THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF MATHEMATICS

Intersection Theory

JACOB KELLER
Spring 2019

A thesis
submitted in partial fulfillment
of the requirements
for a baccalaureate degree
in Mathematics
with honors in Mathematics

Reviewed and approved* by the following:

Jack Huizenga
Assistant Professor of Mathematics
Thesis Supervisor

Ludmil Zikatanov
Professor of Mathematics
Honors Adviser

*Signatures are on file in the Schreyer Honors College.

Abstract

This thesis is intended to give an introduction to intersection theory in algebraic geometry. It is split into three chapters concerning the Chow ring, Schubert Calculus, and Chern classes respectively. It is intended to give the reader a toolkit with which to attack various enumerative problems arising from algebraic geometry. These problems include counting the points in an intersection of two varieties or counting the number of lines contained in a variety, and are central to many facets of modern algebraic geometry.

Table of Contents

Acknowledgements

First, I would like to thank Dr. Huizenga for the years of mentorship and for dedicating so much time to discuss algebraic geometry with me. I would also like to thank my family for all the support throughout my life and my college career.

Chapter 1

Introduction

1.1 Introduction

The purpose of this thesis is to give an introduction to intersection theory with a focus on solving enumerative problems. The subject only became truly rigorous with the introduction of Fulton's book *Intersection Theory* [2] in 1984. This is because the foundational results of the theory are complicated to prove in sufficient generality. Because the focus of this thesis is on solving problems, we will relegate most of the technical issues to the references.

Even though intersection theory only recently acquired solid foundations, its ideas have been around for a very long time. Modern intersection theory is widely believed to have started with the work of Hermann Schubert, who studied intersection theory on Grassmannian varieties. This study is now called Schubert calculus in his honor. Grassmannians are varieties that parameterize linear spaces, and Schubert calculus allows one to study the lines or higher dimensional plances that satisfy certain properties. For instance, in this thesis, we will calculate the number of lines that lie on a smooth cubic surface in three dimensional projective space.

The idea of studying intersection theory on parameter spaces is central to enumerative geometry and intersection theory as a whole. An important example is the following. Suppose we are given five distinct smooth conics in the projective plane: can we calculate how many other conics are tangent to all five? Here tangent means meeting in at least one point at which the tangent spaces of the conics coincide.

A natural approach to this question is as follows. A conic in the projective plane is cut out by an equation of the form

$$AX^2 + BY^2 + CZ^2 + DXY + EXZ + FYZ = 0,$$

where $A, B, C, D, E, F \in \mathbb{C}$, and $[X : Y : Z]$ are the homogeneous coordinates on the projective plane. Two conics, defined by coefficients (A, B, C, D, E, F) and (A', B', C', D', E', F') are equal if and only if there is some non-zero constant $\lambda \in \mathbb{C}$ such that

$$(A, B, C, D, E, F) = (\lambda A', \lambda B', \lambda C', \lambda D', \lambda E', \lambda F').$$

Therefore each conic is uniquely described by the homogeneous coordinates

$$[A : B : C : D : E : F].$$

The collection of all of these homogeneous coordinates is a space, $\mathbb{P}^5$, which is a parameter space for conics in the plane. For any fixed conic C, we can consider a subset

$$\Sigma_C \subseteq \mathbb{P}^5$$

consisting of conics tangent to C. This is closed in the Zariski topology, and is in fact a hypersurface defined by a degree 6 equation. For 5 general conics $C_1, \ldots, C_5$, we note that the set of conics tangent to all 5 of them is the space

$$\Sigma_{C_1} \cap \ldots \cap \Sigma_{C_5}.$$

Thus we have converted the problem of counting conics tangent to five given conics to a problem about intersections of subvarieties of the parameter space of conics.

To finish the problem, one wants to use Bézout's theorem, which would say that

$$|\Sigma_{C_1} \cap \ldots \cap \Sigma_{C_5}| = \deg(\Sigma_{C_1}) \cdot \ldots \cdot \deg(\Sigma_{C_5}) = 6^5.$$

Here $\deg$ stands for the degree of a variety, which we discuss in detail later.

This solution seems tidy, but unfortunately we have over-counted. The true answer is 3264 conics, not the $6^5 = 7776$ predicted by Bézout's theorem. The issue lies in our choice of parameter space.

The space $\mathbb{P}^5$ does parameterize all smooth cubics, but it also parameterizes some degenerate cases, in particular the double lines

$$(\alpha X + \beta Y + \gamma Z)^2 = 0.$$

These objects are best understood through the theory of schemes, in which they are distinct from the lines

$$(\alpha X + \beta Y + \gamma Z) = 0$$

and are in fact tangent to everything they touch.

We would like to form a better parameter space that does not include these objects, but this is another topic entirely. We hope that this example shows the subtlety of enumerative geometry and gives the reader a feel for the difficulties inherent to the subject.

This thesis is intended for someone who has learned most of the material from a first course on algebraic varieties at the level of Harris' book [1]. The fourth chapter requires some familiarity with the concept of a vector bundle.

Chapter 2

Foundations: Bézout's Theorem and the Chow Ring

2.1 Bézout's Theorem

Bézout's theorem is a foundational starting point for intersection theory and enumerative geometry. It effectively describes how two subvarieties of projective space intersect in terms of the degrees of the subvarieties. We recall here the definition of the degree of a subvariety of projective space. This treatment follows [3].

Definition 2.1.1. Let $X \subseteq \mathbb{P}^n$ be a closed subvariety of dimension m. There is a dense open subset of the Grassmannian $\mathbb{G}(n-m, n)$ whose elements all intersect X in exactly d points. We call d the degree of X. More concisely, the degree of X is the number of intersections of X with the general $codim(X)$-plane.

There is some work required to show that the degree is well defined, and much more on this topic can be found in [1]. The following proposition helps justify the word "degree."

Proposition 2.1.2. *If $X \subseteq \mathbb{P}^n$ is a hypersurface defined by a homogeneous polynomial*

$$F(x_0, \ldots, x_n)$$

of degree d, then the degree of X is d.

To state Bézout's theorem in full generality, we would need to introduce the notion of intersection multiplicity. This is a challenging notion to get right, and we will not get into it. To get around this issue, we need the following definitions.

Definition 2.1.3. Two quasi-projective subvarieties Y, Z of a quasi-projective variety X are said to intersect *transversely* at a point $p \in Y \cap Z$ if $X, Y,$ and Z are all smooth at p and the tangent spaces of Y and Z span the tangent space of X at p, i.e. $T_pY + T_pZ = T_pX$.

Definition 2.1.4. Let $X, Y \subseteq Z$ be as above. We say Y and Z intersect generically transversely if they intersect transversely at a general point of each irreducible component of $Y \cap Z$.

If X and Y intersect generically transversely then the intersection multiplicity of each component of $X \cap Y$ is 1 and we have the following result.

Theorem 2.1.5. *(Bézout's Theorem) If X and Y are two smooth projective subvarieties of $\mathbb{P}^n$ intersecting generically transversely, then we have*

$$\deg(X \cap Y) = \deg(X)\deg(Y). \tag{2.1}$$

More generally, if we remove the transversality requirement we always have the inequality

$$\deg(X \cap Y) \leq \deg(X)\deg(Y). \tag{2.2}$$

Proof. [1], Theorem 18.3 □

Because there is only one n-plane in $\mathbb{P}^n$, the degree of a collection of m points in $\mathbb{P}^n$ is m. Therefore as a special case where $dim(X \cap Y) = 0$, we have that

$$|X \cap Y| = \deg(X)\deg(Y).$$

2.2 The Chow Ring

Bézout's theorem gives quite complete information about the intersections of subvarieties of $\mathbb{P}^n$. Part of the goal of intersection theory is to gain a similar understanding of intersections of subvarieties of more general projective varieties. This has been a challenging goal, and only found solid ground to stand on with the book *Intersection Theory* by Bill Fulton [2]. Fulton's book gives a rigorous construction of the Chow ring of a projective variety. We will give a definition of the Chow ring shortly, but the idea is that the Chow ring is an analogue of the cohomology ring of a topological space, and the ring structure reflects how subvarieties intersect. The elements of the Chow ring of a variety are cycles modulo rational equivalence, so we will define those notions now.

Definition 2.2.1. Let X be a smooth quasi-projective variety. A cycle on X is a formal linear combination

$$Z = \sum n_i [Z_i]$$

where $n_i \in \mathbb{Z}$ and Z_i are closed irreducible subvarieties of X. The collection of all cycles on X is a group (the free abelian group generated by closed irreducible subvarieties) denoted $Z(X)$. Additionally, a cycle is called effective if each n_i is positive.

Definition 2.2.2. Two closed irreducible subvarieties $Y, Z \subseteq X$ of a quasi-projective variety X are called rationally equivalent if there is an irreducible closed subvariety $W \subseteq \mathbb{P}^1 \times X$, not contained in any fiber $p \times X$, $p \in \mathbb{P}^1$, such that

$$W \cap ([0:1] \times X) = Y \quad \text{and} \quad W \cap ([1:0] \times X) = Z,$$

where $[u:v]$ denotes the homogeneous coordinates on $\mathbb{P}^1$, and we identify fibers $p \times X$ with X.

In the language of topology, W should be thought of as a homology between the cycles Y and Z. When restricted to divisors, it is reasonable to expect that two divisors are rationally equivalent if and only if they are linearly equivalent. This is true, and the proof can be found in Fulton [2], Proposition 1.6.

We want to use rational equivalence to give an equivalence relation on cycles, which will finally let us define the Chow ring.

Definition 2.2.3. The collection of cycles given by

$$[W \cap ([0:1] \times X)] - [W \cap ([1:0] \times X)].$$

generates a subgroup which we denote by $\text{Rat}(X) \subseteq Z(X)$. The Chow ring is the quotient

$$A(X) = Z(X)/\text{Rat}(X).$$

This definition only gives the group structure of $A(X)$. To give it a ring structure has been a challenge that was only rigorously achieved with Fulton's book [2]. We will not go through the construction here; we just say that

Theorem 2.2.4. *The group $A(X)$ admits a multiplication map $A(X) \otimes_{\mathbb{Z}} A(X) \to A(X)$ (denoted by concatenation) which gives $A(X)$ the structure of a commutative ring. This map is characterized by the following property.*

- *If $Y, Z \subseteq X$ are closed irreducible subvarieties that intersect generically transversely, then*

$$[Y][Z] = [Y \cap Z]. \tag{2.3}$$

Therefore the product structure on $A(X)$ is given by intersection, at least in the nice case of generically transverse intersections.

Because each cycle has a well-defined dimension, you might expect $A(X)$ to admit the structure of a graded ring. In fact this is the case

Proposition 2.2.5. *Rational equivalence on $Z(X)$ respects dimension, and therefore we have an isomorphism of groups*

$$A(X) = \bigoplus_{i=0}^{dim(X)} A^i(X). \tag{2.4}$$

where $A^i(X)$ is the subgroup of cycles of codimension i.

Further, this direct sum decomposition gives $A(X)$ the structure of a graded ring, i.e.

$$A^i(X)A^j(X) \subseteq A^{i+j}(X). \tag{2.5}$$

where $A^i(X)$ is taken to be 0 when $i > dim(X)$ or $i < 0$.

The first statement of Proposition 2.2.5 follows from the definition of rational equivalence along with the principal ideal theorem, and can be found in [3], Proposition 1.4. The second statement follows because if $Y, Z \subseteq X$ intersect generically transversely then their intersection has the expected dimension.

The proof of Theorem 2.2.4 follows directly from the moving lemma which we reproduce here for reference.

Theorem 2.2.6. *(Moving Lemma) Let X be a smooth quasi-projective variety, and Y and Z be any two closed irreducible subvarieties. Then*

- *There exist cycles Y' and Z' representing the classes $[Y]$ and $[Z]$ such that Y' intersects Z' generically transversely. This means that if*

$$Y' = \sum m_i Y_i \quad \text{and} \quad Z' = \sum n_i Z_i$$

then each Y_i intersects each Z_j generically transversely.

- *The class $[Y' \cap Z']$ only depends on the classes $[Y]$ and $[Z]$ (not on the choice of Y' and Z').*

There is a convenient version of the moving lemma for the case where X has sufficient automorphisms which we recall now.

Theorem 2.2.7. *(Kleiman's Theorem) Let G be an algebraic group that acts transitively on a quasi-projective variety X and $Y \subseteq X$ a closed subvariety. Then*

- *For another subvariety $Z \subseteq X$, there is a dense open set $U \subseteq G$ such that for any $g \in U$, gZ intersects Y generically transversely.*

- *If G is affine, then $[gY] = [Y]$ for all $g \in G$.*

Proof. [3], Theorem 1.7. □

If the reader is not familiar with the definition of an algebraic group or the definition of the action of an algebraic group on a variety, that should not be a problem. We will mainly apply the theorem to the case of $PGL(n+1)$ acting on $\mathbb{P}^n$ and on $GL(n)$ acting on the Grassmannian $G(k, n)$.

In order to use this theory to solve problems, we will need to know the structure of some Chow rings. For now we will start with projective spaces and their products.

Theorem 2.2.8. *We can compute the Chow ring $A(\mathbb{P}^n)$ as follows. Let $\zeta \in A^1(X)$ be the class of a hyperplane, which makes sense because any two hyperplanes are rationally equivalent. Then we have an isomorphism*

$$A(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1}). \tag{2.6}$$

Under this identification, we have that if $Y \subseteq \mathbb{P}^n$ is irreducible of dimension m and degree d then

$$[Y] = d\zeta^m.$$

Proof. [3], Theorem 2.1. □

Note that this result subsumes Bézout's theorem in that if Y is of dimesion n and $\deg(Y) = d$, Z is of dimension m and $\deg(Z) = e$, and Y intersects Z transversely then

$$[Y][Z] = d\zeta^n e\zeta^m = de\zeta^{n+m}. \tag{2.7}$$

From here we have an analogue of the Künneth formula for products of projective spaces.

Theorem 2.2.9. *Let $n_1, \ldots, n_m$ be positive integers, and for each $1 \leq i \leq m$, let ζ_i be the hyperplane class in $A(\mathbb{P}^{n_i})$. Then we have a ring isomorphism*

$$A(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}) \cong \mathbb{Z}[\zeta_1, \ldots, \zeta_m]/(\zeta_1^{n_1+1}, \ldots, \zeta_m^{n_m+1}). \tag{2.8}$$

2.3 Examples

Now that we understand some Chow rings we can start to see intersection theory in action.

Question 2.3.1. *([3], Exercise 2.34) Let $\Psi \subseteq \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ be the locus of triples (p, q, r) such that $p, q,$ and r are collinear in $\mathbb{P}^n$. Find the class of $[\Psi] \in A(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$*

To tackle this question, the first step is to show Ψ is a closed subvariety and find its dimension. We note that Ψ is the image of the incidence correspondence

$$P = \{(L, p, q, r) \mid p, q, r \in L\} \subseteq \mathbb{G}(1, n) \times \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n. \tag{2.9}$$

under the projection to $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. Here $\mathbb{G}(1, n)$ is the Grassmannian of lines in $\mathbb{P}^n$, which is equal to $G(2, n+1)$, the Grassmannian of two-planes in affine space $\mathbb{A}^{n+1}$.

We need to show that P is closed. We will do the case where

$$P = \{(L, p) \mid p \in L\} \subseteq \mathbb{G}(1, n) \times \mathbb{P}^n$$

and the case for three points follows immediately. Suppose that L is the projectivization of the two-plane spanned by

$$x = (x_1, \ldots, x_{n+1}) \quad \text{and} \quad y = (y_1, \ldots, y_{n+1}) \in \mathbb{A}^{n+1}.$$

Further if p is the projectivization of the line spanned by

$$p = (p_1, \ldots, p_{n+1})$$

then we see that the condition that p lies in L is equivalent to the statement that $x, y,$ and p are linearly independent, and therefore that the matrix

$$\begin{pmatrix} x_1 & y_1 & p_1 \\ \vdots & \vdots & \vdots \\ x_{n+1} & y_{n+1} & p_{n+1} \end{pmatrix}$$

has rank less than 3. This is equivalent to the statement that each 3×3 minor vanishes. This gives quadratic equations that are homogeneous in the p variables and in the Plücker coordinates of the line. These equations therefore cut out P.

Because morphisms between projective varieties are closed maps, we see that Ψ is closed. The fiber over a line $L \in \mathbb{G}(1, n)$ is the space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of triples of points on L. Because $\mathbb{G}(1, n)$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are irreducible projective varieties, we know that P is an irreducible variety. The dimension of $\mathbb{G}(1, n)$ is $2((n+1)-2) = 2n-2$ and the dimension of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is three. Since P surjects onto $\mathbb{G}(1, n)$, we can see that the dimension of P is $2n - 2 + 3 = 2n + 1$.

If (p, q, r) is in the image of the projection of P to $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, and not all of $p, q,$ and r are equal, then there is a unique line through $p, q,$ and r. The subset

$$\{(L, p, p, p) \mid p \in L\} \subseteq P.$$

is a closed subset of P which is not all of P. Thus its compliment is a dense open subset of P which maps in a 1-1 fashion to Ψ with dense image. This implies that the dimension of Ψ is equal to the dimension of P, which is $2n+1$. In particular, Ψ has codimension $n-1$ inside $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$.

Now to understand the class $[\Psi]$, we start with the case $n = 2$. In this case, Ψ is a hypersurface, and so we have an equality

$$[\Psi] = a\zeta_1 + b\zeta_2 + c\zeta_3.$$

Now we have to determine the integers $a, b,$ and c. To do this we want to multiply $[\Psi]$ with specific classes in $A(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$. For example we can see that

$$\zeta_1\zeta_2^2\zeta_3^2[\Psi] = a\zeta_1^2\zeta_2^2\zeta_3^2 + b\zeta_1\zeta_2^3\zeta_3^2 + c\zeta_1\zeta_2^2\zeta_3^3 = a\zeta_1^2\zeta_2^2\zeta_3^2. \tag{2.10}$$

This last equality is because we have $\zeta_i^3 = 0$ for $i = 1, 2, 3$ according to Theorem 2.2.9. Because $\zeta_1^2\zeta_2^2\zeta_3^2$ is the class of a point, a is the number of points in the intersection of a general cycle representing the class $\zeta_1\zeta_2^2\zeta_3^2$ with Ψ.

The cycles representing $\zeta_1\zeta_2^2\zeta_3^2$ are those of the form $L \times \{p\} \times \{q\}$ where L is an arbitrary line in $\mathbb{P}^2$. This intersects Ψ at the point $(\overline{pq} \cap L, p, q)$. As long as $p \neq q$ then this is a well-defined point, and this shows that $a = 1$. By symmetry, we see that $b = 1$ and $c = 1$ as well.

This strategy of computing classes in $A(X)$—in which one finds the dimension of the subvariety, writes the general form of a class of that dimension, and finds the coefficients by computing intersections—is called the method of undetermined coefficients. If one knows the structure of the relevant Chow ring, this method is a rather powerful computational tool.

Before jumping to the general case, it will be instructive to consider the case $n = 3$. In this case, Ψ is codimension 2, and we have

$$[\Psi] = a\zeta_1^2 + b\zeta_1\zeta_2 + c\zeta_2^2 + d\zeta_1\zeta_3 + e\zeta_2\zeta_3 + f\zeta_3^2 \tag{2.11}$$

By symmetry, the only coefficients we need to calculate are a and b. For a we see that

$$\zeta_1\zeta_2^3\zeta_3^3[\Psi] = a\zeta_1^3\zeta_2^3\zeta_3^3. \tag{2.12}$$

The class $\zeta_1\zeta_2^3\zeta_3^3$ is the class of subvarieties $H \times \{p\} \times \{q\}$ where H is a hyperplane. The line $\overline{pq}$ intersects H in one point, and the class of that point is equal to $\zeta_1\zeta_2^3\zeta_3^3[\Psi]$, showing that $a = 1$.

To calculate b we see that

$$\zeta_1^2\zeta_2^2\zeta_3^3[\Psi] = b\zeta_1^3\zeta_2^3\zeta_3^3. \tag{2.13}$$

The class $\zeta_1^2\zeta_2^2\zeta_3^3$ represents the cycles of the form $L \times L' \times \{q\}$. Where L and L' are lines. For the general cycle of this form, L and L' are skew lines in $\mathbb{P}^3$. Consider the set of points which are collinear with q and some point of L'. These points form a two-plane which by dimension reasons hits L in exactly one point. The class of that point is therefore equal to $\zeta_1^2\zeta_2^2\zeta_3^3[\Psi]$ which implies that $b = 1$.

This suggests our strategy for general n. The locus Ψ has codimension $n - 1$ and so we have

$$[\Psi] = \sum_{a_1+a_2+a_3=n-1} a_{a_1,a_2,a_3}\zeta_1^{a_1}\zeta_2^{a_2}\zeta_3^{a_3}. \tag{2.14}$$

To isolate the coefficient a_{a_1,a_2,a_3}, we need to calculate

$$\zeta_1^{b_1}\zeta_2^{b_2}\zeta_3^{b_3}[\Psi]$$

where $a_i + b_i = n$. The class $\zeta_1^{b_1}\zeta_2^{b_2}\zeta_3^{b_3}$ represents cycles of the form $L_1 \times L_2 \times L_3$ where L_i is a linear space of codimension b_i. We want to calculate the dimension of the linear space spanned by L_2 and L_3. These are the projectivizations of linear spaces $L_2', L_3' \subseteq \mathbb{A}^{n+1}$ of dimension $a_2 + 1$ and $a_3 + 1$ respectively. For the general cycle of the form $L_1 \times L_2 \times L_3$, we have that $L_2' \cap L_3' = \{0\}$, and so their span has dimension $a_1 + a_2 + 2$, and L_2 and L_3 span an $a_2 + a_3 + 1$-plane in $\mathbb{P}^n$. We have that $a_1 + a_2 + a_3 = n - 1$ so $a_2 + a_3 + 1 = n - a_1 = b_1$. A dimension-$b_1$ plane (such as the span of L_2 and L_3) intersects a general codimension-b_1 plane (L_1) in exactly one point this one point represents the unique point in L_1 which lies on a line passing through both L_2 and L_3, and so the class of that point is the desired product

$$\zeta_1^{b_1}\zeta_2^{b_2}\zeta_3^{b_3}[\Psi] = \zeta_1^n\zeta_2^n\zeta_3^n$$

and therefore each $a_{a_1,a_2,a_3} = 1$.

We now use this technique to solve a geometric problem.

Question 2.3.2. *Let A, B, and $C : \mathbb{P}^2 \to \mathbb{P}^2$ be three general automorphisms. For how many points $p \in \mathbb{P}^2$ are the points $p, A(p), B(p)$, and $C(p)$ collinear?*

To do this we need to know the class of the subvariety

$$\Psi = \{(p, q, r, s) \mid p, q, r, \text{ and } s \text{ are collinear}\} \subseteq \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2.$$

because this is so similar to the previous computation, we leave the details to the reader. By the same incidence correspondence we see that it has codimension 2. Its class is then

$$[\Psi] = \sum_{1 \leq i < j \leq 4} \zeta_i \zeta_j \tag{2.15}$$

note in particular there is no ζ_i^2 component for any i.

We will calculate the class

$$\Phi = \{(p, A(p), B(p), C(p)) \mid p \in \mathbb{P}^2\} \subseteq \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \tag{2.16}$$

which has dimension 2. Again we use the method of undetermined coefficients. The general form of a class of dimension 2 is

$$a\zeta_2^2\zeta_3^2\zeta_4^2 + b\zeta_1\zeta_2\zeta_3^2\zeta_4^2 + \dots. \tag{2.17}$$

Here the other terms are obtained from the first two by permuting the coefficients. By symmetry we only need to determine a and b. Therefore we calculate

$$\zeta_1^2[\Phi] = a\zeta_1^2\zeta_2^2\zeta_3^2\zeta_4^2.$$

The class ζ_1^2 represents cycles of the form $\{p\} \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. This intersects $[\Phi]$ at exactly the point $(p, A(p), B(p), C(p)$ and therefore $a = 1$. Now we calculate

$$\zeta_1\zeta_2[\Phi] = b\zeta_1^2\zeta_2^2\zeta_3^2\zeta_4^2.$$

The class $\zeta_1\zeta_2$ represents cycles of the form $L \times L' \times \mathbb{P}^2 \times \mathbb{P}^2$ where L and L' are lines in $\mathbb{P}^2$. We see that as p varies in L, $A(p)$ traces out a general line in $\mathbb{P}^2$. This line hits L' in exactly one point. Thus there is exactly one point $p \in L$ such that $A(p) \in L'$ and therefore $\Phi \cap L \times L' \times \mathbb{P}^2 \times \mathbb{P}^2 = (p, A(p), B(p), C(p))$ and $b = 1$.

Now we need to compute the product $[\Psi][\Phi]$. We have that

$$[\Psi][\Phi] = (\zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_1\zeta_4 + \zeta_2\zeta_3 + \zeta_2\zeta_4 + \zeta_3\zeta_4)$$
$$\times \left(\zeta_1\zeta_2\zeta_3^2\zeta_4^2 + \zeta_1\zeta_3\zeta_2^2\zeta_4^2 + \zeta_1\zeta_4\zeta_2^2\zeta_3^2 + \zeta_2\zeta_3\zeta_1^2\zeta_4^2 + \zeta_2\zeta_4\zeta_1^2\zeta_3^2 + \zeta_3\zeta_4\zeta_1^2\zeta_2^2 + \sum_{1 \leq i < j < k \leq 4} \zeta_i^2\zeta_j^2\zeta_k^2\right).$$

Note that the product

$$(\zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_1\zeta_4 + \zeta_2\zeta_3 + \zeta_2\zeta_4 + \zeta_3\zeta_4)\left(\sum_{1 \leq i < j < k \leq 4} \zeta_i^2\zeta_j^2\zeta_k^2\right)$$

is zero, and therefore we can ignore terms of the form $\zeta_i^2\zeta_j^2\zeta_k^2$. Taking the product of the other terms gives

$$[\Psi][\Phi] = 6\zeta_1^2\zeta_2^2\zeta_3^2\zeta_4^2,$$

and our final answer is that there are 6 points $p \in \mathbb{P}^2$ such that $p, A(p), B(p)$, and $C(p)$ are collinear for general automorphisms A,B, and C.

The fact that this product actually counts these points follows from Kleiman's theorem. In particular, $(\mathbb{P}^2)^4$ carries the natural action of PGL$(3)^4$. If $g = (g_1, g_2, g_3, g_4) \in$ PGL$(3)^4$ and A, B, C are given along with the corresponding cycle Φ then we have that

$$g\Phi = \{(g_1(p), g_2(A(p)), g_3(B(p)), g_4(C(p)) \mid p \in \mathbb{P}^2\}$$

which is equal to

$$\{(p, g_2(A(g_1^{-1}p)), g_3(B(g_1^{-1}p)), g_4(C(g_1^{-1}p)) \mid p \in \mathbb{P}^2\}.$$

Therefore if Φ coresponds to the automorphisms A, B, C then $g\Phi$ corresponds to the automorphisms $g_2Ag_1^{-1}, g_3Bg_1^{-1}, g_4Cg_1^{-1}$ in particular one can obtain any triple of automorphisms in this way. Therefore considering general triples of automorphisms is the same as considering general translates $g\Phi$ of Φ, and so Kleiman's theorem says that Φ and Ψ intersect generically transversely for a general triple of automorphisms A, B, C.

Chapter 3

Schubert Calculus

3.1 Schubert Cycles

In this chapter we will study the Chow ring of the Grassmanian varieties $G(k,n)$. For arbitrary k and n, we will describe the generators of this ring, the Schubert cycles, but we will only see the multiplicative structure for $\mathbb{G}(1,3)$, the Grassmannian of lines in $\mathbb{P}^3$. To describe the multiplicative structure for arbitrary k and n is possible, but not too easy, and there is already a lot to say about $\mathbb{G}(1,3)$. The reason we choose $\mathbb{G}(1,3)$ $(= G(2,4))$ is because it is the first non-trivial example: $G(1,n)$ is equal to $\mathbb{P}^{n-1}$ for all n, and $G(2,3)$ is in a sense dual to $G(1,3)$, in that

$$G(k,V) \cong G(n-k,V^*) \tag{3.1}$$

for an n-dimensional vector space V. The isomorphism of (3.1) is given by taking a k-plane $W \subseteq V$ to the annihilator

$$\mathrm{Ann}(W) = \{\varphi \in V^* \mid \varphi(W) = 0\}. \tag{3.2}$$

Before getting into the structure of $A(\mathbb{G}(1,3))$ we will define the Schubert cycles, which are classes in $A(G(k,V))$ where V is an n-dimensional vector space. To do this we first need to fix a complete flag inside V, i.e. an increasing sequence of subspaces

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_{n-1} \subsetneq V_n = V. \tag{3.3}$$

Such a flag is called complete because we have $\dim(V_i/V_{i+1}) = 1$ and so there are no subspaces W such that $V_i \subsetneq W \subsetneq V_{i+1}$. That is, there are no subspaces you could include in the flag to make it longer.

We can now use this flag to classify the different k-planes inside V. Consider a k-plane $W \subseteq V$. We want to consider the sequence

$$\{0\} = V_0 \cap W \subseteq V_1 \cap W \subseteq \ldots \subseteq V_n \cap W = W. \tag{3.4}$$

By the second isomorphism theorem, we have that

$$(V_{i+1} \cap W) \,/\, (V_i \cap W) \cong (V_i + V_{i+1} \cap W)/V_i \subseteq V_{i+1}/V_i \tag{3.5}$$

which shows that $\dim\left((V_{i+1} \cap W) \,/\, (V_i \cap W)\right) \leq 1$. That is, for each $0 \leq i \leq n-1$, either $V_i \cap W = V_{i+1} \cap W$, or the dimension of $V_i \cap W$ is one greater than the dimension of $V_{i+1} \cap W$.

The Schubert cycles are now described by the integers i such that $\dim(V_{i+1} \cap W)$ is one higher than $\dim(V_i \cap W)$.

Definition 3.1.1. Let $i = i_1, \ldots, i_k$ be a list of integers such that $n-k \geq i_1 \geq \ldots \geq i_k \geq 0$. Given the list i, we define closed subsets

$$\Sigma_{i_1,\ldots,i_k} = \{W \in G(k,n) \mid \dim(V_{n-k+j-i_j} \cap W) \geq j \text{ for all } j = 1, \ldots, k\}. \tag{3.6}$$

and call these Schubert cycles. We sometimes denote this as Σ_i for brevity.

The fact that they are closed in the Zariski topology on $G(k,n)$ is left to the reader. We will also omit the proof of the following, but it is an important feature of the way we defined Schubert cycles.

Proposition 3.1.2. *Let $\Sigma_{i_1,\ldots,i_k}$ be a Schubert cycle. Then the codimension of $\Sigma_{i_1,\ldots,i_k}$ is $i_1+\ldots+i_k$.*

Proof. [3], Theorem 4.1. □

This definition seems quite ugly at first sight, but it has a lot of advantages. We will now try to give some intuition behind it. First consider the set of k-planes $W \subseteq V$ such that the jumps in dimension happen as late as possible, that is

$$\{0\} = W \cap V_0 = W \cap V_1 = \ldots = W \cap V_{n-k} \subsetneq W \cap V_{n-k+1} \subsetneq \ldots \subsetneq W \cap V_n = W.$$

The collection of all such W forms an open subset of the Schubert cycle $\Sigma_{0,\ldots,0} = G(k,n)$. This means that the "expected" jumps in dimension happen at the last k possible spots. The numbers i_j then encode how much earlier than expected each jump is.

To illustrate this, we describe the Schubert cycles for $\mathbb{G}(1,3)$. We visualize this in $\mathbb{P}^3$ as opposed to $\mathbb{A}^4$. In particular, our flag is now

$$\{p\} = V_0 \subseteq V_1 \subsetneq V_2 \subsetneq V_3 = \mathbb{P}^3$$

where p is an arbitrary point, V_1 a line, V_2 a 2-plane. The general line $L \subseteq \mathbb{P}^3$ has jumps like

$$\emptyset = L \cap V_0 = L \cap V_1 \subsetneq L \cap V_2 \subsetneq L \cap V_3 = L.$$

That is, L intersects the 2-plane V_2 but not the line V_1 nor the point V_0.

The general $L \subseteq \Sigma_{1,0}$ jumps like

$$\emptyset = L \cap V_0 \subsetneq L \cap V_1 = L \cap V_2 \subsetneq L \cap V_3 = L. \tag{3.7}$$

This means that L intersects the line V_1, but not the point V_0, and is not contained in the 2-plane V_2.

Note that this is the general line in $\Sigma_{1,0}$, and does not describe every line in $\Sigma_{1,0}$. In general, let $\Sigma_{i_1,\ldots,i_k}$ by a Schubert cycle, and let $\Sigma_{i'_1,\ldots,i'_k}$ be another one such that $i'_j \geq i_j$ for all j. Then the definition implies that

$$\Sigma_{i'_1,\ldots,i'_k} \subseteq \Sigma_{i_1,\ldots,i_k}. \tag{3.8}$$

In fact, (3.8) holds if and only if $i'_j \geq i_j$ for all j. This is one of the benefits of the indexing system we chose.

Definition 3.1.3. Let $i = i_1, \ldots, i_k$ be a list defining a Schubert cycle $\Sigma_{i_1,\ldots,i_k}$ (Σ_i for short). If i and i' are two such lists, then we write $i' \geq i$ if $i'_j \geq i_j$ for all j. Then we can define the Schubert cycle Σ_i as

$$\Sigma_i^\circ = \Sigma_i \setminus \bigcup_{i' \geq i, i' \neq i} \Sigma_i'$$

We will not discuss the Schubert cells in depth, but they are very important. They are central to our understanding of the topology and Chow ring of Grassmannians.

Returning to our description of $\Sigma_{1,0} \subseteq \mathbb{G}(1,3)$, we see that the jumping pattern of (3.7) describes the Schubert cell $\Sigma_{1,0}^\circ$, not the entire cycle. The other lines in $\Sigma_{1,0}$ are also contained in smaller Schubert cycles which we now describe.

We now move to $\Sigma_{1,1}$. The lines $L \in \Sigma^\circ_{1,1}$ are described by

$$\emptyset = L \cap V_0 \subsetneq L \cap V_1 \subsetneq L \cap V_2 = L \cap V_3 = L.$$

In particular L is contained in the 2-plane V_2 and therefore intersects the line V_1, but not at the point V_0.

The lines $L \in \Sigma^\circ_{2,0}$ are described by

$$\{p\} = L \cap V_0 = L \cap V_1 = L \cap V_2 \subsetneq L \cap V_3 = L.$$

This means that to be in $\Sigma^\circ_{2,0}$, L must hit the point p, but not be contained in V_2. The remaining cases of $\Sigma_{2,1}$ and $\Sigma_{2,2}$ are left to the reader.

Definition 3.1.4. Let $\Sigma_{i_1,\ldots,i_k} \subseteq G(k,n)$ be a Schubert cycle. Its class $[\Sigma_{i_1,\ldots,i_k}] \in A^{\sum i_j}(G(k,n))$ is denoted by $\sigma_{i_1,\ldots,i_k}$ and called a Schubert class.

Proposition 3.1.5. *The Schubert classes $\sigma_{i_1,\ldots,i_k}$ do not depend on the choice of flag. That is, if $\Sigma^{\mathcal{V}}_{i_1,\ldots,i_k}$ is a Schubert cycle for the flag $\mathcal{V}$ and $\Sigma^{\mathcal{V}'}_{i_1,\ldots,i_k}$ is the corresponding cycle for the flag $\mathcal{V}'$, then these two cycles are rationally equivalent.*

Proof. [3], Theorem 1.7 (c). This relies on the elementary fact that GL_n acts transitively on flags. □

We have been focusing so much on Schubert cycles due to the following result.

Theorem 3.1.6. *The Chow ring $A(G(k,n))$ is a free abelian group with a basis given by the collection of all Schubert cycles (including $\sigma_{0,\ldots,0}$ which is the fundamental class in $A^0(G(k,n))$, i.e. $\sigma_0 = [G(k,n)]$).*

Proof. [3], Theorem 4.1 shows that the Schubert cycles are the closed strata of an affine stratification of the Grassmannian, and [4], Theorem 1 (Theorem 1.18 in [3]) shows that such cycles form a basis of the Chow ring. □

Therefore to understand the Chow ring of $G(k,n)$, we need to be able to calculate the product of any two Schubert classes. There are formulas that allow one to do this for general k and n, but they are quite tedious. In this thesis we will simply calculate these products for $\mathbb{G}(1,3)$, using the method of undetermined coefficients.

Proposition 3.1.7. *The Schubert classes for $\mathbb{G}(1,3)$ behave according to the following multiplication table.*

$\cdot$	$\sigma_{0,0}$	$\sigma_{1,0}$	$\sigma_{1,1}$	$\sigma_{2,0}$	$\sigma_{2,1}$	$\sigma_{2,2}$
$\sigma_{0,0}$	$\sigma_{0,0}$	$\sigma_{1,0}$	$\sigma_{1,1}$	$\sigma_{2,0}$	$\sigma_{2,1}$	$\sigma_{2,2}$
$\sigma_{1,0}$	$\sigma_{1,0}$	$\sigma_{1,1} + \sigma_{2,0}$	$\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$	0
$\sigma_{1,1}$	$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$	0	0	0
$\sigma_{2,0}$	$\sigma_{2,0}$	$\sigma_{2,1}$	0	$\sigma_{2,2}$	0	0
$\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,2}$	0	0	0	0
$\sigma_{2,2}$	$\sigma_{2,2}$	0	0	0	0	0

Proof. We will verify a few of these, and leave the rest to the reader. First note that $\sigma_{2,2}$ times anything but $\sigma_{0,0}$ gives 0 for dimension reasons. Also note that $\sigma_{0,0}$ is the identity. These two things in addition to the commutativity of the Chow ring mean there is not too much to check.

We start with the product $\sigma_{1,0}\sigma_{1,1}$. The general theory shows that for a general pair of flags $\mathcal{V}$ and $\mathcal{V}'$, the corresponding cycles $\Sigma_{1,0}^{\mathcal{V}}$ and $\Sigma_{1,1}^{\mathcal{V}'}$ intersect generically transversely. Thus we want to describe the lines $L \in \Sigma_{1,0}^{\mathcal{V}} \cap \Sigma_{1,1}^{\mathcal{V}'}$. The $L \in \Sigma_{1,1}^{\mathcal{V}'}$ are exactly the lines contained in the 2-plane of $\mathcal{V}'$, and the $L \in \Sigma_{1,0}^{\mathcal{V}}$ are exactly the lines that touch the line in $\mathcal{V}$. The line in $\mathcal{V}$ intersects the line in $\mathcal{V}'$ in exactly one point p. Thus the lines $L \in \Sigma_{1,0}^{\mathcal{V}} \cap \Sigma_{1,1}^{\mathcal{V}'}$ are the lines contained in the 2-plane of $\mathcal{V}'$ that touch the point p. This is the Schubert cycle $\Sigma_{2,1}$ with respect to the flag

$$\{p\} \subseteq V_1 \subseteq V_2 \subseteq \mathbb{P}^3$$

where V_2 is the 2-plane of $\mathcal{V}'$ and V_1 is any line inside V_2 that contains the point p. This shows that $\sigma_{1,0}\sigma_{1,1} = \sigma_{2,1}$.

We now consider $\sigma_{1,0}\sigma_{1,0}$. If we choose two skew lines L_1 and L_2, then $\sigma_{1,0}^2$ should be the class of the cycle of lines that intersect both L_1 and L_2. This is not in fact a Schubert class so we use the method of undetermined coefficients. The codimension of $\sigma_{1,0}^2$ is 2, and so we have an equality

$$\sigma_{1,0}^2 = a\sigma_{1,1} + b\sigma_{2,0}$$

for some integers a and b. To isolate a we want to intersect $\sigma_{1,0}^2$ with $\sigma_{1,1}$. This is because $\sigma_{1,1}^2 = \sigma_{2,2}$ but $\sigma_{1,1}\sigma_{2,0} = 0$, which can be verified similarly to how we computed $\sigma_{1,1}\sigma_{1,0}$. Thus we need to compute $\sigma_{1,0}^2\sigma_{1,1}$. The class $\sigma_{1,1}$ represents the cycle of lines that are contained in a general hyperplane V_2. The two skew lines L_1 and L_2 from the above description of $\sigma_{1,0}^2$ intersect V_2 in two distinct points. Thus there is precisely one line that is contained in V_2 and intersects both L_1 and L_2. That line is a point in the Grassmannian and therefore is represented by the class $\sigma_{2,2}$. This implies $a = 1$.

Now to isolate b, we intersect $\sigma_{1,0}^2$ with $\sigma_{2,0}$. The class $\sigma_{2,0}$ represents the cycle of all lines that pass through a general point p. Thus $\sigma_{1,0}^2\sigma_{2,0}$ represents the cycle of all lines that pass through p, L_1, and L_2. Consider first the set of all lines that pass through p and L_1. These lines sweep out a 2-plane in $\mathbb{P}^3$ which intersects L_2 in exactly one point, q (because L_1 and p are general we can guarantee that this plane does not contain L_1). There is only one line that passes through p and q, and so the class of that line is $\sigma_{2,2} = \sigma_{1,0}^2\sigma_{2,0}$, and $b = 1$. This establishes that $\sigma_{1,0}^2 = \sigma_{1,1} + \sigma_{2,0}$. □

The most interesting calculations that can be done with Schubert classes require the theory of Chern classes. In the next chapter we will define those and finally see Schubert classes in action.

Chapter 4

Chern Classes

4.1 Motivation and Definitions

The purpose of Chern classes is to describe the vanishing loci of sections of vector bundles. To motivate this, we see how the subset of the Grassmannian consisting of lines contained in a hypersurface of $\mathbb{P}^n$ can be described in this way.

First we must establish some basic vector bundles on $\mathbb{G}(k,n)$.

Definition 4.1.1. The tautological bundle, $\mathcal{S}$ on $\mathbb{G}(k,n)$ is defined by the following incidence correspondence

$$\mathcal{S} = \{(p, L) \mid p \in L\} \subseteq \mathbb{A}^{n+1} \times G(k+1, n+1).$$

Here the projection map is taken to be the projection to the second coordinate. Recall that

$$G(k+1, n+1) = \mathbb{G}(k, n).$$

The proof that $\mathcal{S}$ is a locally trivial bundle can be found in [3], Proposition 3.3. Note that for a fixed $L \in \mathbb{G}(k,n)$, the fiber of $\mathcal{S}$ over L is the $k+1$-plane represented by L in affine space. This is the reason for calling $\mathcal{S}$ the tautological bundle. Some sources refer to it as the canonical bundle, but in algebraic geometry that name is usually reserved for the highest wedge power of the cotangent bundle.

Suppose $X \subseteq \mathbb{P}^n$ is a hypersurface defined by a homogeneous polynomial F of degree d. We claim that F defines a section of the bundle $\mathrm{Sym}^d(\mathcal{S}^*)$, the dth symmetric power of the dual of $\mathcal{S}$.

To justify this, we start with the case $d = 1$ and try to gain a better understanding of the bundle $\mathcal{S}^*$. In particular, we just want to understand the fibers of this bundle. It is a standard fact that if $p \in X$ is a point in a topological space X equipped with a vector bundle $\pi : \mathcal{V} \to X$, then the fiber $\mathcal{V}_p^*$ of $\mathcal{V}^*$ is equal to the dual of the fiber $\mathcal{V}_p$. Thus the points in this fiber are linear forms on $\mathcal{V}_p$. In the case of $\mathbb{G}(k,n)$ with vector bundle $\mathcal{S}$, this means that $\mathcal{S}_L^*$ is naturally isomorphic to $H^0(L, \mathcal{O}(1))$, the space of homogeneous linear polynomials on $L \cong \mathbb{P}^k$. Therefore we see that we have natural isomorphisms $\mathrm{Sym}^d(\mathcal{S}^*)_L \cong \mathrm{Sym}^d(H^0(L, \mathcal{O}(1)) \cong H^0(L, \mathcal{O}(d))$. In other words, a fiber of $\mathrm{Sym}^d(\mathcal{S})$ is a degree d homogeneous polynomial in the homogeneous coordinates of L.

Now return to our hypersurface $X \subseteq \mathbb{P}^n$ defined by a homogeneous polynomial F of degree d. For each k-plane L, we can restrict F to L which gives an element of $H^0(L, \mathcal{O}(d))$. The assignment $L \to F|_L$ is now a section of $\mathrm{Sym}^d(\mathcal{S}^*)$. The essential feature of this section is that its zeros (the k-planes L such that $F|_L = 0$) are exactly the lines contained in X. If we would like to understand the lines contained in X, then we need to understand the zero-locus of this section. It would be great to see its class in the Chow ring, which we can do using the theory of Chern classes.

We will not get into the actual definition of Chern classes, but we will give some properties that uniquely characterize them.

Theorem 4.1.2. *Let X be a smooth quasi-projective variety of dimension n, and $\pi : \mathcal{V} \to X$ a vector bundle. Then to $\mathcal{V}$ we can associate a class*

$$c(\mathcal{V}) = 1 + c_1(\mathcal{V}) + c_2(\mathcal{V}) + \ldots + c_n(\mathcal{V}) \in A(X), \quad c_i(\mathcal{V}) \in A^i(X)$$

which is determined by the following properties

1. *If $\mathcal{V}$ is a line bundle then $c(\mathcal{V}) = 1 + c_1(\mathcal{V})$ where $c_1(\mathcal{V})$ is the divisor determined by $\mathcal{V}$, i.e. the class of the divisor of zeros and poles of a local section of $\mathcal{V}$.*

2. *(Whitney's Formula) Let*
$$0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$$
be a short exact sequence of vector bundles over X. Then there is an equality
$$c(\mathcal{V}')c(\mathcal{V}'') = c(\mathcal{V})$$

3. *Let $\mathcal{V}$ be a vector bundle of rank r and $v_0, \ldots, v_{r-i}$ be global sections. Further let D be the set of points*
$$D = \{x \in X \mid v_0(x), \ldots, v_{r-i}(x) \in \mathcal{V}_x \text{ are linearly dependent.}\}$$
If D has the expected codimension i then $c_i(\mathcal{V}) = [D] \in A^i(X)$.

4. *(Functoriality) Let $f : Y \to X$ be a morphism of smooth quasi-projective varieties, then*
$$f^*(c_i(\mathcal{V})) = c_i(f^*(\mathcal{V})).$$

Proof. [2], Theorem 3.2. □

Note that we have not defined the pullback of classes in $A(X)$, and we will not use the functoriality property. We will use 3, but 1 and 2 will be our main computational tools. To use these properties, consider a vector bundle $\mathcal{V}$ which is a direct sum of line bundles,

$$\mathcal{V} \cong \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r.$$

If we define $\alpha_i = c_1(\mathcal{L}_i)$, then we get that

$$c(\mathcal{V}) = \prod_{i=1}^{r}(1 + c_1(\mathcal{L}_i)).$$

This is a convenient situation, but the general vector bundle does not split as the direct sum of line bundles. Fortunately we can still pretend that this is the case, in particular we have

Theorem 4.1.3. *(Splitting Principle) Suppose an identity holds for the Chern classes $c_i(\mathcal{V})$ assuming that $\mathcal{V}$ splits as a direct sum of line bundles, then that identity holds even if it does not split. The identity can not depend on the Chern classes of the line bundles.*

Proof. [3], Theorem 5.11. □

Example 4.1.4. We will not prove the splitting principle, but we will give an example to elucidate the statement.

Let $\mathcal{L}$ be a line bundle over X and $\mathcal{V}$ a vector bundle over X of rank 2. We calculate the Chern classes of $\mathcal{L} \otimes \mathcal{V}$ in terms of the Chern classes of $\mathcal{V}$ and $\mathcal{L}$. Pretend there are line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{V} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ where $\alpha_1 = c_1(\mathcal{L}_1)$, $\alpha_2 = c_1(\mathcal{L}_2)$. In this case we have that

$$c(\mathcal{V}) = 1 + c_1(\mathcal{V}) + c_2(\mathcal{V}) = 1 + \alpha_1 + \alpha_2 + \alpha_1\alpha_2.$$

We see that

$$\mathcal{L} \otimes \mathcal{V} \cong (\mathcal{L} \otimes \mathcal{L}_1) \oplus (\mathcal{L} \otimes \mathcal{L}_2) \tag{4.1}$$

If $\alpha = c_1(\mathcal{L})$ then note that the isomorphism $A^1(X) \cong Pic(X)$ implies the identity

$$c(\mathcal{L} \otimes \mathcal{L}_1) = 1 + \alpha + \alpha_1, \quad c(\mathcal{L} \otimes \mathcal{L}_2) = 1 + \alpha + \alpha_2.$$

Therefore (4.1) along with Whitney's formula implies the equality

$$c(\mathcal{L} \otimes \mathcal{V}) = (1 + \alpha + \alpha_1)(1 + \alpha + \alpha_2) = 1 + (2\alpha + \alpha_1 + \alpha_2) + (\alpha + \alpha_1)(\alpha + \alpha_2).$$

$$= 1 + (2c_1(\mathcal{L}) + c_1(\mathcal{V})) + (c_1(\mathcal{L})^2 + c_1(\mathcal{L})(\alpha_1 + \alpha_2) + \alpha_1\alpha_2)$$
$$= 1 + (2c_1(\mathcal{L}) + c_1(\mathcal{V})) + (c_1(\mathcal{L})^2 + c_1(\mathcal{L})c_1(\mathcal{V}) + c_2(\mathcal{V})).$$

Because this identity does not depend on α_1 and α_2, the splitting principle implies that it holds even if the $\mathcal{L}_1$ and $\mathcal{L}_2$ do not exist.

We will now use this theory to count the number of lines on a smooth cubic surface in $\mathbb{P}^3$.

Example 4.1.5. Let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface cut out by a homogeneous cubic polynomial F. The polynomial F determines a section s of $\mathrm{Sym}^3(\mathcal{S}^*)$, which is a vector bundle on $\mathbb{G}(1,3)$. If $F_1(X) \subseteq \mathbb{G}(1,3)$ is the variety of lines contained in X (called the Fano variety of lines in X) then $F_1(X)$ is the zero-locus of the section s and the class $[F_1(X)]$ is equal to the Chern class $c_4(\mathrm{Sym}^3(\mathcal{S}^*))$. There are some technicalities, for instance $F_1(X)$ really has a non-trivial scheme structure and it is not guaranteed that the degree of $[F_1(X)]$ is really the number of distinct lines in X. For X smooth, $F_1(X)$ is reduced, and the degree of $[F_1(X)]$ is exactly the number of lines on X. This can be found in [3], Corollary 6.17, and follows from a calculation of the tangent space of the Fano variety.

We start by calculating the Chern classes of $\mathrm{Sym}^3(\mathcal{V})$ of any rank 2 vector bundle $\mathcal{V}$ in terms of the Chern classes of $\mathcal{V}$

To do this, we invoke the splitting principle and assume

$$\mathcal{V} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles with first Chern classes α_1 and α_2 respectively. This decomposition implies that $c_1(\mathcal{V}) = \alpha_1 + \alpha_2$ and $c_2(\mathcal{V}) = \alpha_1\alpha_2$. We have that

$$\mathrm{Sym}^3(\mathcal{V}) = \mathrm{Sym}^3(\mathcal{L}_1 \oplus \mathcal{L}_2)$$

$$\cong (\mathcal{L}_1^{\otimes 3}) \oplus (\mathcal{L}_1^{\otimes 2} \otimes \mathcal{L}_2) \oplus (\mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes 2}) \oplus (\mathcal{L}_2^{\otimes 3}).$$

The total Chern class of this bundle is then

$$(1+3\alpha_1)(1+2\alpha_1+\alpha_2)(1+\alpha_1+2\alpha_2)(1+3\alpha_2).$$

Our goal now is to express this in terms of the Chern classes of $\mathcal{V}$. For the purposes of our problem we only need to compute the codimension-4 term c_4, but we will include a calculation of c_1 and c_2 to give the reader a better sense of these computations. To start on c_1 we expand the product and collect the codimension-1 term, which is

$$c_1(\mathrm{Sym}^3(\mathcal{V})) = 3\alpha_1+2\alpha_1+\alpha_1+3\alpha_2+2\alpha_2+\alpha_2 = 6(\alpha_1+\alpha_2) = 6c_1(\mathcal{V}),$$

Next is the codimension-2 term which is

$$3\alpha_1(2\alpha_1+\alpha_2)+3\alpha_1(\alpha_1+2\alpha_2)+3\alpha_1(3\alpha_2)+(2\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)+(2\alpha_1+\alpha_2)(3\alpha_2)+(\alpha_1+2\alpha_2)(3\alpha_2)$$

$$= 3\alpha_1(3\alpha_1+6\alpha_2)+(2\alpha_1+\alpha_2)(\alpha_1+5\alpha_2)+(\alpha_1+2\alpha_2)(3\alpha_2)$$

$$= 3\alpha_1(3c_1(\mathcal{V})+3\alpha_2)+(\alpha_1+c_1(\mathcal{V}))(c_1(\mathcal{V})+4\alpha_2)+(c_1(\mathcal{V})+\alpha_2)(3\alpha_2)$$

$$= 9\alpha_1c_1(\mathcal{V})+3c_2(\mathcal{V})+\alpha_1c_1(\mathcal{V})+4c_2(\mathcal{V})+c_1(\mathcal{V})^2+4c_1(\mathcal{V})\alpha_2+3c_1(\mathcal{V})\alpha_2+3\alpha_2^2.$$

$$= c_1(\mathcal{V})(10\alpha_1+c_1(\mathcal{V})+7\alpha_2)+7c_2(\mathcal{V})+3\alpha_2^2 = c_1(\mathcal{V})(3\alpha_1+8c_1(\mathcal{V}))+7c_2(\mathcal{V})+3\alpha_2^2.$$

Things at this point are looking ugly, but we can finish the problem by noticing that

$$c_1(\mathcal{V})^2 = \alpha_1^2+2c_2(\mathcal{V})+\alpha_2^2.$$

With this in mind we see

$$c_2(\mathrm{Sym}^3(\mathcal{V})) = 8c_1(\mathcal{V})^2+7c_2(\mathcal{V})+3\alpha_1(\alpha_1+\alpha_2)+3\alpha_2^2$$

$$= 8c_1(\mathcal{V})^2+4c_2(\mathcal{V})+3\alpha_1^2+6c_2(\mathcal{V})^2+3\alpha_2^2 = 11c_1(\mathcal{V})^2+4c_2(\mathcal{V}).$$

We now skip to $c_4(\mathrm{Sym}^3(\mathcal{V}))$.

$$c_4(\mathrm{Sym}^3(\mathcal{V})) = (3\alpha_1)(2\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)(3\alpha_2) = 9c_2(\mathcal{V})(\alpha_1+c_1(\mathcal{V}))(c_1(\mathcal{V})+\alpha_2)$$

$$= 9c_2(\mathcal{V})(2c_1(\mathcal{V})^2+c_2(\mathcal{V}))$$

The Chern classes we are actually interested in are those of $\mathrm{Sym}^3(\mathcal{S}^*)$, and so now we focus on $\mathcal{S}^*$. To calculate $c(\mathcal{S}^*)$ we will be using property 3 of Chern classes, and we will do it for any $G(k,n)$. To calculate $c_i(\mathcal{S}^*)$, our sections v_i will be the ones associated to $k-i+1$ general homogeneous linear polynomials $l_0,\ldots,l_{k-i}$ on $\mathbb{A}^n$. Suppose there is a k -plane $L \subseteq \mathbb{A}^n$ on which the linear forms $v_j(L) = l_j|_L$ are dependent, i.e. there is a linear combination of l_j that vanishes identically on L.

Lemma 4.1.6. *The linear forms $v_j(L) = l_j|_L$ are dependent exactly when L intersects the common vanishing locus*

$$W = V(l_0,\ldots,l_{k-i})$$

in a space of dimension greater than the minimal possible dimension i, i.e. $\dim(L\cap W) \geq i$.

Proof. Consider the annihilator $\mathrm{Ann}(L) \subseteq \mathbb{C}^{n*}$. The set of linear combinations of the l_j that vanish identically on L is exactly the set $\mathrm{Ann}(L) \cap Span(l_0, \dots, l_{k-i})$. Therefore the existence of such a non-zero linear combination is equivalent to $\dim(\mathrm{Ann}(L) \cap Span(l_0, \dots, l_{k-i})) \geq 1$.

We recall from linear algebra that for two subspaces $L, L' \subseteq V$ of a vector space V, we have the equalities

$$\mathrm{Ann}(L \cap L') = \mathrm{Ann}(L) + \mathrm{Ann}(L') \quad \text{and} \quad \mathrm{Ann}(\mathrm{Ann}(L)) = L$$

where the second equivalence is interpreted with respect to the natural isomorphism $\mathbb{C}^n \cong \mathbb{C}^{n**}$. Therefore we have the equivalence

$$\dim(\mathrm{Ann}(L) \cap Span(l_0, \dots, l_{k-i})) \geq 1 \iff \dim(L + \mathrm{Ann}(Span(l_0, \dots, l_{k-i}))) \leq n - 1.$$

We claim that $\mathrm{Ann}(Span(l_0, \dots, l_{k-i}))$ is our set $W = V(l_0, \dots, l_{k-i})$. This follows from expanding the definition,

$$\mathrm{Ann}(Span(l_0, \dots, l_{k-i})) = \{v \in \mathbb{C}^n \mid \sum_{j=0}^{k-i} a_j l_j(v) = 0 \quad \forall a_0, \dots, a_{k-i} \in \mathbb{C}\} = W$$

Thus we come to the conclusion that the $v_j(L)$ are dependent if and only if $dim(L + W) \leq n - 1$. Using the identity

$$\dim(L \cap W) + \dim(L + W) = \dim(L) + \dim(W)$$

we see that

$$\dim(L + W) = -\dim(L \cap W) + k + (n - k + i - 1) = -\dim(L \cap W) + n + i - 1.$$

Thus

$$\dim(L + W) \leq n - 1 \iff -\dim(L \cap W) + n + i - 1 \leq n - 1 \iff \dim(L \cap W) \geq i$$

as desired. □

This computation implies that $c_i(\mathcal{S}^*)$ is the Schubert class $\sigma_{1,\dots,1}$ with i 1's (here we omit the trailing zeros). To justify this, we return to the definition of Schubert cycles as

$$\Sigma_{i_1,\dots,i_k} = \{L \in G(k, n) \mid \dim(V_{n-k+j-i_j} \cap L) \geq j \text{ for all } j = 1, \dots, k\}.$$

We fix an arbitrary complete flag

$$\{0\} = V_0 \subsetneq \dots \subsetneq V_n = \mathbb{C}^n$$

that has W as its $n - k + i - 1$-plane. Then the cycle $\Sigma_{1,\dots,1}$ is equal to

$$\{L \in G(k, n) \mid \dim(V_{n-k+j-1} \cap L) \geq j \text{ for } j = 1, \dots, i,\ \dim(V_{n-k+j} \cap L) \geq j \text{ for } j = i+1, \dots, k\}.$$

This second condition is irrelevant so we really have

$$\Sigma_{1,\dots,1} = \{L \in G(k, n) \mid \dim(V_{n-k+j-1} \cap L) \geq j \text{ for } j = 1, \dots, i\}.$$

and this implies that for $L \in \Sigma_{1,\ldots,1}$ we have $\dim(W \cap L) \geq i$. Conversely, the k-planes L such that

$$\dim(L \cap W) \geq i$$

satisfy (for $m \leq i$)

$$\dim(L \cap V_{n-k+i-1-m}) \geq i - m$$

This is equivalent to the condition that

$$\dim(L \cap V_{n-k+j-1}) \geq j$$

for $j \leq i$, which is what we wanted.

Thus we have finally established that

$$c(\mathcal{S}^*) = 1 + \sigma_1 + \ldots + \sigma_{1,\ldots,1}.$$

In the case of $\mathbb{G}(1,3)$ this implies that $c_1(\mathcal{S}^*) = \sigma_1$ and $c_2(\mathcal{S}^*) = \sigma_{1,1}$. Now we can put the pieces together to say that

$$c_4(\mathrm{Sym}^3(\mathcal{S}^*)) = 9c_2(\mathcal{S}^*)(2c_1(\mathcal{S}^*)^2 + c_2(\mathcal{S}^*)) = 9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1}).$$

At this point we can finally use our description of $A(\mathbb{G}(1,3))$. Reading off the table, we get

$$\sigma_1^2 = \sigma_{1,1} + \sigma_2$$

and therefore

$$c_4(\mathrm{Sym}^3(\mathcal{S}^*)) = 9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1}) = 9\sigma_{1,1}(3\sigma_{1,1} + 2\sigma_2) = 27\sigma_{2,2} + 18\sigma_{1,1}\sigma_2 = 27\sigma_{2,2}.$$

Therefore, we have computed that there are 27 lines on a smooth cubic surface in $\mathbb{P}^3$.

Bibliography

[1] J. Harris, *Algebraic Geometry: A First Course*. Springer-Verlag, New York, 1992.

[2] W. Fulton, *Intersection Theory*, Springer-Verlag, 1984.

[3] J. Harris, D. Eisenbud, *3264 And All That: A Second Course in Algebraic Geometry*. Cambridge University Press, Cambridge, UK, 2016.

[4] B. Totaro, "Chow groups, Chow cohomology, and linear varieties", Forum Math. Sigma 2 (2014), e17, 25 pp.

Academic Vita: Jacob Keller

Education

The Pennsylvania State University, State College, PA
Bachelor of Science, Mathematics.
Minors in Spanish and Physics.
Schreyer Honors College
Expected Graduation: Spring 2019

Research Experience

- **The University of Chicago Mathematics REU 2018**
  - Wrote an exposition on the Beilinson-Bernstein Localization theorem from geometric representation theory.
  - Presented work in 15 minute talk at the end of program.
- **Independent Studies**
  - Étale cohomology (Fall 2018) and Abelian Varieties (Spring 2019) with Dr. Yuri Zarhin.
  - Algebraic Curves (Spring 2018), Intersection Theory (Fall 2018), and Algebraic Surfaces (Spring 2019) with Dr. Jack Huizenga.

Work Experience

- **Tutor** Fall 2016 - Present
  The Morgan Academic Center, State College, PA.
  - Tutor for student athletes in all sports.
  - Tutored students in Mathematics, Physics, and Spanish.
- **Grader** Spring 2018
  The Pennsylvania State University, State College, PA
  - Graded for upper division number theory, Math 465.
  - Worked directly with Professor Wen Ching Li to grade quizzes and exams.

Languages

- **Spanish**
  - Spanish classes since first grade.
  - Studied abroad in Ronda, Spain for six weeks during Summer 2017.
- **Latin**
  - Started learning Summer 2015
  - Received an A in Latin 404, Spring 2018.

Computer Skills

Python. LaTeX. Experience with 3-D modeling and coding video games in JavaScript.

Honors

Dean's list every semester.
Schreyer Honors Scholar.
William B. Forest Honors Scholarship.