rangle = \frac{1}{2^n n!} \left(x^j \frac{\mathrm{d}^{n-1}(x^2-1)^n}{\mathrm{d}x^{n-1}} \bigg|_{-1}^{1} - j \int_{-1}^{1} x^{j-1} \frac{\mathrm{d}^{n-1}(x^2-1)^n}{\mathrm{d}x^{n-1}} \mathrm{d}x \right)$$

Letting $u = x^{j-1}$ and doing another integration by parts yields

$$\langle x^j, P_n(x) \rangle = \frac{1}{2^n n!} \left(x^j \frac{\mathrm{d}^{n-1}(x^2-1)^n}{\mathrm{d}x^{n-1}} \bigg|_{-1}^{1} - j x^{j-1} \frac{\mathrm{d}^{n-2}(x^2-1)^n}{\mathrm{d}x^{n-2}} \bigg|_{-1}^{1} + j(j-1) \int_{-1}^{1} x^{j-2} \frac{\mathrm{d}^{n-2}(x^2-1)^n}{\mathrm{d}x^{n-2}} \mathrm{d}x \right)$$

Iterating integration by parts until the coefficient of the leading x is 1 inductively yields the following relation:

$$\langle x^j, P_n(x) \rangle = \frac{1}{2^n n!} \left(\sum_{k=0}^{j} \frac{j!}{(j-k)!} x^{j-k} \frac{\mathrm{d}^{n-k-1}(x^2-1)^n}{\mathrm{d}x^{n-k-1}} \right) \bigg|_{-1}^{1} \tag{2.1.1.8}$$

Let's focus on $\frac{\mathrm{d}^{n-k-1}(x^2-1)^n}{\mathrm{d}x^{n-k-1}}$. Since the order of the derivative is less than $n \ \forall \ k$, every term will contain (x^2-1). Observe that $((1)^2-1) = ((-1)^2-1) = 0$. Thus, $\frac{\mathrm{d}^{n-k-1}(x^2-1)^n}{\mathrm{d}x^{n-k-1}}$ goes to zero when evaluated, and we get

$$\langle P_m(x), P_n(x)\rangle = \langle x^j, P_n(x)\rangle = 0, \quad j \leq m < n \tag{2.1.1.9}$$

□

Now we must cover the case when $n = m$. We first introduce the following lemma, which will be useful later in proving our claim.

Lemma 2.1.1. *Let $u_n(x) = (x^2-1)^n$. Then,*

$$\frac{\mathrm{d}^{2n}u_n(x)}{\mathrm{d}x^{2n}} = (2n)! \tag{2.1.1.10}$$

Proof. From the binomial theorem, we have

$$u_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{2n-2k}(-1)^k \tag{2.1.1.11}$$

Now, after differentiating $2n$ times, only the x^{2n} term remains. Observe that

$$\frac{\mathrm{d}^{2n}u_n(x)}{\mathrm{d}x^{2n}} = \frac{\mathrm{d}^{2n}x^{2n}}{\mathrm{d}x^{2n}} = (2n)!$$

and we are done. □

Theorem 2.1.2. *Now, we claim that, using the inner-product defined in Eq.* (2.1.1.6)*,*

$$\langle P_n(x), P_n(x)\rangle = ||P_n(x)||^2 = \frac{2}{2n+1} \tag{2.1.1.12}$$

Proof. Observe that using the inner-product defined in Eq. (2.1.1.6) on $P_n(x)$ yields

$$\langle P_n(x), P_n(x)\rangle = \int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} \frac{\mathrm{d}^n(x^2-1)^n}{\mathrm{d}x^n}\frac{\mathrm{d}^n(x^2-1)^n}{\mathrm{d}x^n}\mathrm{d}x \tag{2.1.1.13}$$

Let $u = \frac{\mathrm{d}^n(x^2-1)^n}{\mathrm{d}x^n}$ and do integration by parts to get

$$\langle P_n(x), P_n(x)\rangle = -\frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} \frac{\mathrm{d}^{n-1}(x^2-1)^n}{\mathrm{d}x^{n-1}}\frac{\mathrm{d}^{n+1}(x^2-1)^n}{\mathrm{d}x^{n+1}}\mathrm{d}x \tag{2.1.1.14}$$

where the term that needed evaluated was zero at 1 and -1 due to the (x^2-1) term. Now, do integration by parts again to get

$$\langle P_n(x), P_n(x)\rangle = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} \frac{\mathrm{d}^{n-2}(x^2-1)^n}{\mathrm{d}x^{n-2}}\frac{\mathrm{d}^{n+2}(x^2-1)^n}{\mathrm{d}x^{n+2}}\mathrm{d}x \tag{2.1.1.15}$$

Doing integration by parts n times yields

$$\langle P_n(x), P_n(x)\rangle = \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\frac{\mathrm{d}^{2n}(x^2-1)^n}{\mathrm{d}x^{2n}}\mathrm{d}x \tag{2.1.1.16}$$

However, we can use the result of Lemma 2.1.1 to get

$$\langle P_n(x), P_n(x)\rangle = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\mathrm{d}x = \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n\mathrm{d}x \tag{2.1.1.17}$$

Using Eq. (2.1.5) tells us that $(2n)! = (1/2)_n 2^{2n} n!$ and we now have

$$\langle P_n(x), P_n(x)\rangle = \frac{(1/2)_n}{n!}\int_{-1}^{1}(1-x^2)^n\mathrm{d}x$$

Now observe that we can take advantage of the fact that $(1-x^2)^n$ is an even function. Then, we can do a variable substitution of $t = 1-x^2$ to get

$$\langle P_n(x), P_n(x)\rangle = \frac{2(1/2)_n}{n!}\int_{0}^{1}(1-x^2)^n\mathrm{d}x = \frac{(1/2)_n}{n!}\int_{0}^{1}t^n(1-t)^{-1/2}\mathrm{d}x$$

Recall the Beta function, $B(x, y)$, defined in Eq. (1.9.1.1). Thus, our integral is simply $B(n+1, 1/2) = \Gamma(n+1)\Gamma(1/2)/\Gamma(n+3/2)$, which means we have

$$\begin{aligned}||P_n(x)||^2 = \langle P_n(x), P_n(x)\rangle &= \frac{(1/2)_n}{n!}\frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma(n+3/2)}\\ &= \frac{(1/2)_n\Gamma(1/2)}{\Gamma(n+3/2)} = \frac{\Gamma(n+1/2)}{\Gamma(n+3/2)} = \frac{1}{n+1/2} = \frac{2}{2n+1}\end{aligned}$$

Where the pochhammer ratio relation, Eq. (2.1.7), was used on $(1/2)_n$. □

Finally, we claimed in the introduction that an orthogonal polynomial sequence forms a basis on the vector space of polynomials over the support interval Ω of the orthogonal polynomial. In fact, there is a theorem known as the Stone-Weierstrass theorem, see Theorem 6.24 in [3], that states that any function continuous on a closed and compact set is identical to some polynomial continuous on that set. That is, polynomials are dense on the set of functions continuous on Ω. Thus, on the vector space of measurable (finite Lebesgue integrable) functions, denoted $\mathbf{L}^2\Omega$, an orthogonal polynomial will form a basis. We wish to show that the set of orthogonal Legendre polynomials forms an orthogonal basis on $\mathbf{L}^2[-1, 1]$. Which is equivalent to showing that the set of the normalized Legendre orthogonal polynomials forms an orthonormal basis in $\mathbf{L}^2[-1, 1]$. This means we must show that no other polynomials, except for the polynomial identically zero, in $\mathbf{L}^2[-1, 1]$ is orthogonal to the set of normalized Legendre orthogonal polynomials.

Theorem 2.1.3. *Using the norm induced by the inner-product defined in Eq.* (2.1.1.6)*, let* $e_n = P_n(x)/||P_n(x)||$*. Then,* $\{e_n\}_{n=0}^{\infty}$ *forms an orthonormal basis on* $\mathbf{L}^2[-1, 1]$*. That is, for* $f(x) \in \mathbf{L}^2[-1, 1]$*,*

$$f(x) = \sum_n \langle f(x), e_n\rangle\, e_n$$

If $f(x)$ *polynomial of degree m, we get*

$$f(x) = \sum_{n=0}^{m} \langle f(x), e_n\rangle\, e_n$$

Proof. Assume $f(x)$ polynomial orthogonal to $\{P_n(x)\}_{n=0}^{\infty}$ such that $f(x)$ is not identically zero, and denote $m = \deg f(x)$. Now, $f(x)$ must have the form

$$f(x) = \sum_{j=0}^{m} a_j x^j$$

or

$$x^m = \frac{f(x)}{a_m} - \sum_{j=0}^{m-1} \frac{a_j}{a_m} x^j$$

Using the the inner-product defined in Eq. (2.1.1.6) and the norm induced by it, set $e_n = P_n(x)/||P_n(x)||$. Now, observe that $\langle e_n, e_m \rangle = \frac{\langle P_m(x), P_n(x) \rangle}{||P_n(x)||^2} = \delta_{nm}$. So $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal set. If b_m denotes the leading coefficient of e_m, we get that

$$e_m = b_m x^m + O(x^{m-1})$$

where $O(x^{m-1})$ means a polynomial of degree at most $m-1$. Inserting x^m in terms of $f(x)$ into e_m yields

$$e_m = \frac{b_m f(x)}{a_m} + O(x^{m-1})$$

or

$$f(x) = \frac{a_m e_n}{b_m} + O(x^{m-1})$$

Taking the inner-product reveals

$$\langle f(x), e_m \rangle = \langle \frac{a_m e_m}{b_m} + O(x^{m-1}), e_m \rangle = \frac{a_m}{b_m} \langle e_m, e_m \rangle = \frac{a_m}{b_m} \neq 0 \tag{2.1.1.18}$$

But this contradicts $f(x)$ being orthogonal to $\{P_n(x)\}_{n=0}^{\infty}$. Which means that $f(x)$ is the function identically zero, and thus $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis, or $\{P_n(x)\}_{n=0}^{\infty}$ forms an orthogonal basis, on the vector space of polynomials on $[-1, 1]$. Now, using the Stone-Weierstrass Theorem, we know that for $f(x) \in \mathbf{L}^2[-1, 1]$ there exists polynomial continuous on $[-1, 1]$ that is equivalent to $f(x)$. Thus, using the previous part of the proof, $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis ($\{P_n(x)\}_{n=0}^{\infty}$ forms an orthogonal basis), on $\mathbf{L}^2[-1, 1]$. □

Next, we turn to the ultraspherical polynomials previously mentioned, and show that both the Ultraspherical and Legendre polynomials are two specific types of Jacobi polynomials.

2.1.2 The Ultraspherical Polynomials

Another set of well-studied Jacobi polynomials are the Ultraspherical polynomials, defined as

$$\begin{aligned} C_n^{\lambda}(x) &:= \frac{(2\lambda)_n}{(\lambda + 1/2)_n} P_n^{(\lambda-1/2, \lambda-1/2)}(x) \\ &= \frac{(2\lambda)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+2\lambda \\ \lambda+1/2 \end{matrix} \middle| \frac{1-x}{2} \right) \end{aligned} \tag{2.1.2.1}$$

We will first attempt to find the Rodrigues formula for the Ultraspherical polynomials. Just like the hypergeometric equation and the three-term recurrence relation, the Rodrigues formula for a given orthogonal polynomial sequence is an equivalent way to define them. Physicists often refer to these formulas when using orthogonal polynomial sequences. So, we will turn to the Jacobi polynomials and use its properties for this special case. Recall, if we define

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\alpha+\beta+n+1)(\alpha+\beta+2n+1)} \quad \alpha,\beta > -1 \tag{2.1.2.2}$$

and

$$w(x;\alpha,\beta) := (1-x)^\alpha(1+x)^\beta \tag{2.1.2.3}$$

we have the orthogonality of the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)w(x;\alpha,\beta)\mathrm{d}x = h_n^{(\alpha,\beta)}\delta_{nm} \tag{2.1.2.4}$$

We will also need the adjoint operator

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

This can be obtained using the recurrence relation and hypergeometric form, found in Section 1.9. We will not prove that here and just accept the operator to be true. If we substitute the adjoint operator into the orthogonality and also use the change of indices $\alpha \to \alpha+1, \beta \to \beta+1$, we have

$$\begin{aligned}\int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}\left[P_{n+1}^{(\alpha,\beta)}(x)\right] P_m^{(\alpha+1,\beta+1)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\mathrm{d}x \\ = \frac{n+\alpha+\beta+2}{2}h_n^{(\alpha+1,\beta+1)}\delta_{nm}\end{aligned} \tag{2.1.2.5}$$

Next, if we do integration by parts, with

$$\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}x}\left[P_{n+1}^{(\alpha,\beta)}(x)\right]\mathrm{d}x, u = P_m^{(\alpha+1,\beta+1)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}$$

we have

$$\begin{aligned}\frac{n+\alpha+\beta+2}{2}h_n^{(\alpha+1,\beta+1)}\delta_{nm} = \\ -\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left[P_m^{(\alpha+1,\beta+1)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\right]P_{n+1}^{(\alpha,\beta)}(x)\mathrm{d}x\end{aligned} \tag{2.1.2.6}$$

where the term that needed to be evaluated goes to zero at -1 and 1. Now, let's look at the case when the orthogonality is non-zero (i.e. $n = m$). Let's re-arrange and multiple both sides by $h_{n+1}^{(\alpha,\beta)}$. We have, then,

$$\begin{aligned}h_{n+1}^{(\alpha,\beta)} = -\frac{2h_{n+1}^{(\alpha,\beta)}}{h_n^{(\alpha+1,\beta+1)}(n+\alpha+\beta+2)} \\ \times\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left[P_n^{(\alpha+1,\beta+1)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\right]P_{n+1}^{(\alpha,\beta)}(x)\mathrm{d}x\end{aligned} \tag{2.1.2.7}$$

But, from the orthogonality of the Jacobi polynomials, we know that

$$h_{n+1}^{(\alpha,\beta)} = \int_{-1}^{1} \left(P_{n+1}^{(\alpha,\beta)}(x)\right)^2 (1-x)^\alpha (1+x)^\beta \mathrm{d}x \tag{2.1.2.8}$$

which means, since the orthogonality is unique, that

$$\begin{aligned} -\frac{2h_{n+1}^{(\alpha,\beta)}}{h_n^{(\alpha+1,\beta+1)}(n+\alpha+\beta+2)} \frac{\mathrm{d}}{\mathrm{d}x}\left[P_n^{(\alpha+1,\beta+1)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\right] \\ = P_{n+1}^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \end{aligned} \tag{2.1.2.9}$$

Now, let $n \to n-1$ and re-arrange to get

$$\begin{aligned} -P_n^{(\alpha,\beta)}(x)\frac{h_{n-1}^{(\alpha+1,\beta+1)}(n+\alpha+\beta+1)}{2h_n^{(\alpha,\beta)}} \\ = \frac{1}{(1+x)^\beta(1-x)^\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\left[P_{n-1}^{(\alpha+1,\beta+1)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\right] \end{aligned} \tag{2.1.2.10}$$

Observe that

$$\begin{aligned} \frac{n+\alpha+\beta+1}{2}\frac{h_{n-1}^{(\alpha+1,\beta+1)}}{h_n^{(\alpha,\beta)}} = & \frac{n+\alpha+\beta+1}{2}\frac{2^{\alpha+\beta+3}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(n-1)!\Gamma(\alpha+\beta+n+2)(\alpha+\beta+2n+1)} \\ & \times \frac{n!\Gamma(\alpha+\beta+n+1)(\alpha+\beta+2n+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} = 2n \end{aligned}$$

Which now gives us the following relation between the nth and the $(n-1)$th Jacobi polynomial.

$$-2nP_n^{(\alpha,\beta)}(x) = \frac{1}{(1+x)^\beta(1-x)^\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\left[P_{n-1}^{(\alpha+1,\beta+1)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\right] \tag{2.1.2.11}$$

Iterating (which is equivalent to computing successive integration-by-parts terms in our process) gives us the following relation

$$\begin{aligned} (-1)^k 2^k \frac{n!}{(n-k)!}P_n^{(\alpha,\beta)}(x) = \\ \frac{1}{(1+x)^\beta(1-x)^\alpha}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[P_{n-k}^{(\alpha+k,\beta+k)}(x)(1-x)^{\alpha+k}(1+x)^{\beta+k}\right] \end{aligned} \tag{2.1.2.12}$$

Letting $n = k$ gives us the Rodrigues formula for the Jacobi polynomials.

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!2^n}\frac{1}{(1+x)^\beta(1-x)^\alpha}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] \tag{2.1.2.13}$$

As an example, recall that we define the Legendre polynomials as $P_n(x) = P_n^{(0,0)}(x)$. The Rodrigues formula for the Legendre polynomials is thus

$$P_n(x) = \frac{(-1)^n}{n!2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x^2)^n\right] = \frac{1}{n!2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(x^2-1)^n\right] \tag{2.1.2.14}$$

and we recover the Legendre polynomials as originally define in Section 2.1.1.

Returning to Eq. (2.1.2.12), let $\alpha = \beta = \lambda - 1/2$. That is, we define $C^{(\lambda)}(x) = P^{(\lambda-1/2,\lambda-1/2)}(x)$. So, we also see that, as remarked in Section 2.1.1, $P_n(x) = C^{(1/2)}(x)$. Now, re-arranging Eq. (2.1.2.12) and then multiplying through by the necessary coefficients for the Ultraspherical polynomials, as determined by Eq. (3.1.4.1), shows that

$$C_n^{(\lambda)}(x) = \frac{(\lambda + k + 1/2)_{n-k}(2\lambda)_n}{(2\lambda + 2k)_{n-k}(\lambda + 1/2)_n} \frac{(-1)^k}{(1 - x^2)^{\lambda-1/2}} \frac{(n-k)!}{n!2^k} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left[C_{n-k}^{(\lambda)}(x)(1 - x^2)^{\lambda+k-1/2} \right]$$

Using the ratio and duplication formulas for the pochhammer symbols, the eventual result is

$$(1 - x^2)^{\lambda-1/2} C_n^{(\lambda)}(x) = \frac{(-2)^k(\lambda)_k(n-k)!}{n!(2\lambda + n)_k} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left[C_{n-k}^{(\lambda)}(x)(1 - x^2)^{\lambda+k-1/2} \right] \tag{2.1.2.15}$$

Again letting $n = k$ yields the Rodrigues formula

$$\begin{aligned}
(1 - x^2)^{\lambda-1/2} C_n^{(\lambda)}(x) &= \frac{(-2)^n(\lambda)_n}{n!(2\lambda + n)_n} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(1 - x^2)^{\lambda+n-1/2} \right] \\
&= \frac{(-1)^n(2\lambda)_{2n}}{2^n n!(2\lambda + n)_n(\lambda + 1/2)_n} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(1 - x^2)^{\lambda+n-1/2} \right] \\
&= \frac{(-1)^n(2\lambda)_n}{2^n n!(\lambda + 1/2)_n} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(1 - x^2)^{\lambda+n-1/2} \right]
\end{aligned} \tag{2.1.2.16}$$

where the duplication formula and then the ratio formula for pochhammer symbols were used.

We will explore an alternate derivation for the series representation of the Ultraspherical polynomials, from a rudimentary use of the generating function and the binomial theorem, in the proofs section. Next, we will cover the connection between two other Jacobi polynomials: the Hermite and Laguerre orthogonal polynomials.

2.1.3 Deriving the Hermite Polynomials

The Hermite Polynomials can be derived from the Laguerre polynomials. First, recall the definition and orthogonality of the Laguerre polynomials.

$$L_n^{(\alpha)}(y) := \frac{(\alpha + 1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| y \right) \tag{2.1.3.1}$$

$$\int_0^\infty L_n^{(\alpha)}(y) L_m^{\alpha}(y) \mathrm{e}^{-y} x^\alpha \mathrm{d}y = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{nm}, \quad \alpha > -1 \tag{2.1.3.2}$$

Now, let $y = x^2, \mathrm{d}y = 2x\mathrm{d}x$. We get

$$\begin{aligned}
&2 \int_0^\infty L_n^{(\alpha)}(x^2) L_m^{\alpha}(x^2) \mathrm{e}^{-x^2} x^{2\alpha+1} \mathrm{d}x \\
&\qquad = \int_{-\infty}^\infty L_n^{(\alpha)}(x^2) L_m^{\alpha}(x^2) \mathrm{e}^{-x^2} x^{2\alpha+1} \mathrm{d}x = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{nm}, \quad \alpha > -1
\end{aligned} \tag{2.1.3.3}$$

Now, let's observe two special cases.

$$\begin{aligned}\int_{-\infty}^{\infty} xL_n^{(1/2)}(x^2)xL_m^{(1/2)}(x^2)\mathrm{e}^{-x^2}\mathrm{d}x &= \frac{\Gamma(n+3/2)}{n!}\delta_{nm}, \quad \alpha = 1/2 \\ \int_{-\infty}^{\infty} L_n^{(-1/2)}(x^2)L_m^{(-1/2)}(x^2)\mathrm{e}^{-x^2}\mathrm{d}x &= \frac{\Gamma(n+1/2)}{n!}\delta_{nm}, \quad \alpha = -1/2\end{aligned} \tag{2.1.3.4}$$

Now, the uniqueness of orthogonal polynomials, up to an arbitrary constant, implies that, with domain $\mathbb{R}$ and weight function e^{-x^2}, $\{xL_n^{(1/2)}(x^2), L_n^{(-1/2)}(x^2)\}_{n=0}^{\infty}$ form an orthogonal set. We now attempt to "guess" the correct coefficients. In both of the derivations below, the duplication formula, Eq. (2.1.5), is invoked.

For $\alpha = -1/2$:

$$\begin{aligned}\int_{-\infty}^{\infty} &\left[(-1)^n 2^{2n} n! L_n^{(-1/2)}(x^2)\right] \\ &\times \left[(-1)^m 2^{2m} m! L_m^{(-1/2)}(x^2)\right] \mathrm{e}^{-x^2}\mathrm{d}x \\ &= \Gamma(n+1/2)n!2^{4n}\delta_{nm} \\ &= \frac{\sqrt{\pi}\Gamma(2n)}{2^{2n-1}\Gamma(n)}n!2^{4n}\delta_{nm} \\ &= \frac{\sqrt{\pi}2n\Gamma(2n)}{2^{2n}n\Gamma(n)}n!2^{4n}\delta_{nm} \\ &= \sqrt{\pi}(2n)!2^{2n}\delta_{nm}\end{aligned}$$

For $\alpha = 1/2$:

$$\begin{aligned}\int_{-\infty}^{\infty} &\left[(-1)^n 2^{2n+1} n! x L_n^{(1/2)}(x^2)\right] \\ &\times \left[(-1)^m 2^{2m+1} m! L_m^{(1/2)}(x^2)\right] \mathrm{e}^{-x^2}\mathrm{d}x \\ &= \Gamma(n+3/2)n!2^{4n+2}\delta_{nm} \\ &= \frac{\sqrt{\pi}\Gamma(2n+2)}{2^{2n+1}\Gamma(n+1)}n!2^{4n}\delta_{nm} \\ &= \sqrt{\pi}(2n+1)!2^{2n+1}\delta_{nm}\end{aligned}$$

So, we can now define the Hermite polynomials as

$$\begin{aligned}H_{2n}(x) &= (-1)^n 2^{2n} n! L_n^{(-1/2)}(x^2) \\ H_{2n+1}(x) &= (-1)^n 2^{2n+1} n! x L_n^{(1/2)}(x^2)\end{aligned} \tag{2.1.3.5}$$

which changes our version of Eq. (2.1.3.4) with coefficients to

$$\begin{aligned}
&\int_{-\infty}^{\infty} H_{2n}(x)H_{2m}(x)\mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}(2n)!2^{2n}\delta_{nm} \\
&\int_{-\infty}^{\infty} H_{2n+1}(x)H_{2m+1}(x)\mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}(2n+1)!2^{2n+1}\delta_{nm} \\
&\text{or} \\
&\int_{-\infty}^{\infty} H_{n}(x)H_{m}(x)\mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}n!2^{n}\delta_{nm}
\end{aligned} \tag{2.1.3.6}$$

We must now try to determine a representation of the Hermite Orthogonal polynomials in hypergeometric form. From Eq. (2.1.3.6) and Eq. (2.1.3.1), we can see that

$$\begin{aligned}
H_{2n}(x) &= \frac{(-1)^n 2^{2n} n!(1/2)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k x^{2k}}{(1/2)_k k!} = \frac{(-1)^n 2^{2n} n!(1/2)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k x^{2k}}{(1/2)_k k!} \\
&= (-1)^n 2^{2n} n!(1/2)_n \sum_{k=0}^{n}\frac{(-1)^k x^{2k}}{(n-k)!(1/2)_k k!} = (-1)^n (2n)!\sum_{k=0}^{n}\frac{(-1)^k x^{2k}}{(n-k)!(1/2)_k k!}
\end{aligned}$$

Where we observed that for $k > n, (-n)_k = 0$, then used equations Eq. (2.1.3) and Eq. (2.1.5). Next, using Eq. (2.1.1) shows

$$H_{2n}(x) = (-1)^n (2n)!\sum_{k=0}^{n}\frac{(-1)^k (2x)^{2k}}{(n-k)!(2k)!} = (2n)!\sum_{k=n}^{0}\frac{(-1)^k (2x)^{2n-2k}}{k!(2n-2k)!} = (2n)!\sum_{k=0}^{n}\frac{(-1)^k (2x)^{2n-2k}}{k!(2n-2k)!}$$

where the summing index was reversed to have $k \to n-k$ and then we used the fact that finite summation is associative. This implies, then, that

$$H_n(x) = n!\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!} \tag{2.1.3.7}$$

We now have a series representation of the Hermite polynomials, which are accepted by many, including [7], as the definition of the Hermite polynomials. However, we will continue with manipulations until we reach a hypergeometric form. Now, using Eq. (2.1.3) and Eq. (2.1.5), we find that

$$(-n)_{2k} = \frac{n!}{(n-2k)!} = 2^{2k}(-n/2)_k(-(n-1)/2)_k \tag{2.1.3.8}$$

Inserting into Eq. (2.1.3.7) reveals that

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^n (-n/2)_k (-(n-1)/2)_k}{x^{2k} k!} \\ &= (2x)^n \sum_{k=0}^{\infty} \frac{(-1)^k (-n/2)_k (-(n-1)/2)_k}{x^{2k} k!} \\ &= (2x)^n {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix} \middle| -\frac{1}{x^2} \right) \end{aligned} \tag{2.1.3.9}$$

We have successfully obtained a hypergeometric form for the Hermite orthogonal polynomials. Due to connection between the Hermite and Laguerre polynomials within the Hermite polynomial's definition, we are motivated to prove other connections between the two polynomials in the proofs section. Now that we have introduced how a theory of an orthogonal polynomial can begin, we will explore other special functions that are not orthogonal polynomials

2.1.4 Other Special Functions: The Riemann Zeta Function

We now end this section with a brief foray into other special functions that are of interest, starting with the Riemann Zeta Function. The Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad s \in \mathbb{C} \tag{2.1.4.1}$$

is of major importance in analytic number theory. In the subfield of physics that is thermodynamics, integrals of the form

$$\int_0^{\infty} \frac{x}{\mathrm{e}^x - 1} \mathrm{d}x$$

describe distributions of different particles, such as photons, into the available energy levels. We shall see how such integrals are related to the Gamma and Zeta Functions. Start with the Gamma function,

$$\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-t} t^{s-1} \mathrm{d}t \tag{2.1.4.2}$$

and then let $t = xn$. We now have

$$\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-xn} (nx)^{s-1} n \mathrm{d}x$$

which becomes, after re-arranging,

$$\Gamma(s) \frac{1}{n^s} = \int_0^{\infty} \mathrm{e}^{-xn} x^{s-1} \mathrm{d}x$$

We then sum from sides from 1 to ∞, and then bring the summation within the integral to obtain

$$\Gamma(s)\sum_{n=1}^{\infty}\frac{1}{n^s} = \int_0^{\infty}\sum_{n=1}^{\infty}\mathrm{e}^{-xn}x^{s-1}\mathrm{d}x$$

We can replace the summation on the l.h.s with the Riemann Zeta Function, Eq. (2.1.4.1). The summation within the integral on the r.h.s is simply a geometric series, as

$$|e^{-x}| \leq 1 \quad \forall x \in \mathbb{R}.$$

So now we have

$$\Gamma(s)\zeta(s) = \int_0^{\infty}\frac{x^{s-1}\mathrm{e}^{-x}\mathrm{d}x}{1-\mathrm{e}^{-x}}$$

Multiplying numerator and denominator by e^x gives us our solution

$$\boxed{\Gamma(s)\zeta(s) = \int_0^{\infty}\frac{x^{s-1}\mathrm{d}x}{\mathrm{e}^x-1}} \tag{2.1.4.3}$$

An important value is $s = 2$. In this case, $\Gamma(2) = 1! = 1$, $\zeta(2) = \pi^2/6$, and Eq. (2.1.4.3) becomes

$$\frac{\pi^2}{6} = \int_0^{\infty}\frac{x\mathrm{d}x}{\mathrm{e}^x-1} \tag{2.1.4.4}$$

2.1.5 Other Special Functions: The Bessel Functions

Some of the earliest known special functions were the Bessel Function and the Modified Bessel Function, defined as, respectively,

$$\begin{aligned} J_\nu(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{\nu+2n}}{\Gamma(n+\nu+1)n!} \\ I_\nu(z) &= \sum_{n=0}^{\infty}\frac{(z/2)^{\nu+2n}}{\Gamma(n+\nu+1)n!} \end{aligned} \tag{2.1.5.1}$$

The Bessel functions appear in the solution for the radial component of Laplace's equation in cylindrical coordinates (see Eq. (4.1.1.12) for more information). They can be thought of as one way to generalize sine and cosine waves to the 2D plane. For example, sound waves moving through the air are described using sine and cosines, but the vibrations of a drum skin are described using the Bessel functions. In fact, the Bessel functions have the following direct relation to sine and cosine.

$$\begin{aligned} J_{1/2}(z) &= \sqrt{\frac{2}{\pi z}}\sin(z) \\ J_{-1/2}(z) &= \sqrt{\frac{2}{\pi z}}\cos(z) \end{aligned} \tag{2.1.5.2}$$

The Bessel Functions obey the following integral relations, called Sonine's First and Second Integral relations.

$$\begin{aligned} J_{\alpha+\beta+1}(z) &= \frac{2^{-\beta}z^{\beta+1}}{\Gamma(\beta+1)}\int_0^1 u^{2\beta+1}(1-u^2)^{\alpha/2}J_\alpha(z\sqrt{1-u^2})\mathrm{d}u \\ \frac{x^\nu y^\mu J_{\mu+\nu+1}(\sqrt{x^2+y^2})}{(x^2+y^2)^{(\nu+\mu+1)/2}} &= \int_0^{\pi/2} J_\nu(x\sin\theta)J_\mu(y\cos\theta)\cos^{\mu+1}\theta\sin^{\nu+1}\theta\mathrm{d}\theta \\ &= \int_0^1 J_\nu(x\sqrt{1-u^2})J_\mu(yu)u^{\mu+1}(1-u^2)^{\nu/2}\mathrm{d}u \end{aligned} \tag{2.1.5.3}$$

Many interesting relations for sine and cosine can be obtained from these functions, which are shown in Section 2.2.1.

2.2 Orthogonal Polynomial Relations

In this section, we introduce various proofs concerning orthogonal polynomials, focusing primarily on the Ultraspherical, Laguerre and Hermite polynomials. We will need to make use of several of their properties, all of which are obtained from [7]. Several problems have additional proofs, inspired by the work of Pennsylvania State Erie, the Behrend College alumni Matthew P. Lachesky. These proofs are marked with *.

2.2.1 Series Relations

We will first begin with using the fact that the Laguerre polynomials form a basis on the vector space of polynomials to prove an interesting series relation.

Problem 1.

$$\boxed{x^n = \sum_{k=0}^{n}\frac{(-1)^k n!(\alpha+1)_n}{(n-k)!(\alpha+1)_k}L_k^{(\alpha)}(x)} \tag{2.2.1.1}$$

Proof. Recall that an orthogonal polynomial spans the vector space of polynomials over the support interval, denoted Ω of the orthogonal polynomial. Thus, the orthogonal polynomial forms a basis, and this includes the Laguerre polynomials. For polynomials $f(x)$ and $g(x)$, we define the inner-product

$$\langle f, g\rangle = \int_0^\infty x^\alpha \mathrm{e}^{-x} f(x)g(x)\mathrm{d}x \tag{2.2.1.2}$$

Let $K_n = \Gamma(\alpha+n+1)/n!$ be the associated squared-norm of the Laguerre polynomials. That is,

$$\langle L_n^{(\alpha)}(x), L_m^{(\alpha)}(x)\rangle = \int_0^\infty x^\alpha \mathrm{e}^{-x} L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)\mathrm{d}x = K_n\delta_{nm} \tag{2.2.1.3}$$

This is simply, of course, the orthogonality of the Laguerre polynomials. For any polynomial $\pi(x)$ of degree n, we have,

$$\pi(x) = \sum_{k=0}^{n} c_{n,k}L_k^{(\alpha)}(x), \quad c_{n,k} = \frac{\langle \pi(x), L_k^{(\alpha)}(x)\rangle}{\langle L_k^{(\alpha)}(x), L_k^{(\alpha)}(x)\rangle} \tag{2.2.1.4}$$

where $c_{n,k}$ is the scalar projection of $\pi(x)$ onto $L_k^{(\alpha)}(x)$. Let $\pi(x) = x^n$. Thus, Eq. (2.2.1.4) becomes

$$x^n = \sum_{k=0}^{n} c_{n,k} L_k^{(\alpha)}(x), \quad c_{n,k} = \frac{\langle x^n, L_k^{(\alpha)}(x)\rangle}{\langle L_k^{(\alpha)}(x), L_k^{(\alpha)}(x)\rangle} \tag{2.2.1.5}$$

Now,

$$\langle L_k^{(\alpha)}(x), L_k^{(\alpha)}(x)\rangle = \frac{\Gamma(\alpha+k+1)}{k!} = \frac{\Gamma(\alpha+1)(\alpha+1)_k}{k!}$$

and our summation becomes

$$x^n = \sum_{k=0}^{n} \frac{k!\,\langle x^n, L_k^{(\alpha)}(x)\rangle}{\Gamma(\alpha+1)(\alpha+1)_k} L_k^{(\alpha)}(x) \tag{2.2.1.6}$$

Notice that

$$\Gamma(\alpha+1) = \frac{\Gamma(\alpha+n+1)}{(\alpha+1)_n}$$

so Eq. (2.2.1.6) becomes

$$x^n = \sum_{k=0}^{n} \frac{k!(\alpha+1)_n\,\langle x^n, L_k^{(\alpha)}(x)\rangle}{\Gamma(\alpha+n+1)(\alpha+1)_k} L_k^{(\alpha)}(x) \tag{2.2.1.7}$$

Now we must tackle $\langle x^n, L_k^{(\alpha)}(x)\rangle$, starting by inserting the definition of the Laguerre polynomials:

$$\begin{aligned}
\langle x^n, L_k^{(\alpha)}(x)\rangle &= \int_0^\infty x^{\alpha+n} \mathrm{e}^{-x} L_k^{(\alpha)}(x)\mathrm{d}x \\
&= \frac{(\alpha+1)_k}{k!} \sum_{j=0}^{k} \frac{(-k)_j}{j!(\alpha+1)_j} \int_0^\infty x^{\alpha+n+j}\mathrm{e}^{-x}\mathrm{d}x \\
&= \frac{(\alpha+1)_k}{k!} \sum_{j=0}^{k} \frac{(-k)_j}{j!(\alpha+1)_j} \Gamma(\alpha+n+j+1) \\
&= \frac{(\alpha+1)_k}{k!} \sum_{j=0}^{k} \frac{(-k)_j(\alpha+n+1)_j}{j!(\alpha+1)_j} \Gamma(\alpha+n+1)
\end{aligned}$$

Where the fact that $\Gamma(n+j+\alpha+1) = (\alpha+n+1)_j\Gamma(\alpha+n+1)$ was used in the last step. Now, using the Chu-Vandermande Sum Eq. (2.1.8), we get

$$\begin{aligned}
\langle x^n, L_k^{(\alpha)}(x)\rangle &= \frac{(\alpha+1)_k}{k!} \frac{\Gamma(\alpha+n+1)(-n)_k}{(\alpha+1)_k} \\
&= \frac{\Gamma(\alpha+n+1)(-n)_k}{k!} = \frac{\Gamma(\alpha+n+1)(-1)^k n!}{k!(n-k)!}
\end{aligned}$$

where the relation for negative pochhammer arguments, Eq. (2.1.3), was used in the final substitution. Thus,

$$\langle x^n, L_k^{(\alpha)}(x)\rangle = \frac{\Gamma(\alpha+n+1)(-1)^k n!}{k!(n-k)!} \tag{2.2.1.8}$$

inserting Eq. (2.2.1.8) into Eq. (2.2.1.7) reveals

$$x^n = \sum_{k=0}^{n} \frac{(\alpha+1)_n(-1)^k n!}{(n-k)!(\alpha+1)_k} L_k^{(\alpha)}(x) \tag{2.2.1.9}$$

□

We will present another proof* of this problem, that relies on the uniqueness of orthogonal polynomials.

Proof. We start by modifying a derivation from the previous proof of this problem

$$\begin{aligned}\langle x^n, L_k^{(\alpha)}(x)\rangle &= \frac{\Gamma(\alpha+n+1)(-1)^k n!}{k!(n-k)!} \\ &= \frac{(\alpha+n)!(-1)^k n!(\alpha+k)!}{k!(n-k)!(\alpha+k)!} \\ &= \frac{(\alpha+1)_n(-1)^k n!(\alpha+k)!}{k!(n-k)!(\alpha+1)_k}\end{aligned} \tag{2.2.1.10}$$

Now, define $f(\alpha,n) = \sum_{k=0}^{n} \frac{(\alpha+1)_n(-1)^k n!}{(n-k)!(\alpha+1)_k} L_k^{(\alpha)}(x)$ and observe that, from the orthogonality of the Laguerre polynomials, Eq. (2.2.1.2), we have

$$\begin{aligned}\langle L_k^{(\alpha)}(x), f(\alpha,n)\rangle &= \frac{\Gamma(\alpha+1)(\alpha+1)_k}{k!}\frac{(\alpha+1)_n(-1)^k n!}{(n-k)!(\alpha+1)_k} \\ &= \frac{(\alpha+k)!}{k!}\frac{(\alpha+1)_n(-1)^k n!}{(n-k)!(\alpha+1)_k} \\ &= \langle x^n, L_k^{(\alpha)}(x)\rangle\end{aligned} \tag{2.2.1.11}$$

Since, the orthogonality is unique, we have $x^n = f(\alpha,n)$ or

$$x^n = \sum_{k=0}^{n} \frac{(-1)^k n!(\alpha+1)_n}{(n-k)!(\alpha+1)_k} L_k^{(\alpha)}(x) \tag{2.2.1.12}$$

□

Next, we will derive a different series representation of the Ultraspherical polynomials.

Problem 2.

$$\boxed{C_n^{\lambda} = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(\lambda)_{n-k}(-1)^k(2x)^{n-2k}}{(n-2k)!k!}}$$

Proof. Recall that the generating function for the Ultraspherical polynomials is

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n = (1-(2xt-t^2))^{\lambda} \tag{2.2.1.13}$$

Using the binomial theorem, Eq. (2.2.2.16), twice reveals

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n &= (1-(2xt-t^2))^{\lambda} \\
&= \sum_{j=0}^{\infty} \frac{(\lambda)_j}{j!}(2xt-t^2)^j \\
&= \sum_{j=0}^{\infty} \frac{(\lambda)_j}{j!}(2xt)^j \left(1-\frac{t}{2x}\right)^j \\
&= \sum_{j=0}^{\infty} \frac{(\lambda)_j}{j!}(2xt)^j \sum_{k=0}^{\infty} \frac{(-j)_k}{k!}\left(\frac{t}{2x}\right)^k \\
&= \sum_{k,j=0}^{\infty} \frac{(\lambda)_j(-j)_k(2x)^{j-k}}{j!k!}t^{j+k} \\
&= \sum_{k,j=0}^{\infty} \frac{(\lambda)_j(-1)^k(2x)^{j-k}}{(j-k)!k!}t^{j+k}
\end{aligned}$$

where the equation for negative pochhammer arguments, Eq. (2.1.3), was used in the final step. Comparing the exponents of t in the last series and the first reveals that $n = j + k$. Solving for j and re-indexing gives us the following equation for the generating function.

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \frac{(\lambda)_{n-k}(-1)^k(2x)^{n-2k}}{(n-2k)!k!}\right)t^n$$

which means

$$C_n^{\lambda}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_{n-k}(-1)^k(2x)^{n-2k}}{(n-2k)!k!} \tag{2.2.1.14}$$

Now, notice when $k = 0$ we have our x^n term and when $k = n/2$ we have our constant term. So, as the series iterates, the terms are listed in descending order according to exponent. When $k > \lfloor n/2 \rfloor$, the terms listed do not correspond to any terms in the Ultraspherical polynomials. In fact, the factorial in the denominator would no longer be defined as the poles of the Gamma function are the negative integers. So, with both of these arguments in mind, we must stop our summation at $\lfloor n/2 \rfloor$. Which means we have

$$C_n^{\lambda}(x) = \sum_{n=0}^{\lfloor n/2 \rfloor} \frac{(\lambda)_{n-k}(-1)^k(2x)^{n-2k}}{(n-2k)!k!}t^n \tag{2.2.1.15}$$

and we have found a different series representation for the Ultraspherical polynomials. □

Now, we will present another proof, based upon the following:

Lemma 2.2.1.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} A(k,n-2k)$$

we leave the proof of this to [8]. And now we start another proof* of Eq. (2.2.1.15).

Proof. We begin by factoring the generating function for the ultraspherical polynomials:

$$\begin{aligned}\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n &= (1-2xt+t^2))^{\lambda}\\ &= \frac{1}{(1+t^2)^{\lambda}}\left(\frac{1+t^2}{1-2xt+t^2}\right)^{\lambda}\\ &= \frac{1}{(1+t^2)^{\lambda}}\left(\frac{1-2xt+t^2}{1+t^2}\right)^{-\lambda}\\ &= \frac{1}{(1+t^2)^{\lambda}}\left(1-\frac{2xt}{1+t^2}\right)^{-\lambda}\end{aligned}$$

Now, using the binomial theorem, Eq. (1.6.1), yields:

$$\begin{aligned}\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n &= \frac{1}{(1+t^2)^{\lambda}}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(2x)^n t^n (1+t^2)^{-n}\\ &= \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(2x)^n t^n (1+t^2)^{-(n+\lambda)}\end{aligned}$$

Using the binomial theorem again will yield

$$\begin{aligned}\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{(\lambda)_n}{n!}\frac{(n+\lambda)_k}{k!}(2x)^n t^{n+2k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{(\lambda)_{n+k}}{n!k!}(2x)^n t^{n+2k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{(\lambda)_{n-k}}{(n-2k)!k!}(2x)^{n-2k} t^n\end{aligned}$$

where Eq. (2.1.7) in the form of $(\lambda)_{n+k} = (n+\lambda)_k(\lambda)_n$ and then Lemma 2.2.1 were used in the last two steps. Thus, comparing coefficients, we get

$$C_n^{\lambda}(x) = \sum_{n=0}^{\lfloor n/2\rfloor}\frac{(\lambda)_{n-k}(-1)^k(2x)^{n-2k}}{(n-2k)!k!} \tag{2.2.1.16}$$

□

Finally, we will show how Bessel functions can give rise to various interesting series.

Problem 3.

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1/2} \frac{J_\nu(\sqrt{b^2+\pi^2(n+1/2)^2})}{\left[b^2+\pi^2(n+1/2)^2\right]^{\nu/2}} = \frac{\pi}{2} b^{-\nu} J_\nu(b), \quad b>0, \quad Re(\nu) > -\frac{1}{2}}$$

with a special case of (2.2.1.17)

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1/2} \frac{\sin\sqrt{b^2+\pi^2(n+1/2)^2}}{\sqrt{b^2+\pi^2(n+1/2)^2}} = \frac{\pi}{2}\frac{\sin b}{b}}$$

R. William Gosper computationally proved the convergence of these summations; they were proved using the Sonine Integrals and Bessel functions in [9]. Here we present our approach using similar techniques to solve the problem.

Proof. We shall start with an important Fourier series. Recall that, in general, a function $f(x) \in \mathrm{C}[-\mathrm{L}, \mathrm{L}]$ can be expressed as

$$f(x) = \sum_{n=0}^{\infty} \left(\mathrm{a}_n \cos\left(\frac{n\pi x}{\mathrm{L}}\right) + \mathrm{b}_n \sin\left(\frac{n\pi x}{\mathrm{L}}\right)\right)$$

Recognize that

$$\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right) \qquad \sin(\theta) = -\cos\left(\theta + \frac{\pi}{2}\right)$$

So, if we let $\mathrm{B}_n = -b_n$, and letting $\mathrm{L} = 1$, our Fourier series becomes

$$f(x) = \sum_{n=0}^{\infty} \left(\mathrm{B}_n \cos\left((n+1/2)\pi x\right) + \mathrm{a}_n \sin\left((n+1/2)\pi x\right)\right)$$

Now, let our function be the constant function $\pi/2$. That is, $f(x) = \pi/2$. This function is even, so $a_n = 0$ for all n. So now we have

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \mathrm{B}_n \cos\left((n+1/2)\pi x\right)$$

Now, multiply both sides by $\cos\left((m+1/2)\pi x\right)$ and recall from the orthogonality of cosine, that, when integrating from -1 to 1, all the integrals will be zero, except when $n = m$. Thus we have

$$\int_{-1}^{1} \frac{\pi}{2} \cos\left((n+1/2)\pi x\right) \mathrm{d}x = \mathrm{B}_n \int_{-1}^{1} \cos^2\left((n+1/2)\pi x\right) \mathrm{d}x$$

Solving this integral reveals that

$$\mathrm{B}_n = \frac{(-1)^n}{n+1/2}.$$

Thus, we have the following summation formula:

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1/2} \cos\left((n+1/2)\pi x\right) \;\; x \in [-1, 1] \tag{2.2.1.18}$$

This will be vital later. In the first integral in Eq. (2.1.5.3), Sonine's First Integral relation, let $z = b, \alpha = \nu - 1/2, \beta = -1/2$. We get

$$J_\nu(b) = \sqrt{\frac{2}{\pi}b}\int_0^1 (1-u^2)^{(\nu-1/2)/2} J_{\nu-1/2}\left(b\sqrt{1-u^2}\right)\mathrm{d}u \tag{2.2.1.19}$$

Now, in Sonine's Second integral relation, let $\mu = -1/2, x = b$ and $y = (n+1/2)\pi$.

$$\begin{aligned}&\frac{b^\nu[(n+1/2\pi)]^{-1/2} J_{\nu+1/2}(\sqrt{b^2+\pi^2(n+1/2)^2})}{(b^2+\pi^2(n+1/2)^2)^{(\nu+1/2)/2}}\\ &= \int_0^1 J_\nu\left(b\sqrt{1-u^2}\right) J_{-1/2}(\pi(n+1/2)u)u^{1/2}(1-u^2)^{\nu/2}\mathrm{d}u\end{aligned} \tag{2.2.1.20}$$

We know from Eq. (2.1.5.2) that

$$J_{-1/2}(\pi(n+1/2)u) = \sqrt{\frac{2}{\pi}}\,[\pi(n+1/2)u]^{-1/2}\,u^{-1/2}\cos(\pi(n+1/2)u)$$

Inserting this into Eq. (2.2.1.20), and also letting $\nu \to \nu - 1/2$ gives us

$$\begin{aligned}&\frac{J_\nu(\sqrt{b^2+\pi^2(n+1/2)^2})}{(b^2+\pi^2(n+1/2)^2)^{\nu/2}}\\ &= b^{-\nu}\sqrt{\frac{2}{\pi}b}\int_0^1 J_{\nu-1/2}\left(b\sqrt{1-u^2}\right)(1-u^2)^{(\nu-1/2)/2}\cos(\pi(n+1/2)u)\mathrm{d}u\end{aligned} \tag{2.2.1.21}$$

Now, multiple both sides by $(-1)^n/(n+1/2)$ and sum from 0 to ∞. The r.h.s becomes

$$\begin{aligned}b^{-\nu}\sqrt{\frac{2}{\pi}b}\int_0^1 J_{\nu-1/2}\left(b\sqrt{1-u^2}\right)(1-u^2)^{(\nu-1/2)/2}\\ \times\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1/2)}\cos(\pi(n+1/2)u)\mathrm{d}u\end{aligned} \tag{2.2.1.22}$$

After inserting our summation formula, Eq. (2.2.1.18), we get for the r.h.s

$$b^{-\nu}\frac{\pi}{2}\sqrt{\frac{2}{\pi}b}\int_0^1 J_{\nu-1/2}\left(b\sqrt{1-u^2}\right)(1-u^2)^{(\nu-1/2)/2}\mathrm{d}u$$

Then inserting our version of Sonine's First Integral Formula, Eq. (2.1.5.3), gives us

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1/2}\frac{J_\nu(\sqrt{b^2+\pi^2(n+1/2)^2})}{\left[b^2+\pi^2(n+1/2)^2\right]^{\nu/2}} = \frac{\pi}{2}b^{-\nu}J_\nu(b) \tag{2.2.1.23}$$

Letting $\nu = 1/2$ and then using the identity for $\sin(z)$ in Eq. (2.1.5.2) on both sides of the equation yields

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1/2}\frac{\sin\left(\sqrt{b^2+\pi^2(n+1/2)^2}\right)}{\sqrt{b^2+\pi^2(n+1/2)^2}} = \frac{\pi}{2}\frac{\sin b}{b} \tag{2.2.1.24}$$

Now for an interesting result. Let $b \to 0$. We have the famous limit $\lim_{b\to 0}(\sin b)/b = 1$. Remember also that $\sin(\pi(n+1/2)) = (-1)^n$. We get, after multiplying both sides by π,

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{(n+1/2)^2} = \frac{\pi^2}{2}}$$

□

2.2.2 Limit Relations

We will now prove two different limit relations, one each for the Laguerre and Hermite polynomials.

Problem 4.

$$\boxed{2^n n! \lim_{\beta\to\infty} \beta^{-n} L_n^{(\beta^2/2)}(-\beta x + \beta^2/2) = H_n(x)} \tag{2.2.2.1}$$

This problem is listed as an important relation between the Hermite and Laguerre polynomials within Askey's classification. A combinatorial proof was given by Labelle and Yeh in [10].

Proof. We start by listing the recursion relations for both polynomials.

$$\begin{aligned} xL_n^{(\alpha)}(x) &= -(n+1)L_{n+1}^{(\alpha)}(x) + (2n+\alpha+1)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \\ 2xH_n(x) &= H_{n+1}(x) + 2nH_{n-1}(x) \end{aligned} \tag{2.2.2.2}$$

Our goal is to to show that Eq. (2.2.2.1) satisfies the recursion relation for the Hermite polynomials. We start with the recursion relation for the Laguerre polynomials, letting $\alpha = \beta^2/2$, replacing x with $-\beta x + \beta^2/2$ and multiplying through by $2^n n!\beta^{-n}$.

$$\begin{aligned} 2^n n!(-\beta x + \beta^2/2)\beta^{-n} L_n^{(\beta^2/2)}(-\beta x + \beta^2/2) = \\ &- 2^n(n+1)!\beta^{-n} L_{n+1}^{(\beta^2/2)}(-\beta x + \beta^2/2) \\ &+ 2^n n!\beta^{-n}(2n+1+\beta^2/2)L_n^{(\beta^2/2)}(-\beta x + \beta^2/2) \\ &- (n+\beta^2/2)2^n n!\beta^{-n} L_{n-1}^{(\beta^2/2)}(-\beta x + \beta^2/2) \end{aligned} \tag{2.2.2.3}$$

For ease of notation, let

$$H_n^\star(x) = 2^n n!\beta^{-n} L_n^{(\beta^2/2)}(-\beta x + \beta^2/2).$$

That is, we've reduced the problem to showing that $\lim_{\beta\to\infty} H_n^\star(x) = H_n(x)$. Before we continue, observe that, from its definition, $H_n^\star(x) = 0$ for $n < 0$ and $H_0^\star(x) = 1$. Now, Eq. (2.2.2.3) becomes

$$\begin{aligned} (-\beta x + \beta^2/2)H_n^\star(x) = -\frac{\beta H_{n+1}^\star(x)}{2} + (2n+1+\beta^2/2)H_n^\star(x) \\ - (n+\beta^2/2)\frac{2n}{\beta}H_{n-1}^\star(x) \end{aligned} \tag{2.2.2.4}$$

Simplifying, multiplying by 2 and dividing by $-\beta$ creates

$$2xH_n^\star(x) = H_{n+1}^\star(x) + 2nH_{n-1}^\star(x) - \frac{2}{\beta}\left[(2n+1)H_n^\star(x) - nH_{n-1}^\star(x)\right] \tag{2.2.2.5}$$

Taking the limit of both sides,

$$\begin{aligned} 2x \lim_{\beta\to\infty} H_n^\star(x) &= \lim_{\beta\to\infty} H_{n+1}^\star(x) + 2n \lim_{\beta\to\infty} H_{n-1}^\star(x) \\ &- \lim_{\beta\to\infty} \frac{2}{\beta}\left[(2n+1)H_n^\star(x) - nH_{n-1}^\star(x)\right] \\ &= \lim_{\beta\to\infty} H_{n+1}^\star(x) + 2n \lim_{\beta\to\infty} H_{n-1}^\star(x) \end{aligned} \tag{2.2.2.6}$$

So, $\lim_{\beta\to\infty} H_n^\star(x)$ satisfies the recursion relation for the Hermite polynomials, and thus

$$\lim_{\beta\to\infty} H_n^\star(x) = H_n(x) \tag{2.2.2.7}$$

□

We are now going to present an alternative proof of Problem 4, but we must first go on a bit of a digression.

Theorem 2.2.2. *The Laguerre and Hermite polynomials satisfy the following adjoint operators.*

$$\begin{aligned} \frac{\mathrm{d}H_n(x)}{\mathrm{d}x} &= 2nH_{n-1}(x) \\ \frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x} &= -L_{n-1}^{(\alpha+1)}(x) \end{aligned}$$

Proof. We leave the proof for the Hermite adjoint operator as an exercise for the reader, and shall prove only the Laguerre adjoint operator here. Recall the definition of the Laguerre polynomials, Eq. (2.1.3.1). Taking the derivative, and remembering that the contribution to the sum for $k = 0$ is a constant, and thus shifts the starting index by one after differentiating, reveals

$$\frac{\mathrm{d}}{\mathrm{d}x} L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=1}^{\infty} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^{k-1}}{(k-1)!} \tag{2.2.2.8}$$

We will now switch some terms for Gamma functions, and also use the formula for negative

pochhammer indexes, Eq. (2.1.3).

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}L_n^{(\alpha)}(x) &= -\frac{\Gamma(\alpha+1+n)}{n(n-1)!\Gamma(\alpha+1)}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(n-k+1)_k\Gamma(\alpha+1)}{\Gamma(\alpha+1+k)}\frac{x^{k-1}}{(k-1)!} \\
&= -\frac{\Gamma(\alpha+1+n)}{n(n-1)!\Gamma(\alpha+2)}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(n-k+1)_k\Gamma(\alpha+2)}{\Gamma(\alpha+1+k)}\frac{x^{k-1}}{(k-1)!} \\
&= -\frac{(\alpha+2)_{n-1}}{n(n-1)!}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(n-k+1)_k}{(\alpha+2)_{k-1}}\frac{x^{k-1}}{(k-1)!} \\
&= -\frac{(\alpha+2)_{n-1}}{n(n-1)!}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Gamma(n+1)}{(\alpha+2)_{k-1}\Gamma(n-k+1)}\frac{x^{k-1}}{(k-1)!} \\
&= -\frac{(\alpha+2)_{n-1}}{(n-1)!}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Gamma(n)}{(\alpha+2)_{k-1}\Gamma(n-k+1)}\frac{x^{k-1}}{(k-1)!} \\
&= -\frac{(\alpha+2)_{n-1}}{(n-1)!}\sum_{k=1}^{\infty}\frac{(-(n-1))_{k-1}}{(\alpha+2)_{k-1}}\frac{x^{k-1}}{(k-1)!} \\
&= -\frac{(\alpha+2)_{n-1}}{(n-1)!}\sum_{k=0}^{\infty}\frac{(-(n-1))_k}{(\alpha+2)_k}\frac{x^k}{k!} \\
&= -L_{n-1}^{(\alpha+1)}(x)
\end{aligned}$$

□

Now, we must mention that orthogonal polynomials are the solutions to different differential equations. Historically, this is often the means through which the polynomials were discovered. We shall need the differential equations satisfied by the Hermite and Laguerre polynomials. We leave the proofs to other sources, [7].

Theorem 2.2.3. *The Hermite and Laguerre polynomials satisfy the following differential equations, where $y(x) = H_n(x)$, $f(x) = L_n^{(\alpha)}(x)$ and primes denote derivatives with respect to x.*

$$\begin{aligned}
y''(x) - 2xy'(x) + 2ny(x) &= 0 \\
xf''(x) + (1+\alpha-x)f'(x) + nf(x) &= 0
\end{aligned} \tag{2.2.2.9}$$

We now begin another proof for Problem 4.

Proof. Let $H_n^\star(x) = 2^n n!\beta^{-n}L_n^{(\beta^2/2)}(-\beta x + \beta^2/2)$. Taking the derivative reveals that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}H_n^\star(x) &= 2^n n!\beta^{-n}\frac{\mathrm{d}}{\mathrm{d}x}\left(L_n^{(\beta^2/2)}(-\beta x+\beta^2/2)\right) \\
&= 2^n n!\beta^{-n}\beta\left(L_{n-1}^{(\beta^2/2)+1}(-\beta x+\beta^2/2)\right) \\
&= 2n\left(2^{n-1}(n-1)!\beta^{-(n-1)}L_{n-1}^{(\beta^2/2)+1}(-\beta x+\beta^2/2)\right) \\
&= 2nH_{n-1}^\star(x)
\end{aligned}$$

where the adjoint operator for the Laguerre polynomials was used. This result shows that the adjoint operator for $H_n^\star(x)$ is the same as the Hermite polynomials, and this motivates us to use the differential equations for them. We insert $L_n^{(\beta^2/2)}(-\beta x + \beta^2/2)$ into the differential equation satisfied by the Laguerre polynomials to get

$$(-\beta x + \beta^2/2)L_{n-2}^{(\beta^2/2+2)}(-\beta x + \beta^2/2) - (1 + \beta x)L_{n-1}^{(\beta^2/2+1)}(-\beta x + \beta^2/2) \\ + nL_n^{(\beta^2/2)}(-\beta x + \beta^2/2) = 0$$

Multiplying through by $2^{n+1}n!\beta^{-n}$ and switching to the $H_n^\star(x)$ notation reveals

$$\begin{aligned}
&4n(n-1)(n-2)!2^{n-2}\beta^{-(n-2)}L_{n-2}^{(\beta^2/2+2)}(-\beta x + \beta^2/2) \\
&- 2nx\left(2 * 2^{n-1}(n-1)!\beta^{n-1}L_{n-1}^{(\beta^2/2+1)}(-\beta x + \beta^2/2)\right) + 2n\left(2^n n! L_n^{(\beta^2/2)}(-\beta x + \beta^2/2)\right) \\
&- \left[x\beta^{-(n-1)}2^{n+1}n!L_{n-2}^{(\beta^2/2+2)}(-\beta x + \beta^2/2) + 2^{n+1}n!\beta^{-n}L_{n-1}^{(\beta^2/2+1)}(-\beta x + \beta^2/2)\right] = 0
\end{aligned}$$

or

$$4n(n-1)H_{n-2}^\star(x) - 2x(2nH_{n-1}^\star(x)) + 2nH_n^\star(x) - \frac{1}{\beta}\left[x8n(n-1)H_{n-2}^\star + 4nH_{n-1}^\star(x)\right] = 0$$

Taking the limit of both sides reveals that

$$4n(n-1)\lim_{\beta\to\infty} H_{n-2}^\star(x) - 2x(2n\lim_{\beta\to\infty} H_{n-1}^\star(x)) + 2n\lim_{\beta\to\infty} H_n^\star(x) = 0$$

Using the result for the adjoint operator gives us

$$\lim_{\beta\to\infty} H_n^{\star\prime\prime}(x) - 2x\lim_{\beta\to\infty} H_n^{\star\prime}(x) + 2n\lim_{\beta\to\infty} H_n^\star(x) = 0$$

That is, $\lim_{\beta\to\infty} H_n^\star(x)$ satisfies the differential equation for the Hermite polynomials, Eq. (2.2.2.9). Also, observe that, from our definition at the beginning of the proof,

$$H_n^\star(x) = 0 \text{ for } n < 0 \text{ and } H_0^\star(x) = 1.$$

And thus,

$$\lim_{\beta\to\infty} H_n^\star(x) = H_n(x) \tag{2.2.2.10}$$

□

We will now follow another proof using the generating function*.

Proof. Recall that the generating functions for the Hermite and the Laguerre polynomials are

$$(1-t)^{-\alpha-1}\mathrm{e}^{\frac{xt}{t-1}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n \tag{2.2.2.11}$$

and

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \tag{2.2.2.12}$$

Insert $2^n \beta^{-n} L_n^{(\beta^2/n)}(-\beta x + \frac{\beta^2}{2})$ into Eq. (2.2.2.11) to get

$$\sum_{n=0}^{\infty} L_n^{(\beta^2/n)} \left(-\beta x + \frac{\beta^2}{2}\right) \left(\frac{2t}{\beta}\right)^n = \left(1 - \frac{2t}{\beta}\right)^{-(\beta^2/2+1)} \exp\left(\frac{\left(-\beta x + \frac{\beta^2}{2}\right)\left(\frac{2t}{\beta}\right)}{\frac{2t}{\beta} - 1}\right) \tag{2.2.2.13}$$

Now, focusing on the r.h.s we can change our base to get

$$\exp\left(-(\beta^2/2+1)\ln\left(1 - \frac{2t}{\beta}\right) - \frac{\left(-\beta x + \frac{\beta^2}{2}\right)\left(\frac{2t}{\beta}\right)}{1 - \frac{2t}{\beta}}\right)$$

Now, we have the Taylor Series relation:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

which will will use on $\ln(1 - 2t/\beta)$. We will also use the binomial theorem, Eq. (1.6.1), on $(1 - 2t/\beta)^{-1}$. Inserting and distributing yields

$$\exp\left(-(\beta^2/2+1)\left(\sum_{n=1}^{\infty}\left(\frac{2t}{\beta}\right)^n n^{-1}\right) - \left(\sum_{n=1}^{\infty}\frac{(1)_n}{n!}\left(\frac{2t}{\beta}\right)^{n-1}\right)\left(-\beta x + \frac{\beta^2}{2}\right)\left(\frac{2t}{\beta}\right)\right)$$
$$= \exp\left[\sum_{n=1}^{\infty}\left(\frac{\beta^2}{2n} + \frac{1}{n} + \beta x - \frac{\beta^2}{2}\right)\left(\frac{2t}{\beta}\right)^n\right]$$

Getting a common denominator and isolating the terms that will remain non-zero after taking the limit yields

$$= \exp\left[\sum_{n=1}^{\infty}\left(\frac{nx\beta + \frac{\beta^2}{2}(1-n) + 1}{\beta^n n}\right)(2t)^n\right]$$
$$= \exp\left[\frac{\beta x + 1}{\beta}2t + \frac{2\beta x - \beta^2/2 + 1}{2\beta^2}4t^2 + \sum_{n=3}^{\infty}\left(\frac{nx\beta + \frac{\beta^2}{2}(1-n) + 1}{\beta^n n}\right)(2t)^n\right]$$

Taking the limit as $\beta \to \infty$ leaves only $2xt - t^2$ in the exponent, and we get

$$\lim_{\beta\to\infty}\sum_{n=0}^{\infty} L_n^{(\beta^2/n)}\left(-\beta x + \frac{\beta^2}{2}\right)\left(\frac{2t}{\beta}\right)^n$$
$$= \lim_{\beta\to\infty}\exp\left[\frac{\beta x + 1}{\beta}2t + \frac{2\beta x - \beta^2/2 + 1}{2\beta^2}4t^2 + \sum_{n=3}^{\infty}\left(\frac{nx\beta + \frac{\beta^2}{2}(1-n) + 1}{\beta^n n}\right)(2t)^n\right] \tag{2.2.2.14}$$
$$= \exp\left(2xt - t^2\right) = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n$$

Thus, by comparing coefficients, we get

$$\lim_{\beta\to\infty} 2^n n!\beta^{-n} L_n^{(\beta^2/2)}(-\beta x + \beta^2/2) = H_n(x)$$

□

Problem 5.

$$\boxed{\lim_{\alpha\to\infty} \alpha^{-n} L_n^{(\alpha)}(\alpha x) = \frac{(1-x)^n}{n!}}$$

Just like with Problem 4, this problem was listed within Askey's classification of orthogonal polynomials, and there is a combinatorial proof in [10].

Proof. From the definition of the Laguerre polynomials, Eq. (2.1.3.1), and from Eq. (2.1.7), we have

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\sum_{k=0}^{\infty} \frac{(-n)_k x^k}{k!} \frac{(\alpha+1)_n}{(\alpha+1)_k} = \frac{1}{n!}\sum_{k=0}^{\infty} \frac{(-n)_k x^k}{k!} (\alpha+1+k)_{n-k} \tag{2.2.2.15}$$

Recall that the binomial theorem in pochhammer notation is

$$\sum_{k=0}^{\infty} \frac{(-n)_k x^k}{k!} = (1-x)^n \tag{2.2.2.16}$$

From Eq. (2.2.2.15), we have

$$\alpha^{-n} L_n^{(\alpha)}(\alpha x) = \frac{1}{n!}\sum_{k=0}^{\infty} \frac{(-n)_k x^k}{k!} \frac{(\alpha+1+k)_{n-k}}{\alpha^{n-k}}$$

Now, when distributed

$$(\alpha+1+k)_{n-k} = \alpha^{n-k} + O(\alpha^{n-k-1}).$$

So,

$$\lim_{\alpha\to\infty} \frac{(\alpha+1+k)_{n-k}}{\alpha^{n-k}} = \lim_{\alpha\to\infty} \frac{\alpha^{n-k} + O(\alpha^{n-k-1})}{\alpha^{n-k}} = 1$$

And we have, putting it all together

$$\begin{aligned} \lim_{\alpha\to\infty} \alpha^{-n} L_n^{(\alpha)}(\alpha x) &= \frac{1}{n!}\sum_{k=0}^{\infty} \frac{(-n)_k x^k}{k!} \lim_{\alpha\to\infty} \frac{(\alpha+1+k)_{n-k}}{\alpha^{n-k}} \\ &= \frac{1}{n!}\sum_{k=0}^{\infty} \frac{(-n)_k x^k}{k!} = \frac{(1-x)^n}{n!} \end{aligned} \tag{2.2.2.17}$$

□

2.2.3 Integral Relations

We show here two integral relations involving the Hermite and Laguerre polynomials.

Problem 6. *Let $L_n^{(0)}(x) = L_n(x)$, then*

$$2^{2n}(3/2)_n(-1)^n \int_0^t L_n(u(t-u))du = H_{2n+1}(t/2) \tag{2.2.3.1}$$

or, by letting t=2x,

$$\boxed{2^{2n}(3/2)_n(-1)^n \int_0^{2x} L_n(u(2x-u))du = H_{2n+1}(x)} \tag{2.2.3.2}$$

Proof. From the definition of the Laguerre polynomials, 2.1.3.1,

$$\begin{aligned}
&2^{2n}(3/2)_n(-1)^n \int_0^{2x} L_n(u(2x-u))\mathrm{d}u \\
&= 2^{2n}(3/2)_n(-1)^n \sum_{k=0}^{\infty} \frac{(-n)_k}{k!}\frac{1}{k!}\int_0^{2x} (2xu-u)^k \,\mathrm{d}u \\
&= 2^{2n}(3/2)_n(-1)^n \sum_{k=0}^{\infty} \frac{(-n)_k}{k!}\frac{2^k x^k}{k!}\int_0^{2x} u^k \left(1-\frac{u}{2x}\right)^k \mathrm{d}u
\end{aligned}$$

Now, let $r = u/2x$. Our integral becomes

$$\begin{aligned}
&2^{2n}(3/2)_n(-1)^n \int_0^{2x} L_n(u(2x-u))\mathrm{d}u \\
&\qquad = 2^{2n}(3/2)_n(-1)^n \sum_{k=0}^{\infty} \frac{(-n)_k}{k!}\frac{2^{2k+1}x^{2k+1}}{k!}\int_0^{1} r^k (1-r)^k \,\mathrm{d}r
\end{aligned}$$

Recall the Beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad \mathrm{Re}\, x > 0,\ \mathrm{Re}\, y > 0 \tag{2.2.3.3}$$

So, the integral within the summation is simply

$$B(k+1,k+1) = \Gamma(k+1)^2/\Gamma(2k+2) = (k!)^2/(2k+1)!.$$

Thus, our integral is, using the duplication formula, Eq. (2.1.5) in the third line below,

$$\begin{aligned}
2^{2n}(3/2)_n(-1)^n \int_0^{2x} L_n(u(2x-u))\mathrm{d}u &= 2^{2n}(3/2)_n(-1)^n \sum_{k=0}^{\infty} \frac{(-n)_k}{(2k+1)!}2^{2k+1}x^{2k+1} \\
&= 2^{2n+1}(3/2)_n(-1)^n x \sum_{k=0}^{\infty} \frac{(-n)_k}{2(k+1/2)(2k)!}2^{2k}x^{2k} \\
&= 2^{2n+1}(3/2)_n(-1)^n x \sum_{k=0}^{\infty} \frac{(-n)_k}{2(k+1/2)(1/2)_k k!}x^{2k}
\end{aligned}$$

Now, observe that, from, Eq. (2.1.4),

$$\left(\frac{3}{2}\right)_k = \frac{\Gamma(\frac{3}{2}+k)}{\Gamma(3/2)} = \frac{(\frac{1}{2}+k)\Gamma(\frac{1}{2}+k)}{\frac{1}{2}\Gamma(1/2)} = 2\left(\frac{1}{2}\right)_k\left(\frac{1}{2}+k\right)$$

Our integral is then, after also multiplying by $n!/n!$,

$$\begin{aligned} 2^{2n}(3/2)_n(-1)^n\int_0^{2x} L_n(u(2x-u))\mathrm{d}u &= (-1)^n 2^{2n+1} xn!\frac{(3/2)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k}{(3/2)_k k!}x^{2k} \\ &= (-1)^n 2^{2n+1} n! x L_n^{(1/2)}(x^2) \\ &= H_{2n+1}(x) \end{aligned} \tag{2.2.3.4}$$

where the definition of the Hermite polynomials in terms of the Laguerre defined previously, Eq. (2.1.3.5) was used. Thus, the theorem is true. □

Problem 7. *Let $L_n^{(0)}(x) = L_n(x)$, then*

$$\frac{(-1)^n}{\pi 2^{2n}(1/2)^n}\int_0^t \frac{H_{2n}\left(\sqrt{u(t-u)}\right)}{\sqrt{u(t-u)}}du = L_n(t^2/4) \tag{2.2.3.5}$$

or, by letting t=2x,

$$\boxed{\frac{(-1)^n}{\pi 2^{2n}(1/2)^n}\int_0^{2x} \frac{H_{2n}\left(\sqrt{u(2x-u)}\right)}{\sqrt{u(2x-u)}}du = L_n(x^2)} \tag{2.2.3.6}$$

Proof. We start be writing the Hermite polynomials in terms of the Laguerre, using their definition from Eq. (2.1.3.5). And then, we insert the definition of the Laguerre polynomials, Eq. (2.1.3.1).

$$\begin{aligned} \frac{(-1)^n}{\pi 2^{2n}(1/2)^n}\int_0^{2x} \frac{H_{2n}\left(\sqrt{u(2x-u)}\right)}{\sqrt{u(2x-u)}}\mathrm{d}u &= \frac{n!}{(1/2)^n\pi}\int_0^{2x}\frac{L_n^{(-1/2)}\left(\sqrt{u(2x-u)}\right)}{\sqrt{u(2x-u)}}\mathrm{d}u \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-n)_k}{(1/2)_k k!}\int_0^{2x}(2xu-u^2)^{k-1/2}\mathrm{d}u \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-n)_k(2x)^{k-1/2}}{(1/2)_k k!}\int_0^{2x}u^{k-1/2}\left(1-\frac{u}{2x}\right)^{k-1/2}\mathrm{d}u \\ &\text{letting } r = u/2x \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-n)_k(2x)^{2k}}{(1/2)_k k!}\int_0^1 r^{k-1/2}(1-r)^{k-1/2}\mathrm{d}r \end{aligned}$$

Recall the Beta function, Eq. (2.2.3.3) Thus, the integral within the summation is simply $B(k+1/2, k+1/2) = \Gamma(k+1/2)^2/\Gamma(2k+1)$. Thus, our integral is

$$\begin{aligned} \frac{(-1)^n}{\pi 2^{2n}(1/2)^n}\int_0^{2x} \frac{H_{2n}\left(\sqrt{u(2x-u)}\right)}{\sqrt{u(2x-u)}}\mathrm{d}u &= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-n)_k(2x)^{2k}}{(1/2)_k k!}\frac{\Gamma(k+1/2)^2}{\Gamma(2k+1)} \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-n)_k(2x)^{2k}}{k!}\frac{\Gamma(1/2)\Gamma(k+1/2)}{\Gamma(2k+1)} \end{aligned}$$

The Gamma Duplication formula, Eq. (2.1.5), tells us that

$$\frac{\Gamma(1/2)}{\Gamma(k)} = \frac{2^{2k}\Gamma(k+1/2)}{\Gamma(2k)2}$$

which implies

$$\frac{\Gamma(1/2)}{k!} = \frac{2^{2k}\Gamma(k+1/2)}{\Gamma(2k+1)}$$

So, we now have

$$\begin{aligned}\frac{(-1)^n}{\pi 2^{2n}(1/2)^n}\int_0^{2x}\frac{H_{2n}\left(\sqrt{u(2x-u)}\right)}{\sqrt{u(2x-u)}}\mathrm{d}u &= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-n)_k x^{2k}}{(k!)^2}\Gamma(1/2)^2 \\ &= \frac{(1)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k x^{2k}}{(1)_k k!} \\ &= L_n(x^2)\end{aligned} \tag{2.2.3.7}$$

Thus, we have shown the theorem is true. □

We will now present another proof* of this integral relation, using the Hermite polynomials defined in terms of the Laguerre polynomials.

Proof. Recall from Eq. (2.1.3.5), that the Hermite polynomials are defined in terms of the Laguerre polynomials. So we can write

$$\begin{aligned}H_{2n}\left(\sqrt{x(t-x)}\right) &= (-1)^n 2^{2n} n! L_n^{(-1/2)}(x(t-x)) \\ &= (-1)^n 2^{2n}\left(\frac{1}{2}\right)_n\sum_{k=0}^{\infty}\frac{(-n)_k}{(1/2)_k\, k!}x^k(t-x)^k\end{aligned} \tag{2.2.3.8}$$

inserting into our integral yields

$$\begin{aligned}\int_0^t\frac{H_{2n}\left(\sqrt{x(t-x)}\right)}{\sqrt{x(t-x)}}\mathrm{d}x &= (-1)^n 2^{2n}\left(\frac{1}{2}\right)_n\sum_{k=0}^{\infty}\frac{(-n)_k}{(1/2)_k\, k!}\int_0^t x^{k-1/2}(t-x)^{k-1/2}\mathrm{d}x \\ &= (-1)^n 2^{2n}\left(\frac{1}{2}\right)_n\sum_{k=0}^{\infty}\frac{(-n)_k}{(1/2)_k\, k!}t^{k-1/2}\int_0^t x^{k-1/2}\left(1-\frac{x}{t}\right)^{k-1/2}\mathrm{d}x \\ &= (-1)^n 2^{2n}\left(\frac{1}{2}\right)_n\sum_{k=0}^{\infty}\frac{(-n)_k}{(1/2)_k\, k!}t^{2k}\int_0^1 u^{k-1/2}(1-u)^{k-1/2}\mathrm{d}u \\ &= (-1)^n 2^{2n}\left(\frac{1}{2}\right)_n\sum_{k=0}^{\infty}\frac{(-n)_k}{(1/2)_k\, k!}t^{2k}\frac{\Gamma(k+1/2)^2}{\Gamma(2k+1)}\end{aligned}$$

where we first used a u-sub of $u = x/t$ and then the Beta function, Eq. (1.9.1.1), of the form $B(k+1/2, k+1/2) = \frac{\Gamma(k+1/2)^2}{\Gamma(2k+1)}$. Now we will multiply by $\frac{2^{2k}k!}{2^{2k}k!}$, use Eq. (2.1.4) in the form of

$\Gamma(k+1/2) = (1/2)_k\Gamma(1/2)$ and then use the duplication formula, Eq. (2.1.5),

$$\begin{aligned}
\int_0^t \frac{H_{2n}\left(\sqrt{x(t-x)}\right)}{\sqrt{x(t-x)}}\mathrm{d}x &= (-1)^n 2^{2n}\left(\frac{1}{2}\right)_n \sum_{k=0}^{\infty} \frac{k!2^{2k}(-n)_k}{(1/2)_k\, k!k!}\left(\frac{t}{2}\right)^{2k} \frac{\Gamma(k+1/2)^2}{\Gamma(2k+1)} \\
&= (-1)^n 2^{2n}\pi\left(\frac{1}{2}\right)_n \sum_{k=0}^{\infty} \frac{k!2^{2k}\,(1/2)_k\,(-n)_k}{\Gamma(2k+1)k!k!}\left(\frac{t}{2}\right)^{2k} \\
&= (-1)^n 2^{2n}\pi\left(\frac{1}{2}\right)_n \sum_{k=0}^{\infty} \frac{(2k)!(-n)_k}{\Gamma(2k+1)k!k!}\left(\frac{t}{2}\right)^{2k} \\
&= (-1)^n 2^{2n}\pi\left(\frac{1}{2}\right)_n \sum_{k=0}^{\infty} \frac{(-n)_k}{k!k!}\left(\frac{t}{2}\right)^{2k} \\
&= (-1)^n 2^{2n}\pi\left(\frac{1}{2}\right)_n L^{(0)}(t^2/4)
\end{aligned}$$

thus, we have proven our integral relation. □

Our final integral relation relates the Legendre polynomials and the Hermite polynomials.

Problem 8. *Let $P_n(x)$ denote the nth Legendre polynomial, then*

$$\boxed{P_n(x) = \frac{2}{n!\sqrt{\pi}}\int_0^{\infty} e^{-t^2}t^n H_n(xt)dt} \tag{2.2.3.9}$$

Proof. Let

$$P_n^{\star}(x) = \frac{2}{n!\sqrt{\pi}}\int_0^{\infty} \mathrm{e}^{-t^2}t^n H_n(xt)\mathrm{d}t.$$

Our goal is to show that $P_n^{\star}(x) = P_n(x)$, the nth Legendre polynomials. Please free free to review the Legendre polynomials in the introduction, Section 1.8.

For $H_n(xt)$, we have the modified self-adjoint and recursion relations

$$\begin{aligned}
2nH_{n-1}(xt) &= 2xtH_n(xt) - H_{n+1}(xt) \\
\frac{\mathrm{d}}{\mathrm{d}x}H_n(xt) &= 2ntH_{n-1}(xt)
\end{aligned}$$

Now,

$$\begin{aligned}
P_{n-1}^{\star}(x) &= \frac{2n}{n!\sqrt{\pi}}\int_0^{\infty} \mathrm{e}^{-t^2}t^{n-1}H_{n-1}(xt)\mathrm{d}t \\
(n+1)P_{n+1}^{\star}(x) &= \frac{2}{n!\sqrt{\pi}}\int_0^{\infty} \mathrm{e}^{-t^2}t^{n+1}H_{n+1}(xt)\mathrm{d}t
\end{aligned} \tag{2.2.3.10}$$

Using the recursion relation on $P^\star_{n-1}(x)$ yields

$$\begin{aligned}
P^\star_{n-1}(x) &= \frac{2x}{n!\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2}t^n H_n(xt)\mathrm{d}t - \frac{1}{n!\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2}t^{n-1}H_{n+1}(xt)\mathrm{d}t \\
&= xP^\star_n(x) - \frac{1}{n!\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2}t^{n-1}H_{n+1}(xt)\mathrm{d}t
\end{aligned}$$

so

$$nxP^\star_n(x) = nP^\star_{n-1}(x) + \frac{n}{n!\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2}t^{n-1}H_{n+1}(xt)\mathrm{d}t$$

We will now do an integration by parts, with $u = \mathrm{e}^{-t^2}H_{n+1}$. We get

$$\begin{aligned}
nxP^\star_n(x) = nP^\star_{n-1}(x) + \left.\frac{t^n\mathrm{e}^{-t^2}H_{n+1}}{n!\sqrt{\pi}}\right|_0^\infty - (n+1)x\frac{2}{n!\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2}t^n H_n(xt)\mathrm{d}t \\
+(n+1)\frac{2}{(n+1)!\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2}t^{n+1}H_{n+1}(xt)\mathrm{d}t \\
= nP^\star_{n-1}(x) - (n+1)xP^\star_n(x) + (n+1)P^\star_{n+1}(x)
\end{aligned}$$

where the term that needed evaluated goes to zero at both limits due to the decaying exponential and the t^n component. Simplifying get us

$$(2n+1)xP^\star_n(x) = (n+1)P^\star_{n+1}(x) + nP^\star_{n-1}(x) \tag{2.2.3.11}$$

This is the recursion relation for the Legendre polynomials. However, we must first check beginning values of n for $P^\star_n(x)$, to ensure that they match that for the Legendre polynomials. Since $H_n(x) = 0$ for every $n < 0$, $P^\star_n(x) = 0$ for every $n < 0$. This matches the Legendre, so must now must check $P^\star_0(x)$.

$$P^\star_0(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2}\mathrm{d}t = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\mathrm{e}^{-t}\mathrm{d}t = \frac{\Gamma(1/2)}{\sqrt{\pi}} = 1$$

This matches the Legendre polynomials. Thus,

$$P^\star_n(x) = P_n(x)$$

. □

We will now show another proof* using a connection between the Legendre and Ultraspherical polynomials.

Proof. We start by reminding ourselves of the power series representation, Eq. (2.1.1.4) of the Legendre polynomials, derived from its Rodrigues formula in Section 2.1.1

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k(2x)^{n-2k}\left(\frac{1}{2}\right)_{n-k}}{k!(n-2k)!}$$

which is also, of course, $C_n^{(1/2)}(x)$, as shown in Problem 2. We begin by using the series representation of the Hermite polynomials derived in Section 2.1.3

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!(-1)^k (2x)^{n-2k}}{k!(n-2k)!} \tag{2.2.3.12}$$

We insert this into our integral, Eq. (2.2.3.9), to get

$$\begin{aligned}
\frac{2}{n!\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-t^2} t^n H_n(xt) \mathrm{d}t &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2(-1)^k (2x)^{n-2k}}{\sqrt{\pi} k!(n-2k)!} \int_0^\infty \mathrm{e}^{-t^2} t^{2n-2k} \mathrm{d}t \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{\sqrt{\pi} k!(n-2k)!} \int_0^\infty \mathrm{e}^{-u} u^{n-k-1/2} \mathrm{d}u \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{\sqrt{\pi} k!(n-2k)!} \Gamma(n-k+1/2) \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k} \left(\frac{1}{2}\right)_{n-k}}{k!(n-2k)!}
\end{aligned}$$

where a u-sub of $u = t^2$ was used to turn the integral into the Gamma function, and then the pochhammer Gamma relation, Eq. (2.1.4), in the form of $(1/2)_{n-k} = \Gamma(n-k+1/2)/\Gamma(1/2) = \Gamma(n-k+1/2)/\sqrt{\pi}$. Thus,

$$\frac{2}{n!\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-t^2} t^n H_n(xt) \mathrm{d}t = P_n(x) \tag{2.2.3.13}$$

□

Properties of orthogonal polynomial sequences occur in various contexts, and we shall explore some of these in the next section.

Chapter 3

Methods on Orthogonal Polynomial Sequences

3.1 Inverse Method

The inverse problem in the study of orthogonal polynomial sequences (OPS) is the attempt to obtain the weight function of an OPS given only the recursion formula. As a by-product of this process, the generating function is obtained. We will focus purely on obtaining the generating functions here. We first start with an example.

3.1.1 Fibonacci Sequence

Recall the Fibonacci sequence,

$$1, 1, 2, 3, 5, 8, 13, \ldots \tag{3.1.1.1}$$

We would like to find a generating function for the Fibonacci sequence; an infinite polynomial where the coefficient of t^n is the nth Fibonacci number F_n. That is, a power series of the form

$$1 + t + 2t^2 + 3t^3 + 5t^4 + 13t^5 + \cdots + F_n t^n + \cdots = \sum_{n=0}^{\infty} F_n t^n \tag{3.1.1.2}$$

We start by defining a function $I(t)$

$$I(t) = \sum_{n=0}^{\infty} F_n t^n \tag{3.1.1.3}$$

Now, we are going to use the recursion relation of the Fibonacci numbers to find the generating function. Recall,

$$F_{n+1} = F_n + F_{n-1}, \quad F_0 = 1,\ F_{-1} = 0 \tag{3.1.1.4}$$

Observe that, from Eq. (3.1.1.3),

$$tI(t) = \sum_{n=0}^{\infty} F_n t^{n+1} = \sum_{n=1}^{\infty} F_{n-1} t^n = \sum_{n=0}^{\infty} F_{n-1} t^n \tag{3.1.1.5}$$

where, in the first step, we shifted the index, and, in the second, we used the fact that $F_{-1} = 0$ to pull the sum back to $n = 0$ with impunity. Now, multiply the recursion relation (Eq. (3.1.1.4)) through by t^n and sum from $n = 0$ to ∞ to get

$$\sum_{n=0}^{\infty} F_{n+1} t^n = \sum_{n=0}^{\infty} F_n t^n + \sum_{n=0}^{\infty} F_{n-1} t^n = I(t) + tI(t) \tag{3.1.1.6}$$

where we inserted Eq. (3.1.1.3) and Eq. (3.1.1.5) to obtain the far right of Eq. (3.1.1.6). To take care of the far left, we return to the definition of our function, observing that we can shift the index of Eq. (3.1.1.3) to get

$$I(t) = \sum_{n=-1}^{\infty} F_{n+1} t^{n+1} = 1 + \sum_{n=0}^{\infty} F_{n+1} t^{n+1} \tag{3.1.1.7}$$

(write out the first few terms to convince yourself that the far left of Eq. (3.1.1.7) is the same as the original definition). So we can solve Eq. (3.1.1.7) to get

$$\sum_{n=0}^{\infty} F_{n+1} t^{n+1} = I(t) - 1$$

which upon inserting into Eq. (3.1.1.6) yields

$$I(t) - 1 = tI(t) + t^2 I(t)$$

We can then re-arrange to solve for $I(t)$, which is the generating function of the Fibonacci sequence.

$$I(t) = \frac{1}{1 - (t^2 + t)} = \sum_{n=0}^{\infty} F_n t^n \tag{3.1.1.8}$$

Now that we understand the concept of the inversion method, lets apply it to orthogonal polynomial sequences.

3.1.2 Charlier Polynomials

Recall the Charlier Polynomials have the definition

$$C_n(x;a) := {}_2F_0\left(\begin{matrix} -n,-x \\ - \end{matrix} \middle| -\frac{1}{a} \right) = \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{k!} \left(-\frac{1}{a} \right)^k \tag{3.1.2.1}$$

with recurrence relation

$$-xC_n(x;a) = aC_{n+1}(x;a) - (n+a)C_n(x;a) + nC_{n-1}(x;a) \tag{3.1.2.2}$$

The generating function can be derived from the definition. First, notice

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} C_n(x;a) t^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_k(-x)_k(-1)^k}{k!a^k n!} t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n+k-1)!(-x)_k(-1)^k}{k!a^k n!(-n-1)!} t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-x)_k t^n}{k!(n-k)!a^k} \end{aligned} \tag{3.1.2.3}$$

Shifting the starting index of Eq. (3.1.2.3) yields

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-x)_k t^{n+k}}{k!n!a^k} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-x)_k \frac{t^k}{a^k}}{k!} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-x)_k (\frac{t}{a})^k}{k!} \\ &= e^t {}_1F_0\left(\begin{matrix} -x \\ - \end{matrix} \middle| -\frac{t}{a} \right) \\ &= e^t \left(1 - \frac{t}{a} \right)^x \end{aligned} \tag{3.1.2.4}$$

Which is the generating function for the Charlier polynomials. The inverse method can also be used to generate this function. First, multiple both sides of Eq. (3.1.2.2) by $t^n/n!$ and sum from -1 to ∞, reminding ourselves that $C_n = 0, \forall n \in \mathbf{Z}^-$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{-xC_n(x;a)t^n}{n!} = &\sum_{n=-1}^{\infty} \frac{aC_{n+1}(x;a)t^n}{n!} \\ &- \sum_{n=0}^{\infty} \frac{(n+a)C_n(x;a)t^n}{n!} \\ &+ \sum_{n=1}^{\infty} \frac{nC_{n-1}(x;a)t^n}{n!}\end{aligned} \tag{3.1.2.5}$$

Define a function $F(x;t,a) = F$ such that

$$F = \sum_{n=0}^{\infty} \frac{C_n(x;a)t^n}{n!}. \tag{3.1.2.6}$$

Letting $\frac{\mathrm{d}F}{\mathrm{d}t} = \dot{F}$, we have

$$\dot{F} = \sum_{n=0}^{\infty} \frac{C_n(x;a)t^{n-1}}{(n-1)!} = \sum_{n=-1}^{\infty} \frac{C_{n+1}(x;a)t^n}{n!}. \tag{3.1.2.7}$$

Inserting Eq. (3.1.2.6) and Eq. (3.1.2.7) into Eq. (3.1.2.5), and doing some algebra yields

$$-xF = a\dot{F} - t\dot{F} - aF + tF, \tag{3.1.2.8}$$

and after re-arranging creates

$$\dot{F} - \left(1 + \frac{x}{t-a}\right)F = 0 \tag{3.1.2.9}$$

This is a linear homogeneous ordinary differential equation, which can be solved using the integration technique. If $P(t) = -1 - \frac{x}{t-a}$, and $\mu = \mathrm{e}^{\int P(t)\mathrm{d}t}$, then

$$\mu = (t-a)^{-x}\mathrm{e}^{-t} \tag{3.1.2.10}$$

Multiplying Eq. (3.1.2.9) by μ, and using the product rule in reverse, reveals that $\frac{\mathrm{d}(\mu F)}{\mathrm{d}t} = 0$. Or, $F = c(t-a)^x\mathrm{e}^t$, where $c = c(x;a)$ is a constant. Using that at $t = 0$, $F(x;a) = 1$, we find that the constant is $(-a)^{-x}$. So the generating function for the Charlier polynomial is thus

$$F(x;t,a) = e^t\left(1 - \frac{t}{a}\right)^x \tag{3.1.2.11}$$

Q.E.D We have derived the generating function for the Charlier polynomials.

3.1.3 Legendre

The Legendre (spherical) polynomials are defined as

$$P_n(x,t) := {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix} \middle| \frac{1-x}{2} \right), \tag{3.1.3.1}$$

with the recursion relation

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{3.1.3.2}$$

First, multiple both sides by t^n and sum from 0 to infinity.

$$\sum_{n=0}^{\infty} 2nxt^n P_n(x) + \sum_{n=0}^{\infty} xt^n P_n(x) = \sum_{n=0}^{\infty}(n+1)t^n P_{n+1}(x) + \sum_{n=0}^{\infty} nt^n P_{n-1}(x) \tag{3.1.3.3}$$

Next define $F = F(x,t)$ such that

$$F = \sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=-1}^{\infty} P_{n+1}(x)t^{n+1} = \sum_{n=1}^{\infty} P_{n-1}(x)t^{n-1} \tag{3.1.3.4}$$

and

$$\dot{F} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} = \sum_{n=-1}^{\infty}(n+1)P_{n+1}(x)t^n = \sum_{n=1}^{\infty}(n-1)P_{n-1}(x)t^{n-2} \tag{3.1.3.5}$$

Notice that

$$\dot{F} = \sum_{n=1}^{\infty}(n-1)P_{n-1}(x)t^{n-2} = \sum_{n=1}^{\infty} nP_{n-1}(x)t^{n-2} - \sum_{n=1}^{\infty} P_{n-1}(x)t^{n-2}, \tag{3.1.3.6}$$

which implies

$$\sum_{n=1}^{\infty} nP_{n-1}(x)t^{n-2} = \dot{F} + \sum_{n=1}^{\infty} P_{n-1}(x)t^{n-2} \tag{3.1.3.7}$$

and

$$\sum_{n=1}^{\infty} nP_{n-1}(x)t^n = \dot{F}t^2 + t\sum_{n=1}^{\infty} P_{n-1}(x)t^{n-1} = \dot{F}t^2 + Ft \tag{3.1.3.8}$$

Inserting Eq. (3.1.3.8), Eq. (3.1.3.4) and Eq. (3.1.3.5) into Eq. (3.1.3.3) yields, after rearranging, the following First-Order Linear homogeneous differential equation.

$$\dot{F} + \frac{t-x}{t^2+1-2xt}F = 0 \tag{3.1.3.9}$$

Letting

$$P(t) = (t-x)/(t^2+1-2xt)$$

and integrating via u-substitution and then setting

$$\mu = \mathrm{e}^{\int P(t)\mathrm{d}t}$$

yields the integrating factor

$$\mu = \sqrt{t^2 + 1 - 2xt} \tag{3.1.3.10}$$

Multiplying both sides of Eq. (3.1.3.9) by $Eq.$ (3.1.3.10) and doing the product rule in reverse reveals

$$\frac{\mathrm{d}[F\mu]}{\mathrm{d}t} = 0, \quad \text{or} \quad F = c(\mu)^{-1} \tag{3.1.3.11}$$

where c is a constant. Using that $F(t = 0) = 1$ reveals that the constant is one, and thus

$$F = F(x, t) = \sum_{n=0}^{\infty} P_n t^n = \frac{1}{\sqrt{1 + t^2 - 2xt}} \tag{3.1.3.12}$$

Q.E.D we have derived the generating function for the Legendre Polynomials.

3.1.4 Ultraspherical

The ultraspherical, or Gegenbauer, polynomials are special cases of the Jacobi polynomials (see Section 1.9 for more on the Jacobi polynomials). They have the definition

$$C_n^\nu(x) := \frac{(2\nu)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+2\nu \\ \nu+1/2 \end{matrix} \middle| \frac{1-x}{2} \right), \ \nu \neq 0 \tag{3.1.4.1}$$

with recursion relation

$$2(n + \nu)xC_n^\nu(x) = (n + 1)C_{n+1}^\nu(x) + (n + 2\nu - 1)C_{n-1}^\nu(x) \tag{3.1.4.2}$$

Multiplying both sides of Eq. (3.1.4.2) by t^n and summing from $n = -1$ to ∞, and using the fact that $C_n^\nu(x) = 0, n < 0$ yields the following relation:

$$\sum_{n=0}^{\infty} 2(n + \nu)xt^n C_n^\nu(x) = \sum_{n=-1}^{\infty} (n + 1)t^n C_{n+1}^\nu(x) + \sum_{n=1}^{\infty} (n + 2\nu - 1)t^n C_{n-1}^\nu(x) \tag{3.1.4.3}$$

Now define a new function

$$F = F(x, t; \nu) = \sum_{n=0}^{\infty} t^n C_n^\nu(x) = \sum_{n=-1}^{\infty} t^{n+1} C_{n+1}^\nu(x) = \sum_{n=1}^{\infty} t^{n-1} C_{n-1}^\nu(x) \tag{3.1.4.4}$$

whose derivative is

$$\begin{aligned} \dot{F} = \frac{\mathrm{d}(F(x, t; \nu))}{\mathrm{d}t} &= \sum_{n=0}^{\infty} nt^{n-1} C_n^\nu(x) \\ &= \sum_{n=-1}^{\infty} (n + 1)t^n C_{n+1}^\nu(x) = \sum_{n=1}^{\infty} (n - 1)t^{n-2} C_{n-1}^\nu(x) \end{aligned} \tag{3.1.4.5}$$

Inserting Eq. (3.1.4.4) and Eq. (3.1.4.5) into Eq. (3.1.4.3) yields the following:

$$2x\dot{F}t + 2x\nu F = \dot{F} + t^2\dot{F} + 2\nu F t. \tag{3.1.4.6}$$

Rearranging Eq. (3.1.4.6) yields the following linear ordinary homogeneous differential equation:

$$\dot{F} + \frac{2t\nu - 2x\nu}{1 + t^2 - 2xt}F = 0. \tag{3.1.4.7}$$

Now,

$$P(x, t; \nu)\mathrm{d}t = \frac{2t\nu - 2x\nu}{1 + t^2 - 2xt}\mathrm{d}t$$

Integrating Using the u-substitution technique, where $u(x, t) = 1 + t^2 - 2xt$, yields

$$\int P(x, t; \nu)\mathrm{d}t = \nu \ln\left(1 + t^2 - 2xt\right)$$

Setting integration factor $\mu = \mu(x, t; \nu)$

$$\mu(x, t; \nu) = \mathrm{e}^{\nu \ln\left(1+t^2-2xt\right)} = (1 + t^2 - 2xt)^\nu$$

multiplying Eq. (3.1.4.7) by μ and using the product rule in reverse reveals that

$$\frac{\mathrm{d}(F\mu)}{\mathrm{d}t} = 0 \tag{3.1.4.8}$$

Integrating both sides, which introduces a constant $c(x; \nu)$ on the r.h.s. of Eq. (3.1.4.8), then dividing by μ yields

$$F = \frac{c(x; \nu)}{\mu}$$

Since $F(x, 0; \nu) = 1 = c(x; \nu)$, we get

$$F(x, t; \nu) = F = \frac{1}{(1 + t^2 - 2xt)^\nu} = \sum_{n=0}^{\infty} C_n^\nu(x)t^n. \tag{3.1.4.9}$$

Q.E.D We have derived the generating function for the Gegenbauer polynomials.

Finally, an observation: Notice that

$$\begin{aligned} F(x, t; \nu = 1) = \frac{1}{1 + t^2 - 2xt} &= \sum_{n=0}^{\infty} C_n^1(x)t^n \\ &= \sum_{n=0}^{\infty} U_n(x)t^n = \frac{\sum_{n=0}^{\infty} T_n(x)t^n}{1 - xt} \end{aligned} \tag{3.1.4.10}$$

where

$$\sum_{n=0}^{\infty} U_n(x)t^n$$

is the generating function for the Chebyshev polynomial of the Second Kind, and

$$\sum_{n=0}^{\infty} T_n(x)t^n$$

is the generating function for the Chebyshev polynomial of the First Kind.

3.1.5 Chebyshev Polynomials

There are two types of Chebyshev polynomials, differing starting at the $n = 1$ term.

1. The Chebyshev Polynomials of the First Kind, $T_n(x)$, are defined as

$$T_n(x) := {}_2F_1\left(\begin{matrix} -n,n \\ 1/2 \end{matrix} \middle| \frac{1-x}{2} \right) = \sum_{k=0}^{n} \frac{(-n)_k (n)_k}{(\frac{1}{2})_k k!} \left(\frac{1-x}{2} \right)^k \tag{3.1.5.1.1}$$

with the condition that $T_0 = 1, T_1 = x$.

For future reference,

$$T_2 = 2x^2 - 1$$

The first kind has the recursion relation

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x) \tag{3.1.5.1.2}$$

This becomes, after multiplying through by t^n and summing from $n = 2$ to infinity,

$$\sum_{n=2}^{\infty} 2xT_n(x)t^n = \sum_{n=2}^{\infty} T_{n+1}(x)t^n + \sum_{n=2}^{\infty} T_{n-1}(x)t^n \tag{3.1.5.1.3}$$

Letting $F(x,t) = F$, and using the ability to shift our sum to different starting values, we have the following relation

$$\begin{aligned} F &= \sum_{n=0}^{\infty} T_n(x)t^n = T_0(x) + T_1(x)t + \sum_{n=2}^{\infty} T_n(x)t^n \\ &= 1 + xt + \sum_{n=2}^{\infty} T_n(x)t^n \end{aligned} \tag{3.1.5.1.4}$$

Shifting the sum, we also find that

$$\begin{aligned} F &= \sum_{n=-1}^{\infty} T_{n+1}(x)t^{n+1} = T_0(x) + T_1(x)t + T_2(x)t^2 + \sum_{n=2}^{\infty} T_{n+1}(x)t^{n+1} \\ &= 1 + xt + 2x^2t^2 - t^2 + \sum_{n=2}^{\infty} T_{n+1}(x)t^{n+1} \end{aligned} \tag{3.1.5.1.5}$$

and

$$F = \sum_{n=1}^{\infty} T_{n-1}(x)t^{n-1} = T_0(x) + \sum_{n=2}^{\infty} T_{n-1}(x)t^{n-1} = 1 + \sum_{n=2}^{\infty} T_{n-1}(x)t^{n-1} \tag{3.1.5.1.6}$$

Inserting Eq. (3.1.5.1.4), Eq. (3.1.5.1.5), and Eq. (3.1.5.1.6) into Eq. (3.1.5.1.3), we get

$$2xtF - 2xt - 2x^2t^2 = F - 1 - xt - 2x^2t^2 - t^2 + Ft^2 + t^2 \tag{3.1.5.1.7}$$

After doing some algebra, we find that

$$F(x,t) = F = \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n \tag{3.1.5.1.8}$$

Q.E.D We have derived the generating function for the Chebyshev Polynomials of the First Kind.

2. The Chebyshev Polynomials of the Second Kind, $U_n(x)$, are defined as

$$\begin{aligned} U_n(x) &:= (n+1)\,{}_2F_1\left(\begin{matrix} -n, n+2 \\ 3/2 \end{matrix} \middle| \frac{1-x}{2} \right) \\ &= \sum_{k=0}^{n} \frac{(n+1)(-n)_k(n+2)_k}{(\frac{3}{2})_k k!} \left(\frac{1-x}{2}\right)^k, \end{aligned} \tag{3.1.5.2.1}$$

with the condition that $U_0 = 1, U_1 = 2x$. The second kind has the recursion relation

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x). \tag{3.1.5.2.2}$$

This becomes, after multiplying through by t^n and summing from $n = 0$ to infinity,

$$\sum_{n=0}^{\infty} 2xU_n(x)t^n = \sum_{n=0}^{\infty} U_{n+1}(x)t^n + \sum_{n=0}^{\infty} U_{n-1}(x)t^n \tag{3.1.5.2.3}$$

Letting $F(x,t) = F$, and using the ability to shift our sum to different starting values, we have the following relation

$$\begin{aligned} F = \sum_{n=0}^{\infty} U_n(x)t^n &= U_0(x) + \sum_{n=1}^{\infty} U_n(x)t^n \\ &= 1 + \sum_{n=0}^{\infty} U_{n+1}(x)t^{n+1} = \sum_{n=1}^{\infty} U_{n-1}(x)t^{n-1} \end{aligned} \tag{3.1.5.2.4}$$

Inserting Eq. (3.1.5.2.4) into Eq. (3.1.5.2.3) yields

$$2xtF = F - 1 + Ft^2 \tag{3.1.5.2.5}$$

After solving for F, we find that

$$F(x,t) = F = \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n \tag{3.1.5.2.6}$$

Q.E.D We have derived the generating function for the Chebyshev Polynomials of the Second Kind.

3.1.6 Meixner-Pollaczek polynomials

The Meixner-Pollaczek polynomials are a Sheffer Type-0 OPS (see Section 1.8 for more details), defined as

$$P_n^{\lambda}(x;\phi) := \frac{(2\lambda)_n e^{in\phi}}{n!} {}_2F_1\left(\begin{matrix} -n, \lambda+ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\phi} \right) \tag{3.1.6.1}$$

with the recurrence relation

$$\begin{aligned}(n+1)P_{n+1}^{\lambda}(x;\phi) =& 2[x\sin\phi + (n+\lambda)\cos\phi]P_n^{\lambda}(x;\phi) \\ &- (n+2\lambda-1)P_{n-1}(x;\phi)\end{aligned} \tag{3.1.6.2}$$

Letting

$$P_n = P_n^{\lambda}(x;\phi)$$

and multiplying both sides of Eq. (3.1.6.2) by t^n, distributing and summing from -1 to ∞ yields the following relation:

$$\begin{aligned}\sum_{n=-1}^{\infty}(n+1)t^n P_{n+1} =& 2x\sin\phi \sum_{n=0}^{\infty} P_n t^n + 2\cos\phi \sum_{n=0}^{\infty} nP_n t^n \\ &+ 2\lambda\cos\phi \sum_{n=0}^{\infty} P_n t^n - \sum_{n=1}^{\infty} nP_{n-1}t^n \\ &- 2\lambda \sum_{n=1}^{\infty} t^n P_{n-1} + \sum_{n=1}^{\infty} t^n P_{n-1} = 0\end{aligned} \tag{3.1.6.3}$$

Create a function $H = H(x,t;\lambda,\phi)$ such that:

$$H = \sum_{n=0}^{\infty} P_n t^n = \sum_{n=-1}^{\infty} P_{n+1}t^{n+1} = \sum_{n=1}^{\infty} P_{n-1}t^{n-1} \tag{3.1.6.4}$$

and whose derivative is

$$\dot{H} = \sum_{n=0}^{\infty} nP_n t^{n-1} = \sum_{n=-1}^{\infty}(n+1)P_{n+1}t^n = \sum_{n=1}^{\infty}(n-1)P_{n-1}t^{n-2} \tag{3.1.6.5}$$

Notice that Eq. (3.1.6.5) yields

$$\dot{H} = \sum_{n=1}^{\infty} nP_{n-1}t^{n-2} - \sum_{n=1}^{\infty} P_{n-1}t^{n-2} \tag{3.1.6.6}$$

which becomes, after multiplying both sides by t^2,

$$\dot{H}t^2 = \sum_{n=1}^{\infty} nP_{n-1}t^n - \sum_{n=1}^{\infty} P_{n-1}t^n = \sum_{n=1}^{\infty} nP_{n-1}t^n - Ht \tag{3.1.6.7}$$

which implies

$$\sum_{n=1}^{\infty} nP_{n-1}t^n = \dot{H}t^2 + tH \tag{3.1.6.8}$$

Putting together Eq. (3.1.6.8), Eq. (3.1.6.4), Eq. (3.1.6.5) and Eq. (3.1.6.3), and doing some algebra, reveals the following first order linear homogeneous ordinary differential equation:

$$\dot{H} + 2\frac{\lambda(t - \cos\phi) - x\sin\phi}{1 - 2\cos\phi t + t^2}H = 0. \tag{3.1.6.9}$$

Using the identities

$$\cos(\phi) := \frac{\mathrm{e}^{i\phi} + \mathrm{e}^{-i\phi}}{2}$$

and

$$\sin(\phi) := \frac{\mathrm{e}^{i\phi} - \mathrm{e}^{-i\phi}}{2i}$$

we can re-write Eq. (3.1.6.9) as

$$\dot{H} + \frac{(-\lambda + ix)(\mathrm{e}^{i\phi}) + (-\lambda - ix)(\mathrm{e}^{-i\phi}) + 2t\lambda}{1 - (\mathrm{e}^{i\phi} + \mathrm{e}^{-i\phi})t + t^2}H = 0 \tag{3.1.6.10}$$

Define a function $P(x, t; \phi, \lambda)$ as the coefficients of the second term of the l.h.s in Eq. (3.1.6.10), and integrate:

$$\int P\mathrm{d}t = \int \frac{(-\lambda + ix)(\mathrm{e}^{i\phi}) + (-\lambda - ix)(\mathrm{e}^{-i\phi}) + 2t\lambda}{1 - (\mathrm{e}^{i\phi} + \mathrm{e}^{-i\phi})t + t^2}\mathrm{d}t \tag{3.1.6.11}$$

This is is solvable using partial fraction decomposition.

$$\int P\mathrm{d}t = \int \frac{(-\lambda + ix)\mathrm{e}^{i\phi}\mathrm{d}t}{1 - \mathrm{e}^{i\phi}t} + \int \frac{(-\lambda - ix)\mathrm{e}^{-i\phi}\mathrm{d}t}{1 - \mathrm{e}^{-i\phi}t} \tag{3.1.6.12}$$

Using u-substitution, the integral becomes

$$\int P\mathrm{d}t = -(-\lambda + ix)\ln\left(1 - \mathrm{e}^{i\phi}t\right) - (-\lambda - ix)\ln\left(1 - \mathrm{e}^{-i\phi}t\right) \tag{3.1.6.13}$$

The integrating factor for Eq. (3.1.6.10) is thus

$$\mu = (1 - \mathrm{e}^{i\phi}t)^{-(-\lambda+ix)}(1 - \mathrm{e}^{-i\phi}t)^{-(-\lambda-ix)} \tag{3.1.6.14}$$

Multiplying equation Eq. (3.1.6.10) by μ and doing the product rule in reverse reveals

$$\frac{\mathrm{d}(H\mu)}{\mathrm{d}t} = 0 \quad \text{or} \quad H = c\mu^{-1}$$

where c is a constant. Using the condition $H(t = 0) = 1$, the constant is thus 1 and we have that

$$H = H(n, t, x, \lambda, \phi) = \sum_{n=0}^{\infty} P_n^{\lambda}t^n = (1 - \mathrm{e}^{i\phi}t)^{-\lambda+ix}(1 - \mathrm{e}^{-i\phi}t)^{-\lambda-ix} \tag{3.1.6.15}$$

Q.E.D We have derived the generating function for the Meixner-Pollaczek polynomials.

3.2 Schrödinger Form

The Schrödinger form is an application of orthogonal polynomial sequences to differential equations. Let's jump right into the matter, and the idea of the Schrödinger form will make itself apparent along the way. Recall that an orthogonal polynomial sequence must obey an unrestricted three-term recursion relation of the form

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad P_{-1}(x) = 0, P_0(x) = 1 \tag{3.2.1}$$

If $Q_n(x)$ represents the monic form of $P_n(x)$, then the following monic three term relation must be satisfied.

$$Q_{n+1}(x) = (x - b_n)Q_n(x) - c_n Q_{n-1}(x), Q_{-1}(x) = 0, Q_0(x) = 1 \tag{3.2.2}$$

Also, an orthonormal polynomial sequence must satisfy the following relation:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n + a_n p_{n-1}(x), p_{-1}(x) = 0, p_0(x) = 1 \tag{3.2.3}$$

Since $p_n(x)$ is also monic, we can compare Eq. (3.2.2) and Eq. (3.2.3) to find that $a_n = \sqrt{c_n}$. The relation between $p_n(x)$ and $P_n(x)$ is that $p_n(x)$ is the normalized version of $P_n(x)$, and they are related by the associated squared norm K_n:

$$p_n(x) = \frac{1}{\sqrt{K_n}} P_n(x) \tag{3.2.4}$$

The p_n terms obey an orthonormal relation on some domain Ω, relative to some weight function $w(x)$.

$$\langle p_m(x), p_n(x)\rangle = \int_\Omega p_m(x)p_n(x)w(x)\mathrm{d}x = \delta_{nm} \tag{3.2.5}$$

We now introduce the following equation, followed by a theorem.

$$v(x) = -\ln(w(x)) \tag{3.2.6}$$

Theorem 3.2.1. *The orthonormal forms satisfy the following differential equation:*

$$p_n'(x) = -B_n(x)p_n(x) + A_n(x)p_{n-1}(x) \tag{3.2.7}$$

where $A_n(x)$ and $B_n(x)$ are defined as

$$A_n(x) = a_n \left(\frac{p_n^2(y)w(y)}{y-x}\bigg|_{\partial\Omega} + \int_\Omega \frac{v'(x) - v'(y)}{x-y} p_n^2(y)w(y)dy \right) \tag{3.2.8}$$

$$B_n(x) = a_n \left(\frac{p_n(y)p_{n-1}(y)w(y)}{y-x}\bigg|_{\partial\Omega} + \int_\Omega \frac{v'(x) - v'(y)}{x-y} p_n(y)p_{n-1}(y)w(y)dy \right) \tag{3.2.9}$$

See [7] for a proof. It can also be shown that the following two relations are true:

$$B_n(x) + B_{n+1}(x) = \frac{x - b_n}{a_n} A_n(x) - v'(x) \tag{3.2.10}$$

$$B_{n+1}(x) - B_n(x) = \frac{a_{n+1}A_{n+1}(x)}{x - b_n} - \frac{a_n^2 A_{n-1}(x)}{a_{n-1}(x - b_n)} - \frac{1}{x - b_n} \tag{3.2.11}$$

Where Eq. (3.2.11) is called the string equation, again see [7] for a derivation of these relations.

Now, introducing the Schrödinger form, fundamental to the study of quantum mechanics, but here we use a dimensionless, homogenized version.

Definition 3.2.2. *The Schrödinger form of an OPS is an associated Second-Order Elliptical Partial Differential Equation of the form*

$$\mathbf{\Psi}_n''(x) + V(x;n)\mathbf{\Psi}_n(x) = 0 \tag{3.2.12}$$

whose solutions are

$$\mathbf{\Psi}_n(x) := \frac{e^{-\nu(x)/2}}{\sqrt{A_n(x)}} p_n(x) \tag{3.2.13}$$

and

$$\begin{aligned} V(x;n) =& A_n(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{B_n(x)}{A_n(x)}\right) - B_n(x)(\nu' + B_n(x)) + \frac{a_n}{a_{n-1}}A_n(x)A_{n-1}(x) + \frac{1}{2}\nu''(x) \\ &+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{A_n'(x)}{A_n(x)}\right) - \frac{1}{4}\left(\nu'(x) + \frac{A_n'(x)}{A_n(x)}\right)^2 \end{aligned} \tag{3.2.14}$$

The function $\mathbf{\Psi}_n(x)$ is called the wavefunction, and $V(x;n)$ is called the potential energy term.

The goal here is to solve for the specific wavefunctions and potential energies for different orthogonal polynomials. If we are successful, then that tells us there may exist, somewhere in the universe, a physical scenario where said orthogonal polynomial sequence is part of the solution. And so we begin.

3.2.1 Hermite polynomials

The Hermite polynomials are a Sheffer Type-0 OPS defined as

$$\begin{aligned} H_n(x) &= (2x)^n {}_2F_0\left({}^{-\frac{n}{2},-\frac{n-1}{2}}_{-} \middle| -\frac{1}{x^2}\right) \\ &= (2x)^n \sum_{k=0}^{\infty} \left(-\frac{n}{2}\right)_k \left(\frac{-(n-1)}{2}\right)_k \left(-\frac{1}{x^2}\right)^k \frac{1}{k!} \end{aligned} \tag{3.2.1.1}$$

with the recursion relation

$$xH_n(x) = \frac{1}{2}H_{n+1}(x) + nH_{n-1}(x) \tag{3.2.1.2}$$

and the monic recursion relation, related by $Q_n(x) = H_n(x)/2^n$

$$xQ_n(x) = Q_{n+1}(x) + \frac{n}{2}Q_{n-1}(x) \tag{3.2.1.3}$$

Comparing Eq. (3.2.1.3) to Eq. (3.2.3) and Eq. (3.2.2) reveals that $b_n = 0$, $c_n = n/2$, and $a_n = \sqrt{n/2}$. We must now try to derive these relations. First, we need the orthogonality relation for the Hermite polynomials.

$$\langle H_m, H_n\rangle = \int_{-\infty}^{\infty} H_m(x)H_n(x)\mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}2^n n!\delta_{mn} \tag{3.2.1.4}$$

To match Eq. (3.2.5), we first define an orthonormal polynomial $p_n(x)$ and take note of the weight function $w(x)$ associated with the Hermite polynomials

$$p_n(x) = \sqrt{\frac{1}{2^n n!\sqrt{\pi}}}H_n(x) \tag{3.2.1.5}$$

$$w(x) = \mathrm{e}^{-x^2} \tag{3.2.1.6}$$

So,

$$v(x) = x^2, \tag{3.2.1.7}$$

$$v'(x) = 2x \tag{3.2.1.8}$$

Let the domain $\Omega = (-\infty, \infty)$. We can now solve for $A_n(x)$; after inserting known values, Eq. (3.2.8) becomes

$$A_n(x) = a_n \left(\frac{\mathrm{e}^{-y^2}(H_n(y))^2}{\sqrt{\pi}2^n n!(y-x)}\bigg|_{\partial\Omega} + \int_\Omega \frac{2x-2y}{x-y} \frac{\mathrm{e}^{-y^2}(H_n(y))^2}{\sqrt{\pi}2^n n!} \mathrm{d}y \right) \tag{3.2.1.9}$$

The e^{-y^2} term will dominate at the boundary, so the first term goes to zero. Simplifying, our relation becomes

$$A_n(x) = \frac{a_n}{2^{n-1}n!\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-y^2}(H_n(y))^2 \mathrm{d}y \tag{3.2.1.10}$$

Notice that the integral is a variation of our orthogonality condition for the Hermite polynomials. Re-arranging Eq. (3.2.1.4), inserting into Eq. (3.2.1.10) and simplifying reveals that

$$A_n(x) = 2a_n \tag{3.2.1.11}$$

To solve for $B_n(x)$, let's plug known values into Eq. (3.2.9):

$$B_n(x) = a_n \left(\sqrt{\frac{1}{2^{2n-1}n!(n-1)!}} \frac{\mathrm{e}^{-y^2} H_n(y) H_{n-1}(y)}{\sqrt{\pi}(y-x)}\bigg|_{\partial\Omega} \right. \tag{3.2.1.12}$$

$$\left. + \int_\Omega \frac{2x-2y}{x-y} \sqrt{\frac{1}{2^{2n-1}n!(n-1)!}} \frac{\mathrm{e}^{-y^2} H_n(y) H_{n-1}(y)}{\sqrt{\pi}} \mathrm{d}y \right) \tag{3.2.1.13}$$

Again, the first term is dominated by e^{-y^2}, which goes to zero when evaluated in the limit. Simplifying, the relation becomes

$$B_n(x) = \frac{2a_n\sqrt{2n}}{\sqrt{\pi}2^n n!} \int_{-\infty}^{\infty} \mathrm{e}^{-y^2} H_n(y) H_{n-1}(y) \mathrm{d}y. \tag{3.2.1.14}$$

But the integral is our orthonormal relation, which thus forces the terms to go to zero since the degrees of the polynomial are different. So, $B_n(x) = 0$.

Inserting $A_n(x)$ into Eq. (3.2.10) yields the following relation:

$$0 = \frac{x - b_n}{a_n}(2a_n) - 2x \tag{3.2.1.15}$$

So, after re-arranging, we find $b_n = 0$, which is what we wanted. Using the string equation, Eq. (3.2.11), we get the following relation:

$$0 = \frac{2(a_{n+1})^2}{x - bn} - \frac{2a_n^2 a_{n-1}}{(a_{n-1})(x - b_n)} - \frac{1}{x - b_n} \tag{3.2.1.16}$$

which reveals that $a_n = \sqrt{n/2}$, and thus

$$A_n(x) = \sqrt{2n}$$

Notice that if we compare Eq. (3.2.1.2) to Eq. (3.2.1), we find that $A_n(x) = A_n$ and $B_n(x) = B_n$. Conjecture: this relates to the interval of the Hermite polynomials; that is, it is continuous over the reals.

To solve for the wavefunction, it must first be noted that

$$\mathrm{e}^{-\nu(x)/2} = \mathrm{e}^{-x^2/2} \tag{3.2.1.17}$$

So, plugging all known relations into Eq. (3.2.13), we find that

$$\mathbf{\Psi}_n(x) = \frac{\mathrm{e}^{-x^2/2} p_n(x)}{(2n)^{\frac{1}{4}}} \tag{3.2.1.18}$$

Inserting Eq. (3.2.1.3) into Eq. (3.2.1.18) creates the following solution for the wavefunction:

$$\mathbf{\Psi}_n(x) = \frac{\mathrm{e}^{-x^2/2}}{(2^{2n+1} n (n!)^2 \pi)^{\frac{1}{4}}} H_n(x) \tag{3.2.1.19}$$

This only leaves the potential energy term. As a recap, we know that

$$\nu''(x) = 2, \quad a_n = \sqrt{\frac{n}{2}}, \quad A_n(x) = \sqrt{2n}, \quad A_n'(x) = 0$$
$$\frac{B_n(x)}{A_n(x)} = 0, \quad \frac{A_n'(x)}{A_n(x)} = 0, \quad a_{n-1} = \sqrt{\frac{n-1}{2}}, \quad A_{n-1} = \sqrt{2(n-1)}$$

Plugging in all known relations into Eq. (3.2.14) yields the solution for the potential energy term:

$$V(x; n) = 2n + 1 - x^2 \tag{3.2.1.20}$$

Q.E.D. We have solved for the Schrödinger form for the Hermite Polynomials.

3.2.2 Legendre Polynomials

Recall, that the Legendre (spherical) polynomials are defined as

$$P_n(x, t) := {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix} \middle| \frac{1-x}{2}\right) \tag{3.2.2.1}$$

with the recursion relation

$$(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x) \tag{3.2.2.2}$$

and the monic form, related by $P_n(x) = \binom{2n}{n} \frac{Q_n(x)}{2^n}$

$$x Q_n(x) = Q_{n+1}(x) + \frac{n^2}{(2n-1)(2n+1)} Q_{n-1}(x) \tag{3.2.2.3}$$

Comparing Eq. (3.2.2.3) to Eq. (3.2.3) and Eq. (3.2.2) reveals that

$$b_n = 0, c_n = \frac{n^2}{(2n-1)(2n+1)}, a_n = \frac{n}{\sqrt{(2n-1)(2n+1)}}.$$

We must now try to derive these relations using Eq. (3.2.6)- Eq. (3.2.8). First, we need the orthogonality relation for the Legendre polynomials.

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x) P_n(x) \mathrm{d}x = \frac{2}{2n+1} \delta_{mn} \tag{3.2.2.4}$$

To match Eq. (3.2.5), we first define an orthonormal polynomial $p_n(x)$ and weight function polynomial $w(x)$ such that:

$$p_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x) \tag{3.2.2.5}$$

$$w(x) = 1 \tag{3.2.2.6}$$

So,

$$\nu(x) = \nu'(x) = 0 \tag{3.2.2.7}$$

Let domain $\Omega = (-1, 1)$. We can now solve for $A_n(x)$; after inserting known values, Eq. (3.2.8) becomes

$$A_n(x) = a_n \frac{(2n+1)(P_n(y))^2}{2(y-x)} \bigg|_{\partial\Omega} \tag{3.2.2.8}$$

Evaluating and simplifying, we find that

$$A_n(x) = \frac{a_n(2n+1)}{1-x^2} \tag{3.2.2.9}$$

To solve for $B_n(x)$, let's plug known values into Eq. (3.2.9):

$$B_n(x) = a_n \frac{\sqrt{(2n+1)(2n-1)} P_n(y) P_{n-1}(y)}{2(y-x)} \bigg|_{\partial\Omega} \tag{3.2.2.10}$$

Evaluating and simplifying, we find that

$$B_n(x) = \frac{a_n x \sqrt{(2n+1)(2n-1)}}{1-x^2} \tag{3.2.2.11}$$

Inserting $A_n(x), B_n(x)$ into Eq. (3.2.10) creates the following relation:

$$\frac{a_n x \sqrt{(2n+1)(2n-1)}}{1-x^2} + \frac{a_{n+1} x \sqrt{(2n+3)(2n+1)}}{1-x^2} = \frac{x(2n+1)}{1-x^2} - b_n \frac{2n+1}{1-x^2} \tag{3.2.2.12}$$

which becomes

$$b_n = x \left[1 - \frac{a_n \sqrt{(2n+1)(2n-1)}}{2n+1} - \frac{a_{n+1} \sqrt{(2n+3)(2n+1)}}{2n+1} \right] \tag{3.2.2.13}$$

Since this must hold for all x, $b_n = 0$, which matches the result from observing the orthonormal recursion relation.

Inserting $A_n(x), B_n(x), b_n(x)$ into the string equation, Eq. (3.2.11), and re-arranging, creates the following relation:

$$a_{n+1}x^2\sqrt{(2n+3)(2n+1)} - a_{n+1}^2(2n+3) = a_n x^2\sqrt{(2n+1)(2n-1)} - a_n^2(2n-1) - 1 \quad (3.2.2.14)$$

Since this must hold for all x, we can set $x = 0$, which, after re-arranging, creates the following relation:

$$a_n = \sqrt{\frac{(a_{n+1})^2(2n+3) - 1}{2n-1}} \quad (3.2.2.15)$$

The solution to Eq. (3.2.2.15) is

$$a_n = \frac{n}{\sqrt{(2n-1)(2n+1)}}$$

which is the result from the recursion relation. Now that we have a_n, we can determine several important quantities:

$$A_n(x) = \frac{n}{(1-x^2)}\sqrt{\frac{(2n+1)}{(2n-1)}} \quad (3.2.2.16)$$

and

$$B_n(x) = \frac{xn}{(1-x^2)} \quad (3.2.2.17)$$

Now, we can begin solving for the potential energy term:

$$a_{n-1} = \frac{n-1}{\sqrt{(2n-3)(2n-1)}}, \quad A_{n-1}(x) = \frac{n}{(1-x^2)}\sqrt{\frac{(2n-1)}{(2n-3)}}, \quad \frac{B_n(x)}{A_n(x)} = x\sqrt{\frac{(2n-1)}{(2n+1)}}$$

$$\frac{a_n}{a_{n-1}}A_n(x)A_{n-1}(x) = \frac{n^2}{(1-x^2)^2}, \quad A_n'(x) = \frac{2xn}{(1-x^2)^2}\sqrt{\frac{(2n+1)}{(2n-1)}}, \quad \frac{A_n'(x)}{A_n(x)} = \frac{2x}{1-x^2}$$

Inserting these relations into Eq. (3.2.14) reveals that the potential energy term is

$$V(x;n) = \frac{n+n^2}{1-x^2} + \frac{1}{(1-x^2)^2} \quad (3.2.2.18)$$

Using known relations and Eq. (3.2.13), we can also solve for the wavefunction

$$\mathbf{\Psi}_n(x) = \sqrt{\frac{1-x^2}{2n}}\left(4n^2-1\right)^{\frac{1}{4}} P_n(x) \quad (3.2.2.19)$$

Q.E.D, we have solved for the Schrödinger form for the Legendre Polynomials.

3.2.3 Laguerre Polynomials

Recall, that the Laguerre polynomials are defined as

$$L_n^{(\alpha)}(x) := \frac{(\alpha+1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| x \right) \tag{3.2.3.1}$$

with the recursion relation

$$(n+1)L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \tag{3.2.3.2}$$

and the monic form, related by $Q_n(x) = \frac{n!}{(-1)^n} L_n^{(\alpha)}(x)$

$$Q_{n+1}(x) = (x-2n-\alpha-1)Q_n(x) - n(n+\alpha)Q_{n-1}(x) \tag{3.2.3.3}$$

Comparing Eq. (3.2.3.3) to Eq. (3.2.3) and Eq. (3.2.2) reveals that $b_n = 2n+\alpha+1$, $c_n = n(n+\alpha)$, $a_n = \sqrt{n(n+\alpha)}$. We must now try to derive these relations using Eq. (3.2.6)- Eq. (3.2.8). First, we need the orthogonality relation for the Laguerre polynomials.

$$\langle L_m^{\alpha}, L_n^{(\alpha)} \rangle = \int_0^{\infty} L_n^{(\alpha)}(x) L_m^{\alpha}(x) \mathrm{e}^{-x} x^{\alpha} \mathrm{d}x = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{nm}, \quad \alpha > -1 \tag{3.2.3.4}$$

To match Eq. (3.2.5), we first define an orthonormal polynomial $p_n(x)$ and weight function polynomial $w(x)$ such that:

$$p_n(x) = (-1)^n \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}} L_n^{(\alpha)} \tag{3.2.3.5}$$

$$w(x) = \mathrm{e}^{-x} x^{\alpha} \tag{3.2.3.6}$$

So,

$$v(x) = x - \alpha \ln(x) \quad v'(x) = \frac{x-\alpha}{x} \quad v''(x) = \frac{\alpha}{x^2} \tag{3.2.3.7}$$

Let domain $\Omega = (0, \infty)$. We can now solve for $A_n(x)$; after inserting known values, Eq. (3.2.8) becomes

$$A_n(x) = a_n \frac{\mathrm{e}^{-y} y^{\alpha} (p_n(y))^2}{(y-x)} \bigg|_{\partial\Omega} + \frac{\alpha a_n}{x} \int_{\Omega} p_n^2(y) \mathrm{e}^{-y} y^{\alpha-1} \mathrm{d}y \tag{3.2.3.8}$$

Now, at $y = 0$, the y^{α} term causes the first term to go to zero, and, as $y \to \infty$, e^{-y} causes the term to go to zero again. So, the first term goes away. To solve the second term, we do an integration by parts. Set $u = p_n^2(y)\mathrm{e}^{-y}$, $\mathrm{d}v = y^{\alpha-1}\mathrm{d}y$, and you get

$$A_n(x) = \frac{a_n}{x} \mathrm{e}^{-y} y^{\alpha} (p_n(y))^2 \bigg|_{\partial\Omega} + \frac{a_n}{x} \int_{\Omega} p_n^2(y) \mathrm{e}^{-y} y^{\alpha} \mathrm{d}y - \frac{a_n}{x} \int_{\Omega} 2p_n(y) p_n'(y) \mathrm{e}^{-y} y^{\alpha} \mathrm{d}y$$

The first term goes to zero for the same reason the previous first term went to zero. Now, $p_n'(y)$ is a polynomial of degree at most $n-1$. Therefore, by Theorem 1.5.6, the third term is zero. The integral in middle term is our orthonormal relation, and is 1. Therefore, we find that

$$A_n(x) = \frac{a_n}{x} \tag{3.2.3.9}$$

To solve for $B_n(x)$, let's plug known values into Eq. (3.2.9):

$$B_n(x) = a_n \frac{\mathrm{e}^{-y} y^{\alpha} p_n(y) p_{n-1}(y)}{(y-x)}\bigg|_{\partial\Omega} + \frac{\alpha a_n}{x} \int_{\Omega} p_n(y) p_{n-1}(y) \mathrm{e}^{-y} y^{\alpha-1} \mathrm{d}y \tag{3.2.3.10}$$

The first term goes away again, and we tackle the second term via integration by parts. Set $u = p_n(y)p_{n-1}(y)\mathrm{e}^{-y}$, $\mathrm{d}v = y^{\alpha-1}\mathrm{d}y$, and you get

$$\begin{aligned} B_n(x) =& \frac{a_n}{x} \mathrm{e}^{-y} y^{\alpha} p_n(y) p_{n-1}(y)\bigg|_{\partial\Omega} - \frac{a_n}{x} \int_{\Omega} p_n'(y) p_{n-1}(y) \mathrm{e}^{-y} y^{\alpha} \mathrm{d}y \\ & - \frac{a_n}{x} \int_{\Omega} p_n(y) p_{n-1}'(y) \mathrm{e}^{-y} y^{\alpha} \mathrm{d}y + \frac{a_n}{x} \int_{\Omega} p_n(y) p_{n-1}(y) \mathrm{e}^{-y} y^{\alpha} \mathrm{d}y \end{aligned}$$

The first term goes to zero, the last term goes to zero due to orthogonality, and, since $p_{n-1}'(y)$ is a polynomial of degree at most $n-2$, the third term goes to zero. This only leaves the second term.

$$B_n(x) = -\frac{a_n}{x} \int_{\Omega} p_n'(y) p_{n-1}(y) \mathrm{e}^{-y} y^{\alpha} \mathrm{d}y$$

Now, observe for any orthogonal polynomial $R_n(x)$ that we can write $R_n(x) = r_{n,n}x^n + \pi_n(x)$ where $\pi_n(x)$ is a polynomial of degree at most $n-1$ and $r_{n,n}$ is the coefficient of x^n in R_n. So,

$$\int_{\Omega} (R_n)^2 w(x) \mathrm{d}x = \int_{\Omega} R_n (r_{n,n}x^n + \pi(x)) w(x) \mathrm{d}x = r_{n,n} \int_{\Omega} R_n x^n w(x) \mathrm{d}x$$

That is, we can freely switch between having an OPS or just x^n times some coefficient in our integral. Since $p_n(x) = o_{n,n}x^n + \pi_n(x)$, we have that $p_n'(x) = n o_{n,n} x^{n-1} + \pi_n'(x)$, where $o_{n,n}$ is a coefficient, $\pi_n(x)$ is a polynomial of degree at most $n-1$ and $\pi_n'(x)$ is at most degree $n-2$. Also, $p_n(x) = o_{n-1,n-1}x^{n-1} + \pi_{n-1}(x)$, where $\pi_{n-1}(x)$ is a polynomial of degree at most $n-2$. With all of this in mind, we can first replace $p_n'(x)$ with $n o_{n,n} x^{n-1}$ in our integral, and then replace x^{n-1} with $\frac{p_{n-1}(x)}{o_{n-1,n-1}}$. So now we have

$$B_n(x) = -\frac{n o_{n,n} a_n}{x o_{n-1,n-1}} \int_{\Omega} (p_{n-1}(y))^2 \mathrm{e}^{-y} y^{\alpha} \mathrm{d}y = -\frac{n o_{n,n} a_n}{x o_{n-1,n-1}}$$

Now, the question is, what are our coefficients? Well, looking at the generic recursion relation, Eq. (3.2.3), we see that the coefficient of x^n in $p_n(x)$ is $o_{n,n} = (a_1 a_2 \ldots a_n)^{-1}$ (write out a generic ONPS for yourself to check, a big giveaway is to solve for $p_{n+1}(x)$ and see that the coefficient of x^{n+1} is determined by the $x p_n(x)/a_{n+1}$ term). So, $o_{n,n}a_n/o_{n-1,n-1} = 1$, and we find that

$$B_n(x) = -\frac{n}{x} \tag{3.2.3.11}$$

Inserting $A_n(x), B_n(x)$ into Eq. (3.2.10) creates the following relation:

$$-\frac{n}{x} - \frac{n+1}{x} = \frac{x - b_n}{a_n} \frac{a_n}{x} - \frac{x-\alpha}{x} \tag{3.2.3.12}$$

which becomes

$$b_n = \alpha + 2n + 1$$

Inserting $A_n(x), B_n(x), b_n(x)$ into the string equation, Eq. (3.2.11) creates

$$-\frac{n+1}{x}+\frac{n}{x}=\frac{a_{n+1}^2}{x(x-b_n)}-\frac{a_n^2}{x(x-b_n)}-\frac{1}{x-b_n}$$

Which becomes after re-arranging, the following relation:

$$b_n=a_{n+1}^2-a_n^2=\alpha+2n+1 \tag{3.2.3.13}$$

where the solution for b_n was inserted on the far right. The solution to Eq. (3.2.3.13) is

$$a_n=\sqrt{n(n+\alpha)}$$

which is the result from the recursion relation. Now that we have a_n, we can determine several important quantities:

$$\begin{gathered}
A_n(x)=\frac{\sqrt{n(n+\alpha)}}{x},\quad B_n(x)=-\frac{n}{x},\quad a_{n-1}=\sqrt{(n-1)(n-1+\alpha)}\\
A_{n-1}(x)=\frac{\sqrt{(n-1)(n-1+\alpha)}}{x},\ \frac{B_n(x)}{A_n(x)}=-\sqrt{\frac{n}{n+\alpha}},\quad a_n=\sqrt{n(n+\alpha)}\\
\frac{a_n}{a_{n-1}}A_n(x)A_{n-1}(x)=\frac{n(n+\alpha)}{x^2},\quad A_n'(x)=-\frac{\sqrt{n(n+\alpha)}}{x^2},\quad \frac{A_n'(x)}{A_n(x)}=-\frac{1}{x}
\end{gathered} \tag{3.2.3.14}$$

Inserting these relations and Eq. (3.2.3.7) into Eq. (3.2.14) reveals that the potential energy term is

$$V(x;n)=\frac{-x^2+2x(\alpha+1+2n)+1-\alpha^2}{4x^2} \tag{3.2.3.15}$$

Using known relations and Eq. (3.2.13), we can also solve for the wavefunction

$$\mathbf{\Psi}_n(x)=\frac{x^{\frac{1}{2}(\alpha+1)}\mathrm{e}^{-x/2}L_n^{(\alpha)}(x)}{(n(n+\alpha))^{\frac{1}{4}}}\sqrt{\frac{n!}{\Gamma(n+\alpha+1)}} \tag{3.2.3.16}$$

Q.E.D, we have solved for the Schrödinger form for the Laguerre Polynomials.

3.2.4 Chebyshev Polynomials of the First Kind

Recall that the Chebyshev Polynomials of the First Kind are defined as

$$T_n(x):={}_2F_1\left(\begin{matrix}-n,n\\1/2\end{matrix}\middle|\frac{1-x}{2}\right) \tag{3.2.4.1}$$

with recursion relation

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)\quad n\geq 1,\ T_0(x)=1,\ T_1(x)=x \tag{3.2.4.2}$$

and the monic recursion relation, related by $Q_n(x)=2^{-n}T_n(x)$

$$Q_{n+1}(x)=xQ_n(x)-\frac{1}{4}Q_{n-1}(x)\quad n\geq 1,\ Q_0(x)=1,\ Q_1(x)=\frac{x}{2} \tag{3.2.4.3}$$

Comparing Eq. (3.2.4.3) to Eq. (3.2.3) and Eq. (3.2.2) reveals that $b_n = 0, c_n = 1/4, a_n = 1/2$. We must now try to derive these relations using Eq. (3.2.8) - Eq. (3.2.6). First, we need the orthogonality relation for the Chebyshev polynomials of the First Kind.

$$\int_{-1}^{1} T_n(x)T_m(x)(1-x^2)^{-1/2}\mathrm{d}x = \begin{cases} \frac{\pi}{2}\delta_{nm}, & \text{if } n \neq 0 \\ \pi\delta_{nm}, & \text{if } n = 0 \end{cases} \tag{3.2.4.4}$$

To match Eq. (3.2.5), we first define an orthonormal polynomial $p_n(x)$ and weight function polynomial $w(x)$ such that:

$$p_n(x) = \sqrt{\frac{1}{K_n}}T_n(x) \quad K_n = \begin{cases} \frac{\pi}{2}, & \text{if } n \neq 0 \\ \pi, & \text{if } n = 0 \end{cases} \tag{3.2.4.5}$$

$$w(x) = (1-x^2)^{-1/2} \tag{3.2.4.6}$$

So,

$$v(x) = \frac{1}{2}\ln\left(1-x^2\right),\ v'(x) = \frac{-x}{1-x^2},\ v'' = -\frac{1+x^2}{1-x^2} \tag{3.2.4.7}$$

Let domain $\Omega = (-1, 1)$. We can now solve for $A_n(x)$ and $B_n(x)$. If we try to evaluate Eq. (3.2.8) or Eq. (3.2.9) at -1 and 1, we encounter difficulties such as division by zero. Instead, we shall first take a bit of a detour to prove a useful lemma that will allow us to derive the $A_n(x)$ and $B_n(x)$ an alternate way.[1].

Lemma 3.2.3. *For $n = 1, 2, 3\ldots$*

$$(1-x^2)U_{n-1}(x) = xT_n(x) - T_{n+1}(x)$$

Proof. We invoke the trigonometric identity of the Chebyshev Polynomial of the First Kind, Eq. (1.9.5.3), and also remembering that $x = \cos\theta$,

$$\begin{aligned} T_{n+1}(x) &= \cos(n+1)\theta \\ &= \cos n\theta\cos\theta - \sin n\theta\sin\theta \\ &= xT_n(x) - \frac{\sin n\theta}{\sin\theta}\sin^2\theta = xT_n(x) - (1-x^2)U_{n-1}(x) \end{aligned}$$

where we used Eq. (1.9.5.6) for the final substitution. Now, just arrange for $T_n(x)$. □

Theorem 3.2.4. *The Chebyshev Polynomials of the First Kind, $T_n(x)$, satisfy the following differential equation:*

$$T_n'(x) = \frac{n}{1-x^2}T_{n-1}(x) - \frac{nx}{1-x^2}T_n(x) \tag{3.2.4.8}$$

or, after multiplying through by $\sqrt{1/K_n}$

$$p_n'(x) = \frac{n}{1-x^2}p_{n-1}(x) - \frac{nx}{1-x^2}p_n(x) \tag{3.2.4.9}$$

[1]Proof of Lemma 3.2.3 and Eq. (3.2.4.8) courtesy of Dr. Boon Wee Amos Ong, Professor of Mathematics at The Pennsylvania State University, Behrend

Proof.

$$\begin{aligned}
T_n'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\cos n\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos n\theta) \\
&= \frac{1}{-\sin\theta}(-n\sin n\theta) \\
&= nU_{n-1}(x) = \frac{1}{1-x^2}\Big(n(1-x^2)U_{n-1}(x)\Big) \\
&= \frac{1}{1-x^2}(nxT_n(x) - nT_{n+1}(x)) \\
&= \frac{1}{1-x^2}(nxT_n(x) - 2xnT_n(x) + nT_{n-1}) \\
&= \frac{n}{1-x^2}T_{n-1}(x) - \frac{nx}{1-x^2}T_n(x)
\end{aligned}$$

where Lemma 3.2.3 was used in the fourth line and the Three Term Recurrence Relation, Eq. (1.9.5.9), was used in the fifth line. □

Comparing Eq. (3.2.4.9) to Eq. (4.1.2.1) yields $A_n(x)$ and $B_n(x)$, along with notable terms

$$\begin{aligned}
A_n(x) &= \frac{n}{1-x^2},\ B_n(x) = \frac{nx}{1-x^2},\ A_{n+1}(x) = \frac{n+1}{1-x^2} \\
B_{n+1}(x) &= \frac{(n+1)x}{1-x^2},\ A_{n-1}(x) = \frac{n-1}{1-x^2},\ A_n'(x) = \frac{2xn}{(1-x^2)^2}
\end{aligned} \tag{3.2.4.10}$$

Inserting into Eq. (3.2.10) yields

$$\frac{2xn+x}{1-x^2} = \frac{x-b_n}{a_n}\frac{n}{1-x^2} + \frac{x}{1-x^2} \tag{3.2.4.11}$$

or, after re-arranging,

$$x - b_n = 2xa_n \tag{3.2.4.12}$$

Since this must be valid for all x, including $x = 0$, we get for free that $b_n = 0$ and $a_n = 1/2$, which is what we were looking for. If we want to still use the string equation Eq. (3.2.11) to verify, we get

$$\frac{x}{1-x^2} = \frac{a_{n+1}}{x-b_n}\frac{n+1}{1-x^2} - \frac{a_n^2}{a_{n-1}}\frac{n-1}{1-x^2}\frac{1}{x-b_n} - \frac{1}{x-b_n} \tag{3.2.4.13}$$

which becomes after multiplying by $(x-b_n)(1-x^2)$

$$(x-b_n)x = a_{n+1}(n+1) - \frac{a_n^2}{a_{n-1}}(n-1) + x^2 - 1$$

and then we can insert Eq. (3.2.4.12) to get

$$2x^2a_n = a_{n+1}(n+1) - \frac{a_n^2}{a_{n-1}}(n-1) + x^2 - 1$$

To move ahead, we take a leap of faith and hope that a_n is a constant independent of the index. That is, $a_n = a_{n-1} = a_{n+1} = a$. Thus we have

$$2x^2a = a(n+1) - a(n-1) + x^2 - 1$$

which becomes, after algebra,

$$2a(x^2-1) = x^2-1, \text{ or } a = \frac{1}{2}$$

which means, from Eq. (3.2.4.12), that $b_n = 0$. To find our wavefunction, Eq. (3.2.13)

$$\mathbf{\Psi}_n(x) = \sqrt{\frac{2}{\pi n}}[1-x^2]^{\frac{1}{4}}T_n(x) \tag{3.2.4.14}$$

For the potential energy term, we insert a_n, b_n and the relations found in Eq. (3.2.4.10) into Eq. (3.2.14) to get

$$V(x;n) = \frac{n^2+\frac{1}{2}}{1-x^2} + \frac{3}{4}\frac{x^2}{(1-x^2)^2} \tag{3.2.4.15}$$

Q.E.D. We have solved the Schrödinger form for the Chebyshev Polynomials of the First Kind. Now, let's observe the Chebyshev Polynomials of the Second Kind.

3.2.5 Chebyshev Polynomials of the Second Kind

Recall that the Chebyshev Polynomials of the Second Kind are defined as

$$U_n(x) := (n+1)_2F_1\left(\begin{matrix} -n,n+2 \\ 3/2 \end{matrix} \middle| \frac{1-x}{2}\right) \tag{3.2.5.1}$$

with recursion relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \quad n \geq 0,\ U_0(x) = 1 \tag{3.2.5.2}$$

and the monic recursion relation, related by $Q_n(x) = 2^{-n}U_n(x)$

$$Q_{n+1}(x) = xQ_n(x) - \frac{1}{4}Q_{n-1}(x) \quad n \geq 0,\ Q_0(x) = 1 \tag{3.2.5.3}$$

Comparing Eq. (3.2.5.3) to Eq. (3.2.3) and Eq. (3.2.2) reveals that $b_n = 0, c_n = 1/4, a_n = 1/2$. We now attempt to derive these terms. First, we need the orthogonality relation for the Chebyshev polynomials of the Second Kind.

$$\int_{-1}^{1} U_n(x)U_m(x)(1-x^2)^{1/2}\mathrm{d}x = \frac{\pi}{2}\delta_{nm} \tag{3.2.5.4}$$

To match Eq. (3.2.5), we first define an orthonormal polynomial $p_n(x)$ and weight function polynomial $w(x)$ such that:

$$p_n(x) = \sqrt{\frac{2}{\pi}}U_n(x) \tag{3.2.5.5}$$

$$w(x) = (1-x^2)^{1/2} \tag{3.2.5.6}$$

So,

$$v(x) = -\frac{1}{2}\ln\left(1-x^2\right),\ v'(x) = \frac{x}{1-x^2},\ v'' = \frac{1+x^2}{1-x^2} \tag{3.2.5.7}$$

Let domain $\Omega = (-1, 1)$. We can now solve for $A_n(x)$ and $B_n(x)$. If we try to evaluate Eq. (3.2.8) or Eq. (3.2.9) at -1 and 1, we encounter difficulties such as division by zero. Instead, we shall first take a bit of a detour to prove a useful lemma that will allow us to derive the $A_n(x)$ and $B_n(x)$ an alternate way.[2].

Lemma 3.2.5. *For $n = 1, 2, 3\ldots$*

$$T_n(x) = U_n(x) - xU_{n-1}(x)$$

Proof. We invoke the trigonometric identity of the Chebyshev Polynomial of the Second Kind, Eq. (1.9.5.6), and also remembering that $x = \cos\theta$,

$$\begin{aligned} U_n(x) &= \frac{1}{\sin\theta}\sin(n+1)\theta \\ &= \frac{1}{\sin\theta}(\sin n\theta\cos\theta + \cos n\theta\sin\theta) \\ &= xU_{n-1}(x) + T_n(x) \end{aligned}$$

where we invoked Eq. (1.9.5.3) in the final substitution. □

Theorem 3.2.6. *The Chebyshev Polynomials of the Second Kind, $U_n(x)$, satisfy the following differential equation:*

$$U_n'(x) = \frac{n+1}{1-x^2}U_{n-1}(x) - \frac{nx}{1-x^2}U_n(x) \tag{3.2.5.8}$$

or, after multiplying through by $\sqrt{2/\pi}$

$$p_n'(x) = \frac{n+1}{1-x^2}p_{n-1}(x) - \frac{nx}{1-x^2}p_n(x) \tag{3.2.5.9}$$

Proof.

$$\begin{aligned} U_n'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sin\theta}\sin(n+1)\theta\right) = \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\sin\theta}\sin(n+1)\theta\right) \\ &= \frac{1}{-\sin\theta}\left(\frac{(n+1)\cos(n+1)\theta\sin\theta - \sin(n+1)\theta\cos\theta}{\sin^2\theta}\right) \\ &= \frac{(n+1)\cos(n+1)\theta}{-\sin^2\theta} - \frac{\sin(n+1)\theta\cos\theta}{-\sin\theta\sin^2\theta} \\ &= -\frac{n+1}{1-\cos^2\theta}T_{n+1}(x) + \frac{\cos\theta}{1-\cos^2\theta}U_n(x) \\ &= -\frac{n+1}{1-x^2}T_{n+1}(x) + \frac{x}{1-x^2}U_n(x) \end{aligned}$$

[2]Proof of Lemma 3.2.5 and Eq. (3.2.5.8) courtesy of Dr. Boon Wee Amos Ong, Professor of Mathematics at The Pennsylvania State University, Behrend

Now, using Lemma 3.2.5 shows

$$\begin{aligned}
U_n'(x) &= -\frac{n+1}{1-x^2}(U_{n+1}(x) - xU_n(x)) + \frac{x}{1-x^2}U_n(x) \\
&= \frac{x(n+1)}{1-x^2}U_n(x) + \frac{x}{1-x^2}U_n(x) - \frac{n+1}{1-x^2}U_{n+1}(x) \\
&= \frac{xn+2x}{1-x^2}U_n(x) - \frac{n+1}{1-x^2}(2xU_n(x) - Un-1(x)) \\
&= \frac{n+1}{1-x^2}U_{n-1}(x) - \frac{nx}{1-x^2}U_n(x)
\end{aligned}$$

where the Three Term Recurrence Relation, Eq. (1.9.5.10), was used in the second to last step. □

Comparing Eq. (3.2.5.9) to Eq. (4.1.2.1) yields $A_n(x)$ and $B_n(x)$, along with notable terms

$$\begin{aligned}
A_n(x) &= \frac{n+1}{1-x^2},\ B_n(x) = \frac{nx}{1-x^2},\ A_{n+1}(x) = \frac{n+2}{1-x^2} \\
B_{n+1}(x) &= \frac{(n+1)x}{1-x^2},\ A_{n-1}(x) = \frac{n}{1-x^2},\ A_n'(x) = \frac{2x(n+1)}{(1-x^2)^2}
\end{aligned} \tag{3.2.5.10}$$

Inserting into Eq. (3.2.10) yields

$$\frac{2xn+x}{1-x^2} = \frac{x-b_n}{a_n}\frac{(n+1)}{1-x^2} - \frac{x}{1-x^2} \tag{3.2.5.11}$$

or, after re-arranging,

$$x - b_n = 2xa_n \tag{3.2.5.12}$$

Since this must be valid for all x, including $x = 0$, we get for free that $b_n = 0$ and $a_n = 1/2$, which is what we were looking for. If we want to still use the string equation Eq. (3.2.11) to verify, we get

$$\frac{x}{1-x^2} = \frac{a_{n+1}}{x-b_n}\frac{n+2}{1-x^2} - \frac{a_n^2}{a_{n-1}}\frac{n}{1-x^2}\frac{1}{x-b_n} - \frac{1}{x-b_n} \tag{3.2.5.13}$$

which becomes after multiplying by $(x-b_n)(1-x^2)$

$$(x-b_n)x = a_{n+1}(n+2) - \frac{a_n^2}{a_{n-1}}n + x^2 - 1$$

and then we can insert Eq. (3.2.4.12) to get

$$2x^2a_n = a_{n+1}(n+2) - \frac{a_n^2}{a_{n-1}}n + x^2 - 1$$

To move ahead, we take a leap of faith and hope that a_n is a constant independent of the index. That is, $a_n = a_{n-1} = a_{n+1} = a$. Thus we have

$$2x^2a = a(n+2) - an + x^2 - 1$$

which becomes, after algebra,

$$2a(x^2 - 1) = x^2 - 1, \text{ or } a = \frac{1}{2}$$

which means, from Eq. (3.2.4.12), that $b_n = 0$.

The solution to our wavefunction,is thus, from Eq. (3.2.13),

$$\mathbf{\Psi}_n(x) = \sqrt{\frac{2}{\pi(n+1)}}[1 - x^2]^{\frac{3}{4}} U_n(x) \tag{3.2.5.14}$$

For the potential energy term, we insert a_n, b_n and the relations found in Eq. (3.2.4.10) into Eq. (3.2.14) to get

$$V(x; n) = \frac{(n+1)^2}{1 - x^2} + \frac{\frac{1}{2}x^2 + \frac{1}{4}}{(1 - x^2)^2} \tag{3.2.5.15}$$

Q.E.D. We have solved the Schrödinger form for the Chebyshev Polynomials of the Second Kind. We are now ready to state a conjecture.

3.3 Conjecture

Before continuing, we would like to focus on an important development. Begin by noticing that the domain for the Legendre Polynomials are

$$\Omega = (-1, 1)$$

and we have

$$\begin{aligned} A_n(x) &= \frac{n}{(1 - x^2)}\sqrt{\frac{(2n+1)}{(2n-1)}} \\ B_n(x) &= \frac{xn}{(1 - x^2)} \\ V(x; n) &= \frac{n + n^2}{1 - x^2} + \frac{1}{(1 - x^2)^2} \end{aligned}$$

which are all discontinuous at -1 and 1. We also have the wavefunction,

$$\Psi_n(x; n) = \sqrt{\frac{1 - x^2}{2n}}\left(4n^2 - 1\right)^{\frac{1}{4}} P_n(x)$$

which goes to zero as x goes to 1 or -1.

Also, notice that for the Hermite Polynomials,

$$\Omega = (-\infty, \infty)$$

and we have

$$B_n(x) = 0, \qquad A_n(x) = \sqrt{2n}, \quad V(x; n) = 2n + 1 - x^2$$

which are all continuous everywhere. We also have the wave function

$$\mathbf{\Psi} = \frac{\mathrm{e}^{-x^2/2}}{(2^{2n+1}n(n!)^2\pi)^{\frac{1}{4}}}H_n(x)$$

which goes to zero in the limit as x goes to infinity from the decaying exponential term.

For the Laguerre Polynomials,

$$\Omega = (0, \infty)$$

and we have

$$B_n(x) = -\frac{n}{x} \qquad A_n(x) = \frac{\sqrt{n(n+\alpha)}}{x}$$
$$V(x;n) = \frac{-x^2 + 2x(\alpha + 1 + 2n) + 1 - \alpha^2}{4x^2}$$

which are all discontinuous at 0. We also have the wave function

$$\mathbf{\Psi}_n(x) = \frac{x^{\frac{1}{2}(\alpha+1)}\mathrm{e}^{-x/2}L_n^{(\alpha)}(x)}{(n(n+\alpha))^{\frac{1}{4}}}\sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}$$

which goes to zero in the limit as x goes to infinity from the decaying exponential term, and is also zero when evaluated at $x = 0$.

For the Chebyshev Polynomials of the First Kind

$$\Omega = (-1, 1)$$

and we have

$$A_n(x) = \frac{n}{1-x^2}, \quad B_n(x) = \frac{nx}{1-x^2}$$
$$V(x;n) = \frac{n^2 + \frac{1}{2}}{1-x^2} + \frac{3}{4}\frac{x^2}{(1-x^2)^2}$$

which are all discontinuous at 1 and -1. We also have the wavefunction

$$\Psi_n(x;n) = \sqrt{\frac{2}{\pi n}}[1-x^2]^{\frac{1}{4}}T_n(x)$$

which goes to zero as x goes to 1 or -1.

And finally for the Chebyshev Polynomials of the Second Kind

$$\Omega = (-1, 1)$$

and we have

$$A_n(x) = \frac{n+1}{1-x^2}, \quad B_n(x) = \frac{nx}{1-x^2}$$
$$V(x;n) = \frac{(n+1)^2}{1-x^2} + \frac{\frac{1}{2}x^2 + \frac{1}{4}}{(1-x^2)^2}$$

which are discontinuous at 1 and -1. We also have the wavefunction

$$\Psi_n(x;n) = \sqrt{\frac{2}{\pi(n+1)}}[1-x^2]^{\frac{3}{4}}U_n(x)$$

which goes to zero as x goes to 1 or -1.

Conjecture 3.3.1. *The variable coefficients $A_n(x)$ and $B_n(x)$ and the potential energy $V(x;n)$ for any OPS are discontinuous at the boundary of the domain Ω of the polynomial, and the wavefunction $\Psi(x;n)$ goes to zero as x approaches either boundary term.*

This remains to be proven.

Chapter 4

Applications in Physics and Numerical Analysis

4.1 Applications

4.1.1 A Physicist's Introduction

Orthogonal polynomial sequences make the occasional appearance within the contexts of physics. Here, we will cover the two instances that should be familiar with every undergraduate physicist. Note: dots above a function correspond to time derivatives, primes some other, usually position, derivative.

Electrostatics

An object with charge density ρ generates an electric field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho\,\mathbf{r}\,\mathrm{d}\tau}{||\mathbf{r}||^3} \tag{4.1.1.1}$$

where ϵ_0 is the permittivity of free space, and τ represents some volume. Now, the electric field is a conservative vector field; that is, it is the gradient of some scalar function. That function is the (negative of the) electric potential $V(\mathbf{r})$. That is, if ∇ represents the gradient,

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}) \tag{4.1.1.2}$$

Now, when calculating the flux of the electric field, we encounter Gauss's Law, the first of what are known as Maxwell's Equations, named after James Clerk Maxwell. If we construct some Gaussian surface, with outward normal unit vector $\hat{n}$ and surface area A, enclosing some charge q_{enc}, we get, where $\cdot$ is the dot product,

$$\int_A \mathbf{E}(\mathbf{r})\cdot\hat{n}\mathrm{d}A = \frac{q_{enc}}{\epsilon_0} \tag{4.1.1.3}$$

Now, we take note that if the Gaussian surface is enclosing some charge density ρ, we have

$$q_{enc} = \int_\tau \rho\mathrm{d}\tau \tag{4.1.1.4}$$

and, from Green's Theorem,

$$\int_\tau \nabla\cdot\mathbf{E}(\mathbf{r})\mathrm{d}\tau = \int_A \mathbf{E}(\mathbf{r})\cdot\hat{n}\mathrm{d}A \tag{4.1.1.5}$$

where τ is the volume enclosed by the Gaussian surface.

Definition 4.1.1. *If* $\mathbf{v}$ *vector,* $\nabla\cdot\mathbf{v}$ *is called the divergence of* $\mathbf{v}$*. In Cartesian coordinates:*

$$\begin{aligned}\mathbf{v} &= \mathbf{v}(x, y, z)\\ \nabla\cdot\mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\end{aligned} \tag{4.1.1.6}$$

In Spherical coordinates:

$$\begin{aligned}\mathbf{v} &= \mathbf{v}(r, \theta, \phi)\\ \nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\end{aligned} \tag{4.1.1.7}$$

Thus, replacing the l.h.s of Eq. (4.1.1.3) with Eq. (4.1.1.5), and the r.h.s with Eq. (4.1.1.4), we get

$$\int_\tau \nabla\cdot\mathbf{E}(\mathbf{r})\mathrm{d}\tau = \frac{1}{\epsilon_0}\int_\tau \rho\mathrm{d}\tau$$

or

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = \frac{\rho}{\epsilon_0} \tag{4.1.1.8}$$

Eq. (4.1.1.8) is fundamental to the study of electromagnetism, and is referred to as the Differential Form of Gauss's Law. Now, inserting Eq. (4.1.1.8) into Eq. (4.1.1.2) yields

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho}{\epsilon_0} \tag{4.1.1.9}$$

Definition 4.1.2. *The operator $\nabla^2 = \nabla\cdot\nabla$ is called the Laplace operator, and, if f scalar function, $\nabla^2 f$ is the Laplacian of f.*

In Cartesian coordinates:

$$\begin{aligned} f &= f(x, y, z) \\ \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned} \tag{4.1.1.10}$$

In Spherical coordinates:

$$\begin{aligned} f &= f(r, \theta, \phi) \\ \nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \end{aligned} \tag{4.1.1.11}$$

Eq. (4.1.1.9) is called the Poisson Equation, and is an Elliptical Partial Differential Equation. If we set $\rho = 0$, we have

$$\nabla^2 V(\mathbf{r}) = 0 \tag{4.1.1.12}$$

Eq. (4.1.1.12) is called Laplace's Equation. Note: It is made clear in [11] that this does not state the charge density is 0, otherwise, there would not be an electric potential! This is simply stating that we are looking in a charge-free region for an electric potential that is generated by charges located *somewhere else*.

Now, given initial values and boundary conditions, the solutions to Eq. (4.1.1.12) are unique. For one dimension, the solution is a linear equation. For two dimensions, which is solved using the separation of variables technique, the solution is usually a Fourier Series tampered by a decaying exponential. We shall focus on the three-dimensional Laplacian. We will concern ourselves with situations that have azimuthal symmetry, where the electric potential V is independent of ϕ, which suggests using spherical coordinates. Thus, Laplace's Equation becomes, via Eq. (4.1.1.11),

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \tag{4.1.1.13}$$

Now, we use separation of variables

$$V(r, \theta) = R(r)\Theta(\theta)$$

to get

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) \tag{4.1.1.14}$$

The l.h.s of Eq. (4.1.1.14) is a function of r, and the r.h.s is a function of θ. They can only be equal if they both equal a constant.

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \ell(\ell+1), \qquad \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = -\ell(\ell+1) \tag{4.1.1.15}$$

The radial equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = R\ell(\ell+1) \tag{4.1.1.16}$$

has solution

$$R(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \tag{4.1.1.17}$$

and the angular equation

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = -\Theta\sin\theta\ell(\ell+1) \tag{4.1.1.18}$$

has solution

$$\Theta(\theta) = P_\ell(\cos\theta) \tag{4.1.1.19}$$

where $P_\ell(\cos\theta)$ are the Legendre Orthogonal Polynomials! A physicist defines them not in terms of hypergeometric functions, but via the Rodrigues formula

$$P_\ell(x) = \frac{1}{2^\ell\ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell(x^2-1)^\ell \tag{4.1.1.20}$$

Thus, the general solution is, via the Principle of Superposition,

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \tag{4.1.1.21}$$

We will now work through one example.

Consider a sphere of radius R with the following electric potential, written in terms of both θ and $x = \cos\theta$:

$$\begin{aligned} V(r,\theta) &= \begin{cases} V_0, & 0 < \theta < \frac{\pi}{4}, \quad r = R \\ 0, & \frac{\pi}{4} < \theta < \frac{3\pi}{2}, \quad r = R \\ V_0, & \frac{3\pi}{2} < \theta < \pi, \quad r = R \\ 0, & r \to \infty \end{cases} \\ V(r,x) &= \begin{cases} V_0, & 1 < x < \frac{1}{\sqrt{2}}, \quad r = R \\ 0, & \frac{1}{\sqrt{2}} < x < \frac{-1}{\sqrt{2}}, \quad r = R \\ V_0, & \frac{-1}{\sqrt{2}} < x < -1, \quad r = R \\ 0, & r \to \infty \end{cases} \end{aligned} \tag{4.1.1.22}$$

Find the first three non-zero terms in the explicit equation for the electric potential.

We begin by observing that the condition $V(r \to \infty) = 0$ implies that A in Eq. (4.1.1.21) is zero, as that term will blow-up otherwise. Thus, we start with

$$V(r,\theta) = \sum_{m=0}^{\infty} \frac{B_m}{r^{m+1}} P_m(\cos\theta) \tag{4.1.1.23}$$

Define $V_m(\theta)$ as

$$V_m(\theta) = V(R,\theta) = \sum_{\ell=0}^{\infty} \frac{B_m}{R^{m+1}} P_m(\cos\theta) \tag{4.1.1.24}$$

Now, upon observing Eq. (4.1.1.22) we see that the electric potential is even. Using the orthogonality of the Legendre polynomials, Eq. (1.9.6.2), we have

$$\begin{aligned}\int_0^{\pi} P_n(\cos\theta) V_m(\theta) \sin\theta \mathrm{d}x &= \int_0^{\pi} P_n(\cos\theta) \sum_{m=0}^{\infty} \frac{B_m}{R^{m+1}} P_m(\cos\theta)\mathrm{d}x \\ &= \frac{2B_n}{(2n+1)R^{n+1}} \delta_{nm}\end{aligned}$$

or, re-arranging for B_n, integrating over x, and, since $V_0(\theta)$ is even, multiplying by a factor of 2 and integrating over half the domain yields

$$B_n = (2n+1)R^{n+1} \int_0^1 P_n(x) V_m(x)\mathrm{d}x = (2n+1)V_0 R^{n+1} \int_{\frac{1}{\sqrt{2}}}^1 P_n(x)\mathrm{d}x \tag{4.1.1.25}$$

Since $V_0(\theta)$ is even, the first three non-zero terms co-efficients are B_0, B_2 and B_4 So, we need $P_0(x), P_2(x)$ and $P_4(x)$.

$$\begin{aligned}P_0(x) &= 1 \\ P_2(x) &= \frac{3x^2-1}{2} \\ P_4(x) &= \frac{35x^4 - 30x^2 + 3}{8}\end{aligned}$$

Thus, we have

$$\begin{aligned}B_0 &= V_0 R \int_{\frac{1}{\sqrt{2}}}^1 \mathrm{d}x = V_0 R\left(1 - \frac{1}{\sqrt{2}}\right) \\ B_2 &= 5V_0 R^3 \int_{\frac{1}{\sqrt{2}}}^1 \frac{3x^2-1}{2}\mathrm{d}x = \frac{5}{2} V_0 R^3 \left(x^3 - x\right)\Big|_{\frac{1}{\sqrt{2}}}^1 = \frac{5}{4\sqrt{2}} V_0 R^3 \\ B_4 &= 9V_0 R^5 \int_{\frac{1}{\sqrt{2}}}^1 \frac{35x^4 - 30x^2 + 3}{8}\mathrm{d}x = \frac{9}{8} V_0 R^5 \left(7x^5 - 10x^3 + 3x\right)\Big|_{\frac{1}{\sqrt{2}}}^1 \\ &= \frac{9}{32\sqrt{2}} V_0 R^5\end{aligned} \tag{4.1.1.26}$$

Thus, our electric potential to three terms (which is practical enough) is

$$\begin{aligned} V(r,\theta) =& V_0\left[\frac{R}{r}\left(1-\frac{1}{\sqrt{2}}\right)+\frac{R^3}{r^3}\frac{5}{4\sqrt{2}}\left(1-\frac{3}{2}\sin^2\theta\right)\right. \\ &\left.+\frac{R^5}{r^5}\frac{9}{256\sqrt{2}}\left(35\cos^4\theta-30\cos^2\theta+3\right)\right] \end{aligned} \tag{4.1.1.27}$$

We now cover the second usage of orthogonal polynomials in introductory physics.

Quantum Harmonic Oscillator

The journey to the quantum harmonic oscillator is one that stretches 19th and early 20th century physics.

Definition 4.1.3. *The Lagrangian $J = J(q,\dot{q})$ of a system with kinetic energy T and potential energy V is defined with generalized coordinates $q, \dot{q}$ as*

$$J(q,\dot{q}) := T(\dot{q}) - V(q)$$

whose action is defined as

$$S = \int J dt$$

If there are multiple objects within the system, the kinetic and potential terms, T and V respectively, may be sums. Lagrangians were introduced in the 18th century by Joseph-Louis Lagrange; however, they really came into importance in the mid-19th century due to the following principle by Irish mathematician Sir William Rowan Hamilton.

Theorem 4.1.4 (Hamilton's Principle)**.** *Given two different states that describe the coordinates of a physical system, the variation between the states is zero. Thus, the action is minimized, that is the functional derivative of the action is*

$$\delta S = \int \delta J dt = 0 \tag{4.1.1.28}$$

Corollary 4.1.4.1. *Due to Hamilton's Principle, the Euler-Lagrange Equation is true, where the Euler-Lagrange Equation is defined as*

$$\frac{\partial J}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial J}{\partial \dot{q}}\right] \tag{4.1.1.29}$$

Proof. Now, the functional derivative of the Lagrangian is, by definition,

$$\delta J = \frac{\partial J}{\partial q}\delta q + \frac{\partial J}{\partial \dot{q}}\delta \dot{q} \tag{4.1.1.30}$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial J}{\partial \dot{q}}\delta q\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial J}{\partial \dot{q}}\right]\delta q + \frac{\partial J}{\partial \dot{q}}\delta \dot{q} \tag{4.1.1.31}$$

Inserting Eq. (4.1.1.31) and Eq. (4.1.1.30) into Eq. (4.1.1.28) reveals

$$\begin{aligned}\int \delta J \mathrm{d}t &= \int \left[\frac{\partial J}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial J}{\partial \dot{q}}\right]\right]\delta q \mathrm{d}t + \int \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial J}{\partial \dot{q}}\delta q\right]\mathrm{d}t \\ &= \int \left[\frac{\partial J}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial J}{\partial \dot{q}}\right]\right]\delta q \mathrm{d}t = 0\end{aligned} \tag{4.1.1.32}$$

where the second term went to zero because the functional derivative, evaluated at the boundary of the Lagrangian, is zero as a consequence of Hamilton's Principle. Now, the r.h.s of Eq. (4.1.1.32) can only be zero if

$$\frac{\partial J}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial J}{\partial \dot{q}}\right] \tag{4.1.1.33}$$

□

We will take as an axiom here that the momentum of the system, linear if q is a translational coordinate, angular otherwise, is

$$p_q(t) = \frac{\partial J}{\partial \dot{q}} \tag{4.1.1.34}$$

which means, after inserting Eq. (4.1.1.34) into the Euler-Lagrange equation (4.1.1.29) yields

$$\dot{p_q}(t) = \frac{\partial J}{\partial q} \tag{4.1.1.35}$$

Since the Lagrangian is a manifestation of the energy of the system, Eq. (4.1.1.34) and Eq. (4.1.1.35) hint at something bigger.

Theorem 4.1.5. *The generalized momentum $p_q(t)$ and the corresponding generalized coordinate q are conjugate pairs.*

Before we can offer more evidence of our theorem, we must review Hamilton's continuation of Lagrangian mechanics: The Hamiltonian.

Definition 4.1.6. *The Hamiltonian $H = H(q, p_q)$ is*

$$H = \sum_i \dot{q}_i p_q - L \tag{4.1.1.36}$$

where $\sum_i \dot{q}_i p_q$ is summing over all objects within the system.

Theorem 4.1.7. *If the system is conservative, the Hamiltonian is the energy of the system.*

Proof. If the system is conservative, then all outside forces are represented within the potential energy term (as the sum of the forces within the system would be the negative gradient of the potential energy, in the exact same way the electric field was the negative gradient of the electric potential in Section 4.1.1). Observe that

$$\sum_i \dot{q}_i p_q = 2T$$

(For example, if $q_i = x$, then $\dot{q}_i = v$ and $p_q = mv$. So, $\dot{q}_i p_q = mv^2$, which is twice the translational kinetic energy.) From here, we have

$$H = 2T - L = 2T - T + V = T + V$$

Thus, the Hamiltonian is the energy of the system. □

Proof of Theorem 4.1.5. Observe that the functional of the Hamiltonian is, by definition,

$$\delta H = \sum_i \frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p_q}\delta p_q \tag{4.1.1.37}$$

So, taking the functional of Eq. (4.1.1.36) reveals

$$\delta H = \sum_i \delta[\dot{q}_i p_q] - \delta L = \sum_i \delta\dot{q}_i p_q + \dot{q}_i \delta p_q - \frac{\partial J}{\partial q_i}\delta q_i - \frac{\partial J}{\partial \dot{q}_i}\delta\dot{q}_i \tag{4.1.1.38}$$

Inserting Eq. (4.1.1.34) and Eq. (4.1.1.35) into Eq. (4.1.1.38) yields

$$\delta H = \sum_i \delta\dot{q}_i p_q + \dot{q}_i \delta p_q - \dot{p}_q \delta q_i - p_q \delta\dot{q}_i = \sum_i \dot{q}_i \delta p_q - \dot{p}_q \delta q_i \tag{4.1.1.39}$$

comparing Eq. (4.1.1.39) to Eq. (4.1.1.37) reveals

$$\frac{\partial H}{\partial p_q} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_q \tag{4.1.1.40}$$

Since the Hamiltonian is the energy of our system, equations 4.1.1.35, 4.1.1.34 and 4.1.1.40 are direct statements that momentum and position are conjugate pairs. □

Albert Einstein revealed that the energy at the smallest scales is quantized, particularly in the case of photons. This supported the fact that photons are waves. That is, if ν is the frequency a photon and h is Planck's constant, we have, where $\hbar = h/2\pi$

$$E = h\nu = \hbar\omega \tag{4.1.1.41}$$

Louis DeBrogile revealed that all matter are waves, and that the momentum of quantized matter is, where λ is the wavelength and $k = 2\pi/\lambda$

$$p = \frac{h}{\lambda} = \hbar k \tag{4.1.1.42}$$

If matter is a wave, then it must solve the wave-equation, an Elliptical Partial Differential Equation of the form:

$$\frac{\partial^2 \Psi(r,t)}{\partial t^2} = c^2 \frac{\partial^2 \Psi(r,t)}{\partial r^2} \tag{4.1.1.43}$$

Where $\Psi = \Psi(r,t)$ is called the wavefunction and c is the speed of the wave (often the speed of light). Erwin Schroedinger, through a famous thought experiment, took Eq. (4.1.1.43), Eq. (4.1.1.41), Eq. (4.1.1.42) and Eq. (4.1.1.36) to create the Schroedinger equation

$$i\hbar\frac{\partial \Psi}{\partial t} = \left(\frac{-\hbar^2}{2m}\nabla^2\Psi + V\Psi\right) = \hat{H}\Psi \tag{4.1.1.44}$$

where

$$\hat{H} = \left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)$$

is called the Hamiltonian operator. Observe that if we claim the momentum is also an operator, defined as

$$p = -i\hbar\nabla \tag{4.1.1.45}$$

we get

$$\hat{H} = \left(\frac{p^2}{2m} + V\right) = T + V$$

which is the Hamiltonian of classical mechanics, Eq. (4.1.1.36), for a conservative system. If we separate variables and let $\Psi = \psi(r)\phi(t)$, we get the time-independent Schroedinger equation, with (eigen) energy E:

$$\hat{H}\psi = E\psi \tag{4.1.1.46}$$

Consider the Taylor expansion of the potential energy term, where we are at a small displacement from a minima x_0, which, since we are free to set out coordinate system, we make the origin.

$$V(x) = V(x_0) + V'(x_0)x + \frac{1}{2}V''(x_0)x^2 + \cdots \tag{4.1.1.47}$$

$V(x_0)$ is the background energy, and, since we set the coordinate system, we make it zero. Since we are at a minima, $V'(x_0) = 0$ and, since our displace is small, $x^n = 0,\ n \geq 3$. This leaves only the second order term. Let $k = V''(x_0)$ be called the spring constant, to yield the potential energy:

$$V(x) = \frac{1}{2}kx^2 \tag{4.1.1.48}$$

which is the potential energy for an idealized spring undergoing harmonic motion, the classic simple harmonic oscillator. The force $F(x) = F$ with said potential energy is Hooke's Law, where, if we have a mass m on the end of a spring with spring constant k

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x} = -kx = m\ddot{x} \tag{4.1.1.49}$$

Where the r.h.s is from Newton's Second Law. If we set

$$\omega = \sqrt{\frac{k}{m}} \tag{4.1.1.50}$$

we get the Second Order Linear Homogeneous Ordinary Differential Equation

$$\ddot{x} + \omega^2 x = 0$$

with solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \tag{4.1.1.51}$$

Now, using Eq. (4.1.1.50) our potential energy term becomes

$$V(x) = \frac{1}{2}m\omega^2 x^2 \tag{4.1.1.52}$$

This potential energy is applicable to any mass undergoing oscillatory motion under the constraints of the derivation of Eq. (4.1.1.48), and this includes diatomic molecules. So, under this motivation, if we insert Eq. (4.1.1.52) into the time-independent Schroedinger equation (4.1.1.46) we get the *quantum harmonic oscillator*

$$\left(\frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2 x^2\right)\psi = E\psi \tag{4.1.1.53}$$

We will follow the algebraic technique of [12] to solve this system, with a result that we hope is surprising.

We begin by reminding ourselves that the momentum in one dimension is $p = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}$, and thus our Hamiltonian is

$$\hat{H} = \frac{1}{2m}[p^2 + (m\omega x)^2] \tag{4.1.1.54}$$

Since operators do not commute, we can not factor the Hamiltonian in the hopes of simplifying the problem. Introduce the ladder operators:

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\big(\mp ip + m\omega x\big) \tag{4.1.1.55}$$

We next introduce the commutator, which 'measures' the degree operators failure to commute. For momentum and position operators p and x,

$$[x, p] = xp - px = i\hbar \tag{4.1.1.56}$$

(see [12] for the proof). Now, we find that when we calculate $a_- a_+$

$$\begin{aligned} a_- a_+ &= \frac{1}{2\hbar m\omega}\big(ip + m\omega x\big)\big(-ip + m\omega x\big) \\ &= \frac{1}{2\hbar m\omega}[p^2 + (m\omega x)^2 - im\omega[x, p]] = \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2} \end{aligned}$$

where we used Eq. (4.1.1.56). Also, upon calculating, $a_+ a_-$, we find

$$a_+ a_- = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}$$

Thus, $[a_, a+] = 1$, and

$$\hat{H} = \hbar\omega\left(a_\mp a_\pm \mp \frac{1}{2}\right) \tag{4.1.1.57}$$

We call $a_\pm$ the ladder operators because

$$\begin{aligned} \hat{H}a_+\psi &= (E + \hbar\omega)\psi \\ \hat{H}a_-\psi &= (E - \hbar\omega)\psi \end{aligned} \tag{4.1.1.58}$$

That is, if you know your current energy level E, applying the appropriate ladder operator $a_\pm$ will raise or lower your energy to $E \pm \hbar\omega$ [12].

There must be some energy level ψ_0 where we can go no lower. Which would mean

$$a_-\psi_0 = 0$$

So we get

$$\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x} + m\omega x\right)\psi_0 = 0$$

or,

$$\frac{\mathrm{d}\psi_0}{\mathrm{d}x} = -\frac{m\omega}{\hbar}x\psi_0 \tag{4.1.1.59}$$

This is a non-linear differential equation, but we can solve it by separation of variables. The solution is

$$\psi_0 = A_0 \mathrm{e}^{-\frac{m\omega}{2\hbar}x^2}$$

Now, to determine the arbitrary coefficient A_0, we use a process called *normalization*. We begin by taking note that we are operating in a Hilbert probability space. This is a type of inner-product space (see Section 1.3) with the requirement that inner-products correspond to probabilities and must be finite. In particular, if we take the inner product over the domain of the inner-product, we must get 1. Here, the inner-product is an integral, and the domain is the reals. Thus,

$$\langle\psi_0, \psi_0\rangle = 1 = |A_0|^2 \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{m\omega}{\hbar}x^2}\mathrm{d}x = |A_0|^2\sqrt{\frac{\pi\hbar}{m\omega}}$$

Thus, $A_0^2 = \sqrt{m\omega/\pi\hbar}$, and we have

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{m\omega}{2\hbar}x^2} \tag{4.1.1.60}$$

Now, for the eigen energy E_0, via Eq. (4.1.1.57), and keeping in mind that $a_-\psi_0 = 0$

$$\hat{H}\psi = \hbar\omega(a_+a_-\psi_0 + \frac{1}{2}\psi_0) = E_0\psi_0 \tag{4.1.1.61}$$

Thus,

$$E_0 = \frac{1}{2}\hbar\omega$$

The general solution for any n^{th} eigen energy is thus obtained using the ladder operations.

$$\psi_n(x) = A_n(a_+)^n\psi_0(x), \text{ with } E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

To find the coefficient A_n, we assume that, if $\sim$ means proportional to,

$$a_+\psi_n \sim \psi_{n+1} \quad a_-\psi_n \sim \psi_{n+1}$$

We then introduce constants of proportionality, and find those constants by normalizing the calculations of $\langle a_+\psi_n, a_+\psi_n\rangle$ and $\langle a_-\psi_n, a_-\psi_n\rangle$. We get that

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1}, \quad a_-\psi_n = \sqrt{n}\psi_{n-1}$$

which means that

$$\psi_n(x) = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0(x) \tag{4.1.1.62}$$

Great! So we solved the problem (for the computations of the last few equations, see [12]). But, the exasperated reader asks, that is great and all, but why is this here? Fret not! For starters, the inclusion of the ladder operator makes calculations cumbersome. Second, had we used another approach involving power series instead of algebraic operators, we would have obtained a different solution. The goal now is to "bridge-the-gap" to that solution. We begin writing out ψ_0 explicitly and multiplying by $\frac{\sqrt{2^n}}{\sqrt{2^n}}$ in Eq. (4.1.1.62). We shall keep the denominator $\sqrt{2^n}$, but combine the numerator $\sqrt{2^n}$ and $(a_+)^n$ to create a new operator: $H_n(y)$, where

$$y = \sqrt{\frac{m\omega}{\hbar}}x$$

So, in light of these modifications, Eq. (4.1.1.62) becomes

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(y)\mathrm{e}^{-\frac{y^2}{2}} \tag{4.1.1.63}$$

The solution for $H_n(y)$ is

$$H_n(y) = (-1)^n\mathrm{e}^{y^2}\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^n\mathrm{e}^{-y^2} \tag{4.1.1.64}$$

which is the Rodrigues formula for the Hermite Orthogonal Polynomials! That is, $H_n(y)$ *are* are the Hermite polynomials! Surprised?

4.1.2 Numerical Analysis

The most applicable use of orthogonal polynomial sequences is in numerical analysis, due to an OPS serving as a basis on the vector space of polynomials. Their main use comes from approximating integrals. We first begin with a quick introduction for the prerequisite numerical analysis concepts needed.

Introduction to Interpolation

We first begin by taking a set of numbers, called the *nodes*, $\{t_i\}_{i=0}^n$ and construct a polynomial of degree $n+1$ whose roots are those numbers. We assume that the nodes are ordered, that is $t_0 < t_1 < \cdots < t_n$.

$$P_n(x) = \prod_{i=0}^{n}(x - t_i) \tag{4.1.2.1}$$

Notice that the derivative evaluated at one of the nodes is

$$P_n'(t_k) = \lim_{x \to t_k} \frac{P_n(x)}{x - t_k} = \prod_{\substack{i=0 \\ i \neq k}}^{n} (t_k - t_i) \tag{4.1.2.2}$$

We now construct a polynomial of degree n using Eq. (4.1.2.2) and Eq. (4.1.2.1).

$$l_k(x) = \frac{P_n(x)}{(x - t_k)P_n'(t_k)} = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - t_i)}{(t_k - t_i)} \tag{4.1.2.3}$$

Notice that l_k has the orthogonality property

$$l_k(t_j) = \delta_{k_j} \tag{4.1.2.4}$$

Which means if the polynomial l_k is evaluated at one of the nodes, it will be one if the index of the node matches the index of the polynomial, and zero otherwise.

Now, if we have another set of numbers, called the *ordinates*, $\{f(t_k)\}_{k=0}^{n}$, we would like to create a polynomial of degree at most n that passes through $(t_k, f(t_k))$, $k \in \{0, 1, 2, ...n\}$. That polynomial is the Lagrange Interpolation Polynomial:

$$L_n(x) = \sum_{k=0}^{n} l_k(x) f(t_k) \tag{4.1.2.5}$$

and has the orthogonality property due to Eq. (4.1.2.4)

$$L_n(t_j) = \sum_{k=0}^{n} \delta_{k_j} f(t_k) = f(t_j) \quad j \in \{0, 1, 2, ...n\} \tag{4.1.2.6}$$

To continue, we must introduce a foundational theorem from Analysis.

Theorem 4.1.8. *The Weierstrass-Approximation Theorem Let $f = f(x), x \in \mathbf{R}$ be a continuous function on some interval $[a, b]$. Then, $\forall \epsilon > 0$ there exists some polynomial $p(x) = p$ continuous on $[a, b]$ such that $|f(x) - p(x)| < \epsilon$*

Say we have some phenomenon in science or engineering that is governed by some function $f(x)$, but we do not know what this function is; all we can do is record data at different nodes t to receive the ordinates $f(t)$. From the Weierstrass Approximation Theorem, we know that for a function on any compact interval $[a, b]$ (where, for us, $a = t_0$ and $b = t_n$), we can construct a polynomial to approximate that function. Hence, the motivation for this process. Let's cover an example: Say we are given the follow data:

t	$f(t)$
0.5	0.4862
0.6	0.5543
0.7	0.6321

Create an interpolation polynomial to fit this data.
First, we find our l_k polynomials.

$$l_0(x) = \frac{(x-0.6)(x-0.7)}{(0.5-0.6)(0.5-0.7)}$$
$$l_1(x) = \frac{(x-0.5)(x-0.7)}{(0.6-0.5)(0.6-0.7)}$$
$$l_2(x) = \frac{(x-0.5)(x-0.6)}{(0.7-0.5)(0.7-0.6)}$$

Thus, our interpolation polynomial is

$$L_2(x) = 0.4862l_0 + 0.5543l_1 + 0.6321l_2$$

Introduction to Gaussian Quadrature

We now introduce the concept of quadrature. The word "quadrature" means to rectify, or to "square", a shape. The ancient Greeks would use quadrilaterals and triangles to accomplish this, and their hope was that they could find the area of any shape using inscribed quadrilaterals and triangles. Of course, it was later proven that quadrature of a circle is impossible, hence the phrase "You can't square a circle." In modern mathematics, a common problem is trying to approximate an integral as a finite series. That is, if $f(x)$ continuous on some interval $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{i=0}^{n} f(x_i)\Delta x$$

Since integrals are related to the area under a function on a graph, this process is called *Gaussian quadrature*. There are many methods of doing this: Rectangular, Midpoint, Trapezoidal, Simpson's etc. Here, we will observe another method involving orthogonal polynomials and interpolation polynomials.

Now, let $\pi(x)$ be any polynomial of degree at most $2n-1$, and we re-introduce the positive-definite operator $\mathcal{L}$, where, if $w(x)$ is some weight function on a domain Ω

$$\mathcal{L}[f(x)] = \int_\Omega f(x)w(x)\mathrm{d}x \tag{4.1.2.7}$$

(see Eq. (1.5.2) in the introduction for a refresher on this topic). Now, of course, every orthogonal polynomial $\{P_n(x)\}_{n=0}^{\infty}$ sequence has an associated $\mathcal{L}$. That is, if K_n is some constant called the squared norm, then

$$\mathcal{L}[P_n(x), P_m(x)] = \int_\Omega P_n(n)P_m(x)w(x)\mathrm{d}x = K_n\delta_{nm} \tag{4.1.2.8}$$

Now, create the Lagrange interpolation polynomial of degree $n-1$ for our polynomial $\pi(x)$.

$$L_n(x) = \sum_{k=1}^{n} \pi(t_k)l_k(x), \quad l_k(x) = \frac{P_n(x)}{(x-t_k)P_n'(t_k)} \tag{4.1.2.9}$$

where $P_n(x)$ is our orthogonal polynomial of degree n and $\{t_k\}_{k=1}^{n}$ is the set of its roots.

Theorem 4.1.9. $\mathcal{L}[\pi(x)] = \mathcal{L}[L_n(x)]$. *That is,*

$$\int_\Omega \pi(x)w(x)dx = \mathcal{L}[\pi(x)] = \mathcal{L}[L_n(x)] = \sum_{k=1}^{n} \lambda_k \pi(t_k)$$

See [1] for more details on the proof.

Definition 4.1.10. *The Christoffel Numbers λ_k are defined as*

$$\lambda_k = \mathcal{L}[l_k(x)] = \int_\Omega \frac{P_n(x)w(x)dx}{(x-t_k)P_n'(t_k)} \tag{4.1.2.10}$$

Corollary 4.1.10.1. *If $f(x)$ continuous and differentiable on Ω, then*

$$\int_\Omega f(x)w(x)dx \approx \sum_{k=1}^{n} \lambda_k f(t_k)$$

Let's see an example using a polynomial. Evaluate $\int_{-\infty}^{\infty}(3x^2 - 2x^2)\mathrm{e}^{-x^2}\mathrm{d}x$. We begin by setting $\pi(x) = 3x^3 - 2x^2$, $\Omega = (-\infty, \infty)$, and $w(x) = \mathrm{e}^{-x^2}$. We recognize that this weight function and domain are associated with the Hermite polynomials, $\{H_n\}_{n=0}^{\infty}$ (Eq. (1.8.3.2)). Since $\deg(\pi(x)) = 3$, we need $H_3(x), H_3'(x)$ and its roots.

$$H_3(x) = x^3 - \frac{3}{2}x, \quad H_3'(x) = 3x^2 - \frac{3}{2}$$

with roots:

$$x_1 = 0, x_2 = \sqrt{\frac{3}{2}}, x_3 = -\sqrt{\frac{3}{2}}$$

Now, we evaluate both $H_3'(x)$ and $\pi(x)$ with the roots of $H_3(x)$.

Index (k)	x_k	$H_3'(x_k)$	$\pi(x_k)$
1	0	$\frac{3}{2}$	0
2	$\sqrt{3/2}$	3	$3\left(\left[3/2\right]^{3/2} - 1\right)$
3	$-\sqrt{3/2}$	3	$-3\left(\left[3/2\right]^{3/2} + 1\right)$

Next, we evaluate the Christoffel Numbers using the values in the table and The Gamma Function (see Eq. (1.5.3) in the introduction for more details).

$$\lambda_1 = \int_{-\infty}^{\infty} \frac{(x^3 - \frac{3}{2}x)\mathrm{e}^{-x^2}\mathrm{d}x}{(-\frac{3}{2})(x-0)} = \frac{2\pi}{3}$$

$$\lambda_2 = \int_{-\infty}^{\infty} \frac{(x^3 - \frac{3}{2}x)\mathrm{e}^{-x^2}\mathrm{d}x}{3\left(x - \sqrt{\frac{3}{2}}\right)} = \frac{\sqrt{\pi}}{6}$$

$$\lambda_3 = \int_{-\infty}^{\infty} \frac{(x^3 - \frac{3}{2}x)\mathrm{e}^{-x^2}\mathrm{d}x}{3\left(x + \sqrt{\frac{3}{2}}\right)} = \frac{\sqrt{\pi}}{6}$$

Thus, we get

$$\int_{-\infty}^{\infty}(3x^2 - 2x^2)\mathrm{e}^{-x^2}\mathrm{d}x = \sum_{k=1}^{3}\lambda_k\pi(x_k) = -\sqrt{\pi} \tag{4.1.2.11}$$

Now we have a computational method to solve intricate integrals. But, the reader asks, how does one approximate any integral $f(x)$ on $[a, b]$? That is, $\int_a^b f(x)\mathrm{d}x$? First, we must find an orthogonal polynomial whose weight function is $w(x) = 1$. We see that there is: the Legendre Polynomials (Eq. (1.9.6.1)). But, the domain is $[-1, 1]$. So, we must do a change of variables to change the limits of integration $a \to -1, b \to 1$. The diligent reader could find numerous ways, but the most straightforward is a linear line. That is,

$$t = \left(\frac{2}{b-a}\right)x - \frac{b+a}{b-a} = \frac{2x-b-a}{b-a}, \quad \mathrm{d}t = \left(\frac{2}{b-a}\right)\mathrm{d}x \tag{4.1.2.12}$$

So our integral becomes

$$\int_a^b f(x)\mathrm{d}x = \int_{-1}^{1} f\left(\frac{(b-a)t + (b+a)}{2}\right)\left(\frac{b-a}{2}\right)\mathrm{d}t \approx \sum_{k=1}^{n}\lambda_k f(t_k) \tag{4.1.2.13}$$

where P_n is the nth Legendre Polynomials, $\{t_k\}_{k=1}^{n}$ is the set of its roots and the Christoffel Numbers λ_k are

$$\lambda_k = \int_{-1}^{1}\frac{P_n(x)\mathrm{d}x}{(x-t_k)P_n'(t_k)} \tag{4.1.2.14}$$

The approximation sign in Eq. (4.1.2.13) is an equality if $f(x)$ is a polynomial of degree at most $2n - 1$. We now see one of the importances of orthogonal polynomials in numerical analysis.

Bibliography

[1] Theodore S. Chihara. *An Introduction to Orthogonal Polynomials*. 1978.

[2] Daniel J. Galiffa. *On the Higher-Order Sheffer Orthogonal Polynomial Sequences*. Springer Briefs in Mathematics. Springer, 2013.

[3] Joseph Muscat. *Functional Analysis*. Springer, 2014.

[4] David R. Mazur. *Combinatorics:A Guided Tour*. The Mathematical Association of America, 2010.

[5] J. Meixner. Orthogonale polynomsystme mit einer besonderen gestalt der erzeugenden funktion. *Journal of the London Mathematical Society*, 9, 1934.

[6] Roelof Koekoek and Rene F. Swarttouw. The askey-scheme of hypergeometric orthgonal polynomials and its q-analogue.

[7] Mourad E. H. Ismail. *Classical and Quantum Orthogonal Polynomials in One Variable*, volume 98 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2009.

[8] E.D. Rainville. *Special Functions*. Macmillan, 1960.

[9] Mourad E.H. Ismail R. Wlliam Gosper and Ruiming Zhang. On some strange summation formulas. *Illinois Journal of Mathematics*, 37, 1994.

[10] J. Labella and Y.N. Yeh. Combinatorial proofs of some limit formulas involving orthogonal polynomials. *Discrete Mathematics*, 79, 1989/90.

[11] David J. Griffiths. *Introduction to Electrodynamics*. Pearson, 4th edition, 2013.

[12] David J. Griffiths. *Introduction to Quantum Mechanics*. Cambridge University Press, 2nd edition, 2017.

Curriculum Vitae for

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Education

The Pennsylvania State University | Erie, PA **December 2018**
Bachelor of Science in Physics Dean's List
Bachelor of Science in Mathematics
Concentration: Focus in Pure Mathematics
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Honors Education

Behrend Honors Program Spring 2016
Penn State Schreyer's Honors College December 2018

Behrend's Pi Mu Epsilon Chapter Spring 2016
Behrend's Sigma Pi Sigma Chapter Spring 2018
Society of Physics Students Spring 2018
American Physical Society Fall 2017

Supplemental Education

Penn State Orion Wilderness Hiking | State College, PA Summer 2014
Institute for the International Education of Students (IES) Study Abroad | Berlin, Germany Summer 2015

Skills

Proficient in German
Can code in C++ and python
Extensive use of latex coding language for documents
Over three years of educational instruction experience (see below)
Interests and experience in a variety of fields, including political science and STEM
Microsoft Office Products (word, powerpoint, excel)
Comfortable with speaking in front of large groups of people

Honors/Awards

Received the following scholarships

- Anonymous Friend Trustee Scholarship
- McMannis Scholarship
- The Federal Pell Grant
- Lake Erie Trust Fund Scholarship
- Phillips Scholarship for Schreyer's Scholars
- School of Science Scholarship
- Gilman Scholarship for studying abroad in Berlin, Germany

Recipient of Behrend's Academic Excellence in Physics Award for the graduating class of 2018

Leadership and Community Involvement

Volunteered for Penn State Behrend's "Spend a Summer Day" program

Aided Penn State Behrend Honors and Schreyer Honors Programs July 22 2016

- Penn State Behrend Astronomy Open House volunteer

Volunteered for "Science Olympiad" at Penn State Behrend during Spring 2016

President of the Astronomy/Physics club at Penn State Behrend

- Presented at speaking engagement on the topic of gravitational waves to members of the student body

Vice President of the Math Club at Penn State Behrend

- Organized monthly speaker series collaboration between professors and students

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U.S. Government Internship Experience

Sandia National Laboratories, a branch of the Department of Energy, in Albuquerque 12 Weeks | Summer 2017

- Created a protocol to detect context dependence in quantum computing processors.
- Results published in the report CCR Summer Proceedings 2017 by The Center for Computing Research at Sandia National Laboratories
- Presented Findings at the APS March Meeting 2018 in LA, in a talk titled "A Procedure in Detecting Crosstalk and Drift in Quantum Information Processors"
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- Results submitted to Physical Review X. Draft is available on the arXiv under the title

"Probing context-dependent errors in quantum processors" by Rudinger, et. al

Penn State Research Experience

Researched orthogonal polynomials Fall 2017 - Present

- Supervisor: Dr. Galiffa, tenured professor of mathematics, djg34@psu.edu
- Presented findings at the 2018 Mathematical Association of America (MAA) Allegheny Mountain Section Spring Conference in a talk titled "Schrödinger Form Derivation for Orthogonal Polynomial Sequences"
- Research became foundation for undergraduate thesis titled "Theory, Methods and Applications of the Classical Hypergeometric Orthogonal Polynomial Sequences of Sheffer and Jacobi"
- Received grant of $1200 to continue research for thesis Summer 2018
- Gave defense of thesis in talk open to the public November 2018
- Additional findings involving orthogonal polynomials currently being prepared for submission to journals for publication as two papers

Exo-moon Detection Research Summer 2016

- $1200 Research Grant to study exomoons under the auspices of Dr. Darren Williams, Professor of Astronomy and Physics, dmw145@psu.edu
- Gave four talks on gravitational waves, one on Dyson Spheres, and one on the history of calculus for Dr. Williams

International treaties and the Comprehensive Statistical Database on Multilateral Treaties Spring 2015

- Dr. John Gamble, Professor of Political Science and International Law, jkg2@psu.edu
- Pursued the European Journal of International Law and American Journal of International Law for uses of graphs
- Results used in a paper titled "Teaching International Law Better and to More People: Goals and Prospects" presented at the 77th Biennial Conference of the International law Association (ILA) 8–11 August 2016, Sandton Convention Centre Johannesburg, South Africa

Presented derivation of the Schrödinger Equation from the Klein-Gordon for Dr. Blair Tuttle, Department of Physics

Tutoring and Lecturing Experience

Teaching Assistant, Physics Department, Penn State

- Flipped classroom structure funded by NSF grant
- Physics 211 (kinematics, energy, introductory physics) Spring 2016, 2017 and Fall 2017
- Physics 212 (electromagnetism) Fall 2016, 2018 and Spring 2018
- Supervisor: Dr. Yeung, Department Chair of Physics and Professor of Physics cxy11@psu.edu

Planetarium Assistant at Yahn Planetarium Summer 2016 – Spring 2017

- Presented astronomical subjects and the layout of the constellations to the average person
- Yahn Planetarium director: Jim Gavio, jvg10@psu.edu

Penn State Behrend Learning Resource Center January 2016 – Present

- Individual tutoring Spring 2016 – Present
- Physics Study Sessions – instruct students as a group in a classroom
- Weekly Physics 211 Study Session Spring 2016
- Weekly Physics 212 Study Session Fall 2016
- Weekly Physics 211 Study Session and a Physics 212 study session Spring 2017 – Fall 2018
- Barbara Hido, Program Coordinator, for more information - bjh29@psu.edu

Penn State Behrend School of Humanities and Social Sciences

- Teaching assistant to Political Science 3: Introduction to Comparative Politics Fall 2015
- Teaching assistant to Political Science 20: Politics to Western Europe Fall 2017
- Supervisor: Dr. John Gamble, jkg2@psu.edu