CONVERSION GAIN AND SENSITIVITY IN MARGINAL OSCILLATORS:
CONTINUOUS AND SAMPLED-DATA NEGATIVE RESISTANCE CONVERTERS

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Abstract

A marginal oscillator is an instrument used to detect changes in the losses of a tuned circuit. It consists of a parallel RLC circuit driven by a dependent current source that is controlled by the voltage across the RLC circuit. The dependent current source implements a nonlinear negative resistance converter that injects just enough energy to overcome the resistive losses and thereby produce a steady-state oscillation in the voltage across the circuit. The main figure of merit for a marginal oscillator is its conversion gain, which relates the change in the amplitude of oscillation to the change in the losses of the tuned circuit. Viswanathan showed that the conversion gain depends on the shape of the nonlinear negative resistance converter. In general, a higher conversion gain is desired. An earlier study that simulated the marginal oscillator yielded estimates of the conversion gain that were not consistent with theoretical predictions. The objectives of this thesis are threefold. First, we want to reconcile the difference between theory and simulation by choosing an appropriate integration algorithm and its parameters. This thesis shows that by appropriately choosing the integration algorithm and its parameters, the simulation results are in agreement with the theoretical predictions. The second objective is to investigate the feasibility of implementing the dependent current source using a data-sampled system in place of analog circuitry in order to facilitate the implementation of nonlinear characteristics that maximize conversion gain. Specifically, this study uses numerical simulation to determine the effects of quantization in time and amplitude on the the conversion gain. The third objective is to numerically determine the sensitivity of the marginal oscillator, which is the smallest change in losses that can be detected in the presence of thermal noise for a given conversion gain.
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Chapter 1

Marginal Oscillators

1.1 Applications

A marginal oscillator is an instrument for detecting losses in the electrical properties of a tuned circuit. Roberts and Rollins credit Pound with developing the marginal oscillator in the 1940s for use in nuclear resonance spectroscopy experiments [1, 2]. Pound demonstrated that a marginal oscillator can detect the absorption of radio frequency energy by the nuclei of a solid material [3]. By measuring the change in absorption as a function of frequency, Pound was able to investigate the molecular structure of the solid.

Recent applications of marginal oscillators include characterization of defects in silicon [4], measurement of the penetration depth of superconductors [5], ion cyclotron resonance spectroscopy [6, 7], and investigation of the mechanical properties of thin-films [8]. Additionally, they can be used for monitoring the curing of plastics [9] and precision measurement of both capacitances [10] and very low temperatures [11].

1.2 Conceptual Model

Figure 1.1 represents a marginal oscillator as a parallel resonant circuit driven by a voltage-controlled current source. A parallel RLC network represents the resonant circuit. The inductance $L$ and capacitance $C$ of the network determine the frequency of oscillation, and the losses within the network are represented by the resistance $R$. To maintain a steady-state sinusoidal oscillation, the dependent current source must supply just enough energy to counter these losses. In other words, the dependent current source must appear as a negative resistance with value $-R$, so that the parallel LC network sees an open circuit. Therefore, the dependent current source represents a negative resistance converter (NRC). The overall circuit is called a marginal oscillator because the dependent source provides just enough energy to sustain oscillation.

As will be shown in Chapter 2, it is desirable to implement the dependent current source so
that the current is a nonlinear function of the input voltage $v(t)$. In short, the gain between the output current and the input voltage is dependent on the amplitude of $v(t)$. Pound was the first to implement a marginal oscillator. His implementation used the nonlinear characteristic of a vacuum tube to produce $G(v)$ [3]. As the input voltage to the vacuum tube increases, its gain, and hence the feedback current $i(t)$ to the RLC circuit, decreases.

### 1.3 Figures of Merit

The main figure of merit of a marginal oscillator is the sensitivity of the amplitude of oscillation with respect to the losses in the resonance circuit. This sensitivity is defined as

$$S_{AR}^A = \frac{\% \text{ change in } A}{\% \text{ change in } R} = \frac{\Delta A/A}{\Delta R/R},$$

(1.1)

where $A$ and $R$ represent the nominal amplitude and losses, while $\Delta A$ represents the change in amplitude due to a change $\Delta R$ in losses. In the limit as $\Delta R$ approaches zero,

$$S_{AR}^A = \frac{\partial A}{\partial R} \frac{R}{A}.$$  

(1.2)

This definition is consistent with signal processing and control systems usage. However, physicists often use the term sensitivity to indicate the smallest change in root mean square (rms) voltage that can be seen above the noise floor in a circuit. In order to avoid confusion between the two terms, this paper uses the convention of Viswanathan [12] and refers to the sensitivity defined in Eq. (1.1) as the conversion gain

$$G_c = S_{AR}^A.$$  

(1.3)

In general, we want the conversion gain to be as large as possible.

To understand why a large conversion gain is useful, consider the conversion gain of a Q-meter, where the dependent current source in Figure 1.1 is replaced by an independent sinusoidal
current source whose frequency is $1/\sqrt{LC}$. Suppose that the current source is chosen so that
\[ i(t) = I_0 \cos \left( \frac{1}{\sqrt{LC}} t \right), \quad (1.4) \]
independent of $v(t)$. Using phasor analysis, the sinusoidal steady-state voltage across the resonant circuit is
\[ v(t) = RI_0 \cos \left( \frac{1}{\sqrt{LC}} t \right) = A \cos \left( \frac{1}{\sqrt{LC}} t \right), \quad (1.5) \]
where $A$ is the oscillation amplitude. It follows that the conversion gain of a Q-meter is
\[ G_c = \frac{R \partial A}{A \partial R} = \frac{R \partial R}{RI_0 \partial R} = 1. \quad (1.6) \]
In a typical magnetic resonance experiment, the normalized change in $R$, $\Delta R/R$, is on the order of $10^{-6}$. As the nominal amplitude $A$ of oscillation is about 1 V and the conversion gain is unity, the change in $A$ is about 1 $\mu$V. In order to improve the signal-to-noise ratio (SNR) when measuring a change in amplitude $\Delta A$, a higher conversion gain is needed.

In 1950, Pound was the first to recognize that significantly larger conversion gains could be achieved by replacing the independent current source in the Q-meter with a dependent current source. The resulting feedback path not only generates a sinusoidal signal but also increases the conversion gain. However, Pound did not analyze how the form of the feedback $G(v)$ affects the conversion gain. It was not until the mid-1970s that Viswanathan analytically computed the conversion gain and showed it is determined by the shape of $G(v)$. Conversion gains on the order of 10 to 100 have been achieved in experiments.

Another figure of merit of the marginal oscillator is the sensitivity, that Adler defined as the minimum change in oscillation amplitude that can be detected in the presence of thermal noise due to losses in the resonant circuit [13]. This method does not directly take into consideration the conversion gain and how it affects the sensitivity. For this reason, this thesis introduces another definition of the sensitivity of a marginal oscillator. The new definition is motivated by the concept of noise factor (NF) and the noise figure, $10 \log_{10}(\text{NF})$, which are figures of merit for amplifiers. The NF specifies the amount of noise introduced by a device. For an amplifier, the NF is the ratio of the SNR of the input to the SNR of the output. Ideally, the NF of an amplifier is unity, but in practice, because amplifiers always add noise, the NF is always greater than unity. This study defines the NF for a marginal oscillator in a similar manner by comparing the SNR of a marginal oscillator to the SNR of a Q-meter. The reason for doing this is the SNR for a marginal oscillator is dependent on its conversion gain so we want to compare that SNR to the SNR of a Q-meter whose conversion gain is unity. For both a marginal oscillator and a Q-meter, define the SNR as the ratio of the rms envelope voltage due to changes in the losses of the resonant circuit to the rms envelope voltage due to thermal noise. Using this definition of SNR, the NF for a marginal oscillator is defined as the ratio of the SNR of a Q-meter to the SNR of the marginal oscillator. Unlike the case for an amplifier, the restriction that NF is greater
than unity may not hold due to the nonlinearities of the marginal oscillator.

1.4 Thesis Contributions

The specific aims of this thesis are threefold. The first aim is to compare the simulated amplitude, frequency, and conversion gain with theoretical predictions. This goal requires selection of an appropriate solver and solver parameters, along with defining a protocol for estimating conversion gain from simulation results. The second aim is to use simulation results to determine the viability of implementing a marginal oscillator where the NRC is realized using a digital signal processing (DSP) system. In particular, the aim is to determine how the sample rate and quantization of the output levels affect the conversion gain of a marginal oscillator that uses a sampled-data NRC. The third aim is to numerically determine the sensitivity of the marginal oscillator by including a noise source that models the thermal noise due to the losses of the resonant circuit.

Miller made an early attempt to compare the conversion gains from theory, simulation, and experiment [14], but he was unsuccessful for two reasons. First, the simulation used a poor choice of integration algorithm. A nonstiff solver was used while a marginal oscillator is a stiff system. Second, Miller did not take into account that the amplitude $A$ is a nonlinear function of the losses $R$ when calculating the conversion gain using Eq. (1.1). The ratio $\Delta A/\Delta R$ used in that equation needs a small $\Delta R$ in order to accurately approximate the partial derivative in Eq. (1.2).

This thesis shows that Viswanathan’s theoretical prediction closely matches Matlab simulation results. This result was achieved in two steps. First, by reviewing properties of integration algorithms and methods for doing simulation, this thesis shows how to accurately simulate the behavior of the model. Second, taking into account the strong nonlinear relationship of $A$ and $R$, a protocol for calculating conversion gain is realized. Using the protocol, the simulation matches Viswanathan’s prediction. Furthermore, this study revealed that it is possible to simulate the marginal oscillator in Simulink rather than an m-file exclusively, resulting a decrease in simulation time by a factor of 100 due to the reduced number of function calls needed when using Simulink.

The second aim of the thesis is to show the feasibility of implementing the NRC using a sampled-data system. This is desirable as it facilitates NRC functions tailored to increase the conversion gain. As a first step, this thesis shows it is possible to replace the continuous-time feedback $i(t) = G(v)$ with a sampled-time implementation $i(k) = G(v(k))$, where time is quantized but amplitude is not. The second step is to investigate what happens when the amplitude is also quantized, for a sampled-data implementation. In short, larger gains are needed in the feedback path when the NRC is implemented using a sampled-data system. This result was found by simulation, not theory. Finally, this thesis investigates how to implement the NRC using a discrete-time DSP system.

The third aim is to take into account the effect of thermal noise on the marginal oscillator by including a white noise generator in the simulation model. The noise source accounts for the thermal noise of the losses in the resonant circuit, and for simplicity, ignores the contribution...
of noise by the NRC. This facilitates numerical calculation of the noise factor of the marginal oscillator, as previously defined. In addition, incorporation of thermal noise enables a study of the trade-offs among measurement bandwidth, conversion gain, and noise factor.

The remainder of the thesis is organized as follows. Chapter 2 describes the design of a marginal oscillator and the figures of merit. Chapter 3 describes the implementation of the simulation model and discusses the simulation results. Chapter 4 provides a summary and discussion of the work.
Design and Figures of Merit

For the purpose of analysis, it is convenient to represent the marginal oscillator as the feedback network in Figure 2.1. This form separates the marginal oscillator into a linear time-invariant resonant circuit represented by the transfer function $H(s) = V(s)/I(s)$ and a feedback block represented by the memoryless nonlinearity $G(v)$.

Section 2.1 motivates the designs of the resonant circuit and the NRC by investigating a model of the marginal oscillator as a nonlinear, second-order ordinary differential equation (ODE). Section 2.2 shows how to represent the linear RLC circuit using a transfer function and state-space representation. Section 2.3 shows that the nonlinear feedback element realizes a NRC and discusses techniques for realizing a NRC. A discussion of the conversion gain is given in Section 2.4. Section 2.5 defines the sensitivity and noise figure of a marginal oscillator.

2.1 Marginal Oscillator

The RLC circuit in Figure 1.1 with an input current $i(t)$ and a voltage $v(t)$ across it, can be represented by the ODE

$$\frac{d^2 v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{1}{LC} v = \frac{1}{C} \frac{di}{dt}.$$  \hspace{1cm} (2.1)
Substituting \( i(t) = G(v) \), which represents the NRC, and defining

\[ g(v) = \frac{dG(v)}{dv} \quad (2.2) \]

leads to the following representation of the marginal oscillator,

\[ \frac{d^2v}{dt^2} + \frac{1}{C} \left( \frac{1}{R} - g(v) \right) \frac{dv}{dt} + \frac{1}{LC}v = 0. \quad (2.3) \]

Note that the conductance looking into the dependent current source, at a particular voltage \( v \), is

\[ -\frac{di}{dv} = -\frac{dG(v)}{dv} = -g(v). \quad (2.4) \]

If the dependent source is implemented as \( G(v) = v/R \), then the conductance looking into it is \(-1/R\) and the representation of the marginal oscillator in Eq. (2.3) reduces to the undamped second-order ODE

\[ \frac{d^2v}{dt^2} + \frac{1}{LC}v = 0 \quad (2.5) \]

that admits a steady-state sinusoidal oscillation with a frequency of \( 1/\sqrt{LC} \).

In practice, it is difficult to have \( g(v) \) match \( 1/R \) exactly. When \( g(v) \) is less than \( 1/R \), the amplitude of oscillation decays exponentially, and when \( g(v) \) is greater than \( 1/R \), the amplitude of oscillation tends to infinity. Instead, one uses a nonlinear function \( G(v) \) that represents a conductance that varies with the voltage \( v(t) \). In this case, the marginal oscillator achieves a steady-state oscillation whose amplitude is determined by both the losses \( R \) in the RLC circuit and the shape of the nonlinear function \( G(v) \) [12].

### 2.2 Resonant Circuit

The transfer function of the resonant circuit follows directly by taking the Laplace transform of Eq. (2.1)

\[ H(s) = \frac{V(s)}{I(s)} = \frac{\frac{1}{C}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}, \quad (2.6) \]

and represents the impedance looking into the RLC circuit. The transfer function \( H(s) \) has a Bode magnitude plot with a bandpass filter shape. The peak magnitude occurs at the natural frequency \( \omega_n = 1/\sqrt{LC} \) and is \( H(j\omega_n) = R \). The 3 dB bandwidth is \( \beta = 1/RC \). It is useful to characterize the transfer function by the quality factor, or Q-factor, which is defined as

\[ Q = 2\pi \times \frac{\text{Energy stored}}{\text{Energy dissipated per cycle}}. \quad (2.7) \]

Using the above definition, it is straightforward to show that \( Q = R/\omega_n L \) for the resonant circuit. In signal processing, the Q-factor is defined as the peak frequency divided by the 3 dB

bandwidth. For the resonant circuit at the peak frequency, the two definitions are identical, and

\[ Q = \frac{R}{\omega_n L} = \frac{\omega_n}{\beta}. \]  

(2.8)

In general, a high Q-factor is desirable because it indicates a narrow bandwidth, decreasing the effect of thermal noise. It is useful to write the transfer function as

\[ H(s) = \frac{1}{s^2 + \frac{\omega_n^2}{Q}s + \omega_n^2}, \]  

(2.9)

in terms of the Q-factor and the natural frequency \( \omega_n \).

For ease of numerical simulation, it is useful to represent the resonant circuit by an all-integrator block diagram. The all-integrator block diagram is derived from a state-space representation using the physical variable definition, where the state variables are chosen to represent the stored energy in the resonant circuit. In particular, scaled versions of the inductor current and the capacitor voltage are used, where the scaling is chosen so that the system matrix can written in terms of the natural frequency \( \omega_n \) and the Q-factor. Choosing \( x_1 = LCi_L(t) \), where \( i_L(t) \) is the inductor current, and \( x_2 = Cv(t) \) achieves this goal and results in the state-space representation

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-\omega_n^2 - \frac{\omega_n}{Q}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
0 \\
1
\end{pmatrix}
i(t)
\]  

(2.10)

\[ v(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \]  

(2.11)

Using the state-space representation in Eqs. (2.10) and (2.11), the corresponding all-integrator block diagram is shown in Figure 2.2. In terms of the inductor current and capacitor voltage, the initial conditions are

\[ x_1(0) = LCi_L(0) \]  

(2.12)

\[ x_2(0) = Cv(0). \]  

(2.13)

### 2.3 Negative Resistance Converter

This thesis considers a NRC realized by a dependent current source \( i = G(v) \) in Figure 2.3a, whose current-voltage (i-v) characteristics are shown in Figure 2.3b. This particular form is useful for two reasons. First, it is a good representation of the i-v characteristics of marginal oscillator feedback circuits made using vacuum tubes and FETs. Second, this piecewise-linear form facilitates analytical calculation of conversion gain. As shown in Figure 2.3b, when the
magnitude of the input voltage is less than some threshold $v_T$, the dependent source appears as a gain block with gain $g_A$. However, when the magnitude exceeds $v_T$, the gain rolls off to $g_B$.

To understand why the dependent current source realizes a NRC, use the definition of $\bar{i}$ in Figure 2.3a

$$\bar{i} = -G(v).$$

The conductance looking into the dependent current source at a given input voltage $v^*$ is

$$\left.\frac{\Delta \bar{i}}{\Delta v}\right|_{v^*} = \left.\frac{di}{dv}\right|_{v^*} = -\left.\frac{dG(v)}{dv}\right|_{v^*} = -g(v^*).$$

For $G(v)$ of the form shown in Figure 2.3b, the slope $g(v)$ is shown in Figure 2.4 By the definition in Eq. (2.15), the conductance is negative for all voltages. Therefore, the resistance is also negative, and the dependent current source is a NRC.

In order to implement the NRC using either an analog circuit or a sampled-data system using a dedicated signal processing chip, it is useful to represent the NRC using the dependent voltage source $G(v)$ in series with the feedback resistance $R_f$ in Figure 2.5a rather than a dependent

**Figure 2.2.** All-integrator block diagram of resonant circuit.

**Figure 2.3.** (a) Dependent current source (b) Current-voltage (i-v) characteristic.
current source. For the circuit in Figure 2.5a,

\[ \bar{i} = \frac{v - G(v)}{R_f}. \]  

(2.16)

The function \( G(v) \) is chosen so that the networks in Figures 2.3a and 2.5a have identical i-v characteristics. Using Eq. (2.14) to eliminate \( \bar{i} \) in Eq. (2.16) leads to

\[ G(v) = G(v)R_f + v. \]  

(2.17)

Defining the slope of \( G(v) \) as

\[ \bar{g}(v) = \frac{dG(v)}{dv}, \]  

(2.18)

it follows that

\[ \bar{g}(v) = g(v)R_f + 1. \]  

(2.19)

Given the above expression for \( \bar{g}(v) \) and the plot of \( g(v) \) in Figure 2.4, it follows that \( G(v) \) must have the shape shown in Figure 2.5b.

Using the NRC in Figure 2.5a that is realized with a voltage-controlled voltage source, Section

Figure 2.5. (a) NRC with a dependent voltage source (b) Current-voltage (i-v) characteristic of dependent voltage source.
2.3.1 shows how to develop a continuous-time implementation of the NRC using an analog circuit. Then, Section 2.3.2 shows how to extend that result to obtain a sampled-data implementation using a lookup table (LUT) and analog-to-digital and digital-to-analog converters (ADC and DAC).

### 2.3.1 Analog Implementation

In order to obtain an analog circuit implementation using operational amplifiers, first consider the case where \( g_A = g_B \), that is the desired relationship between voltage \( v \) and current \( i \) is linear and given by

\[
\bar{i} = -g_A v. \tag{2.20}
\]

Consider the implementation in Figure 2.6a. By comparison with Figure 2.5a we can see that \( G(v) \) is implemented by the non-inverting amplifier with a gain of \( \alpha+1 \). It follows from Eq. (2.16) that

\[
\bar{i} = \frac{v - G(v)}{R_f} = \frac{v - (1-\alpha)v}{R_f} = -\frac{\alpha}{R_f}v. \tag{2.21}
\]

Comparing this result to Eq. (2.20),

\[
g_A = \frac{\alpha}{R_f}. \tag{2.22}
\]

Observe that a desired value of \( g_A \) can be obtained by varying the gain of the operational amplifier using \( \alpha \) or by varying the feedback resistance \( R_f \).

Now consider the case where the desired i-v characteristic is given by the piecewise-linear curve in Figure 2.3b. The slope needs to decrease from \( g_A \) to \( g_B \) for \( |v| > v_T \). To accomplish this, a voltage divider is used to decrease the value of \( \alpha \). This implementation is shown in

![Figure 2.6. (a) Linear analog NRC circuit (b) Current-voltage (i-v) characteristic.](image)

---

**Note:** The diagrams in the text are placeholders and do not include actual images. The description is based on the text provided.
For $|v| < v_T$, Tricou determined the conductance looking into this circuit is 
\begin{equation}
-g_A = -\frac{\alpha}{R_f + R_2},
\end{equation}
(2.23)
The conductance for $|v| > v_T$ is
\begin{equation}
-g_B = -\frac{\alpha - R_2}{R_f + R_2 + \frac{R_f R_2}{R_3}},
\end{equation}
(2.24)
Suppose that $R_f >> R_2$, $\alpha << 1$, and $R_3 >> R_2$, then the above conductances reduce to
\begin{equation}
-g_A = -\frac{\alpha}{R_f},
\end{equation}
(2.25)
\begin{equation}
-g_B = -\frac{\alpha}{R_f} \left( 1 - \frac{R_2}{\alpha R_3} \right).
\end{equation}
(2.26)
This representation shows that $g_A$ can be varied by adjusting either $\alpha$ or $R_f$ as before. Furthermore, the value of $g_B$ can be adjusted by varying $R_2$. In the circuit implementation, potentiometers are used to realize $R_f$ and $R_2$, which adjust $g_A$ and $g_B$, respectively. Figure 2.8 shows a plot of the i-v characteristics of the NRC shown in Figure 2.7. Tricou also shows how to choose the control voltage $v'_T$ to achieve a desired $v_T$.

This analysis shows that even a simple i-v characteristic described by the three parameters $g_A$, $g_B$, and $v_T$ leads to a complicated analog circuit. Studies currently underway to find feedback functions $i = G(v)$ that improve conversion gain will likely lead to even more complicated designs. To simplify the hardware representation of the feedback $i = G(v)$, it would be useful to implement the NRC using a sampled-data system.
2.3.2 Sampled-Data Implementation

It is not straightforward to directly implement the dependent current source as a digital system, because DACs behave as voltage sources. Instead, this thesis creates a sampled-data NRC based on the circuit in Figure 2.5a with a dependent voltage source. Figure 2.9 shows the implementation, where the dependent voltage source is replaced by a sampled-data system. Denoting the sample period as $T$, the value of the terminal voltage $v(t)$ at the sample instance $t = kT$ is $v(k)$, where $k$ is an integer. The lookup table (LUT) transforms the sampled input voltage to a sampled output voltage $v_1(kT)$. In turn, this voltage is converted to an analog voltage by a zero-order hold (ZOH) DAC such that

$$v_1(t) = v_1(k) \quad \text{for} \quad kT \leq t < (k+1)T. \quad (2.27)$$

The output waveform $v_1(t)$ is a staircase signal that is used to approximate the i-v characteristic in Figure 2.5b. It follows from Eq. (2.16) that

$$\bar{i}(t) = \frac{v(t) - v_1(t)}{R_f} = \frac{v(t) - v_1(k)}{R_f} \approx \frac{v(t) - G(v)}{R_f}, \quad (2.28)$$

where $v_1(k) = G(v)$ at time $t = kT$. Therefore, the signal $v_1(t)$ is a staircase approximation of the analog signal $v_2(t)$ in Figure 2.7. However, because the LUT is used instead of an analog circuit, it is easy to implement arbitrary i-v characteristics.

The remaining two sections of the chapter describe performance parameters for a marginal oscillator. Section 2.4 shows how the i-v characteristics of the feedback function affect the conversion gain of the marginal oscillator. Section 2.5 defines sensitivity, noise factor, and noise figure for a marginal oscillator, which describe the characteristics of a marginal oscillator in the presence of noise.
2.4 Conversion Gain

The conversion gain is the sensitivity of the oscillation amplitude with respect to losses in the resonant circuit and is defined as

\[ G_c \equiv \frac{R}{A} \frac{\partial A}{\partial R}. \]  

(2.29)

As discussed in Section 1.3, the conversion gain of a Q-meter is unity. A high conversion gain is desirable, because a small change in the losses of the resonant circuit is easily discernible by a larger change in the amplitude of the oscillation. A conversion gain much higher than unity, on the order 10 to 100, can be achieved with a marginal oscillator by choosing an appropriate nonlinear \( G(v) \).

Viswanathan was the first researcher to calculate the conversion gain of a marginal oscillator. She considered the nonlinear feedback function \( G(v) \) shown in Figure 2.3b. The conversion gain for the marginal oscillator with that feedback was determined by Viswanathan to be [12]

\[ G_c = \frac{\pi}{2(1 - g_B/g_A)g_A R \sin \theta}, \]

(2.30)

where the angle \( 0 \leq \theta \leq \pi/2 \) satisfies

\[ \sin 2\theta + 2\theta = \frac{\pi}{1 - g_B/g_A} \left( \frac{1}{g_A R} - \frac{g_B}{g_A} \right). \]

(2.31)

This angle also relates oscillation amplitude \( A \) to the threshold voltage \( v_T \) defined in Figure 2.3b by

\[ v_T = A \sin \theta. \]

(2.32)

When \( g_A R \) is sufficiently greater than unity, the conversion gain depends only on \( g_B R \). This limit is given by

\[ \lim_{g_A R \to \infty} G_c = \lim_{\theta \to \infty} \left[ \frac{1}{2} \left( \frac{\sin 2\theta + \sin \theta}{\sin 2\theta} \right) \frac{1}{1 - g_B R} \right] = \frac{1}{1 - g_B R}. \]

(2.33)

For a hard limiter with \( g_B = 0 \) and a sufficiently large \( g_A R \), the conversion gain is unity.
Given a marginal oscillator with given values of $R$, $L$, and $C$, it is necessary to determine $g_A$, $g_B$, and $v_T$ to obtain an amplitude $A$ and a conversion gain $G_c$. Unfortunately, in the three Eqs. (2.30) through (2.32), $A$ and $G_c$ are known, while $g_A$, $g_B$, $v_T$, and $\theta$ are unknown. To obtain a solution, first choose $g_A$ so that $g_A R$ is slightly greater than unity, as required for oscillation. Given $A$, $G_c$, and $g_A R$, it is possible to solve for $\theta$ using Eqs. (2.30) and (2.31). First, solve Eq. (2.30) for $g_B/g_A$,

$$\frac{g_B}{g_A} = 1 - \frac{\pi}{2G_c g_A R \sin \theta}.$$  \hspace{1cm} (2.34)

Substitute Eq. (2.34) into Eq. (2.31) to get

$$\pi = 2\theta + (1 + 2G_c(g_A R - 1)) \sin 2\theta.$$  \hspace{1cm} (2.35)

The latter equation can be solved for $\theta$, where $\theta \in [0, \pi/2]$. Once $\theta$ is obtained, $g_B$ can be found by using Eq. (2.30)

$$g_B = g_A - \frac{\pi}{2RG_c \sin 2\theta}.$$  \hspace{1cm} (2.36)

Finally, knowledge of $\theta$ and $A$ allows the calculation of the threshold voltage using Eq. (2.32).

### 2.5 Sensitivity and Noise Figure

Before sensitivity and noise figure can be discussed, the effect of noise as well as the SNR of a marginal oscillator and a Q-meter need to be defined. This thesis assumes that all of the noise is due to thermal noise in the resistance $R$ of the resonant circuit. Thermal noise is assumed to be Gaussian white noise with a power spectral density (PSD) of

$$S_n(\omega) \equiv 2kT \text{ watts per Hz} \quad \text{for} \quad |\omega| \ll 2\pi kT/h,$$  \hspace{1cm} (2.37)

where $T$ is the temperature of the conducting medium in Kelvin, $k$ is Boltzmann’s constant, and $h$ is Planck’s constant. Figure 2.10 shows that the thermal noise of the resistance $R$ is modeled

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{marginal_oscillator.png}
\caption{Schematic representation of a marginal oscillator with addition of thermal noise.}
\end{figure}
as a current source \( i_n(t) \) in parallel with \( R \), with a PSD of

\[
S_i(\omega) = \frac{2kT}{R}.
\] (2.38)

The SNR of a marginal oscillator as well as that of a Q-meter is defined as the ratio of the rms envelope voltage due to changes in the losses of the resonant circuit to the rms envelope voltage due to thermal noise.

First consider the Q-meter system shown in Figure 2.11. As the independent current source \( i(t) \) is noiseless, only the current source \( i_n(t) \) contributes to the noise of the envelope of \( v(t) \). From Eq. (2.9) the transfer function that relates the noise current \( i_n(t) \) to the voltage \( v(t) \) is

\[
H(j\omega) = \frac{1}{C} \frac{j\omega}{(j\omega)^2 + \frac{\omega_n}{Q} j\omega + \omega_n^2}.
\] (2.39)

Assuming the rms value of the independent current signal is much larger than the rms value of the noise signal, it can be shown that the mean-square value of the envelope of \( v(t) \) due to the noise source \( i_n(t) \) is [17]

\[
v_{\text{rms}}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_v(\omega) d\omega,
\] (2.40)

where

\[
S_v(\omega) = |H(j\omega)|^2 S_i(\omega).
\] (2.41)

Stremeler shows that Eq. (2.40) can be expressed equivalently as [16]

\[
v_{\text{rms}}^2 = \frac{1}{\pi} |H(j\omega_0)|^2 S_i(\omega) B_{eq} = \frac{1}{\pi} R^2 \frac{2kT}{R} \frac{\pi}{2RC} = \frac{kT}{C},
\] (2.42)

where \( |H(j\omega_0)|^2 = R^2 \) is the midband system voltage gain and

\[
B_{eq} = \frac{\pi}{2RC} \text{ rad/s}
\] (2.43)

is the equivalent bandwidth, as derived in Appendix A. It follows from Eq. (2.42) that the rms

![Figure 2.11. Schematic representation of a Q-meter with addition of thermal noise.](image)
envelope noise is

\[ v_{rms} = \sqrt{\frac{kT}{C}}. \]  

(2.44)

Note that the rms envelope noise is independent of the resistance \( R \). This is because even though the PSD of the noise current is inversely proportional to \( R \), the bandwidth of the resonant circuit is proportional to \( R \). These two effects cancel, yielding the above result.

Because the conversion gain of a Q-meter is unity, a change in the losses causes a proportionally equal change in the envelope voltage. For a change in the losses \( \Delta R \), the change in the rms amplitude is

\[ \Delta A_{rms} = \frac{\Delta R}{R} A_{rms} = \xi A_{rms}. \]  

(2.45)

Therefore, the SNR of a Q-meter is

\[ \text{SNR}_{Q-meter} = \frac{\Delta A_{rms}}{v_{rms}} = \xi A_{rms} \sqrt{\frac{C}{kT}}, \]  

(2.46)

where \( \xi \) is the normalized change in the losses \( \Delta R/R \).

In a marginal oscillator, because of the nonlinear feedback element, the rms noise is also effected by the form of \( G(v) \). Viswanathan gives the rms noise of the marginal oscillator as \[12\]

\[ v_{rms} = \sqrt{\frac{kT G_c}{C}}. \]  

(2.47)

The conversion gain \( G_c \) of a marginal oscillator indicates that a change in the losses causes a change in the envelope voltage that is proportionally \( G_c \) times as much. Therefore, for a change in the losses \( \Delta R \), the change in the rms amplitude is

\[ \Delta A_{rms} = \frac{\Delta R}{R} G_c A_{rms} = \xi G_c A_{rms}. \]  

(2.48)

This means the SNR of a marginal oscillator is

\[ \text{SNR}_{m.o.} = \frac{\Delta A_{rms}}{v_{rms}} = \xi A_{rms} \sqrt{\frac{C G_c}{kT}}. \]  

(2.49)

This thesis defines the sensitivity of a marginal oscillator as the change in the normalized change in the losses \( \xi = \Delta R/R \) that leads to a unity SNR in the presence of thermal noise. It follows that the sensitivity is

\[ S_{m.o.} = \xi = \frac{1}{A_{rms}} \sqrt{\frac{kT}{C G_c}}, \]  

(2.50)

Notice that increasing the conversion gain causes the value of the sensitivity to decrease.

This thesis defines the NF as

\[ \text{NF} = \frac{\text{SNR}_{Q-meter}}{\text{SNR}_{m.o.}}. \]  

(2.51)
Using this definition, the NF of a marginal oscillator is

\[ \text{NF} = \frac{1}{\sqrt{G_c}}. \]  

(2.52)

The noise figure, \(10 \log_{10} \text{(NF)}\), is

\[ \text{noise figure} = -5 \log_{10}(G_c). \]  

(2.53)

Typically, conversion gains are greater than unity, so the NF of a marginal oscillator is less than unity. This is different from the NF of an amplifier, which is never less than unity. This indicates that as the conversion gain increases, the effect of thermal noise on the output voltage of the envelope decreases.
Simulations

The marginal oscillator is simulated using Simulink, a simulation tool in MATLAB. Within Simulink, the marginal oscillator is represented by the all-integrator block diagram in Figure 3.1. Performing the simulation using an m-file exclusively is undesirable, because a necessary function call leads to significantly longer simulation times.

3.1 Marginal Oscillator Parameter Values

Numerical simulations in this study use circuit values of an experimental marginal oscillator. For this system, the nominal frequency of oscillation \( f_n = 3.308 \, \text{MHz} \) is obtained using an inductance \( L \) of 22.8 \( \mu \text{H} \) and a capacitance \( C \) of 101.53 pF. The losses of the resonant circuit are established from measurements of the system 3 dB bandwidth obtained using a vector network analyzer.

![Figure 3.1. Block diagram of marginal oscillator.](image-url)
Using Eq. (2.8), the nominal Q-factor of the resonant circuit is
\[ Q = \frac{f_n}{3 \text{ dB BW}} = \frac{3.308 \text{ MHz}}{26.464 \text{ kHz}} = 125.0. \] (3.1)

Using Eq. (2.8) it follows that the nominal losses of the resonant circuit is
\[ R = Q\omega_n L = 59.237 \text{ kΩ}. \] (3.2)

The experiment is designed with a fixed value of \( g_A \) chosen so that \( g_A R = 1.0344 \). It follows that the nominal value of \( g_A \) is
\[ g_A = \frac{1.0344}{R} = 17.462 \mu\text{Ω}. \] (3.3)

The nominal values \( L, C, R, \) and \( g_A \) will be used in subsequent simulations.

In practice, given \( L, C, R, \) and \( g_A \), the user then specifies the conversion gain \( G_c \) and the amplitude \( A \), and then solves for \( g_B \) and \( v_T \) using Eqs. (2.32), (2.35), and (2.36). In particular, given \( G_c \) and \( g_A R \), Eq. (2.35) is solved for \( \theta \). Once \( \theta \) is known, \( g_B \) is found using Eq. (2.36) and \( v_T \) is found using Eq. (2.32). This study uses a conversion gain \( G_c \) of 10 and a nominal amplitude \( A \) of 0.25 V. This results in \( g_B = 14.795 \mu\text{Ω} \) and \( v_T = 0.1670 \text{ V} \).

Additional necessary parameters are the initial values \( x_1(0) \) and \( x_2(0) \) of the integrator outputs. To derive these values, suppose the voltage has the form
\[ v(t) = a \cos(\omega t), \] (3.4)
with \( a > 0 \). This means the initial conditions on the voltage are \( v(0) = a \) and \( \dot{v}(0) = 0 \). From Eq. (2.13),
\[ x_2(0) = Ca. \] (3.5)

Now consider the inputs and output of the summation block in Figure 2.2
\[ \dot{x}_2(t) = i(t) - \frac{1}{RC} x_2(t) - \frac{1}{LC} x_1(t). \] (3.6)

At time \( t = 0 \), this simplifies to
\[ x_1(0) = LC \left[ G(a) - \frac{a}{R} \right], \] (3.7)
where
\[ G(a) = \begin{cases} g_A a, & a < v_T \\ g_A v_T + (a - v_T)g_B, & \text{otherwise}. \end{cases} \] (3.8)

If \( a < v_T \), the initial conditions simplify to
\[ x_1(0) = \frac{LCa}{R} (g_A R - 1) \] (3.9)
\( x_2(0) = Ca. \)  \hspace{1cm} (3.10)

This study chooses \( a = 0.1 \) V. It follows that \( x_1(0) = 1.3442 \times 10^{-022} \) and \( x_2(0) = 1.0153 \times 10^{-011} \).

Using simulations, we ultimately want to determine the conversion gain \( G_c \). Using the approximation given by Miller [8],

\[
G_c \approx \frac{\Delta A}{\Delta R A}, \tag{3.11}
\]

where \( A \) is a function of \( R \). This thesis shows that this expression is only valid if \( \Delta R \) is small.

Using Eq. (2.35) is then solved for \( \theta \). Once \( \theta \) is known, \( g_B \) is determined using Eq. (2.36). The threshold voltage \( v_T \) is determined so that for the nominal value of \( R = 59.237 \) k\( \Omega \), the amplitude \( A \) is 1 V. As \( R \) is varied, the fixed values of \( g_A, g_B, \) and \( v_T \) are used.

Figure 3.2 shows a plot of the amplitude \( A \) versus the losses \( R \) for conversion gains \( G_c \) of 10, 25, and 50. Note that for all curves, \( A = 1 \) V at \( R = R \) by design. Also note that as \( G_c \) gets larger, the curves become more nonlinear with respect to \( R \). Therefore, the estimation of \( \partial A/\partial R \) by \( \Delta A/\Delta R \) must use a smaller \( \Delta R \) as \( G_c \) increases.

The effect of the nonlinear relation between \( A \) and \( R \) on the approximation of \( G_c \) by Eq. (3.11) is now considered. Let

\[
\hat{G}_c = \frac{\Delta A}{\Delta R} \frac{\bar{R}}{A} = \frac{\bar{A} - A}{\bar{R} - R A}, \tag{3.12}
\]

where \( \bar{A} \) and \( \bar{R} \) represent the nominal amplitude of oscillation and losses, respectively. In order to obtain the estimate \( \hat{G}_c \), in experiment and simulation, the nominal losses \( \bar{R} \) is perturbed by an amount \( \Delta R \) using a calibration circuit [18]. When the calibration circuit is turned on, additional losses are introduced into the circuit so that

\[ R = \bar{R} - \Delta R. \]  \hspace{1cm} (3.13)

A decrease in the losses leads to a decrease \( \Delta A \) in \( A \) so that

\[ A = \bar{A} - \Delta A. \]  \hspace{1cm} (3.14)

The question arises as to how small \( \Delta R \) must be so that \( \hat{G}_c \) is a good estimate of \( G_c \). To determine this, \( \hat{G}_c \) is calculated as a function of \( \Delta R \) using Eq. (3.12). More specifically, using the \( \bar{R} \) specified above, \( \bar{A} = 1 \) V, and \( G_c = 25 \), the amplitude \( A \) is calculated as a function of \( R \) using the procedure that generated Figure 3.2 and the estimate \( \hat{G}_c \) is calculated using Eq. (3.12). Figure 3.3 shows the estimated conversion gain \( \hat{G}_c \) versus \( \Delta R/R \). The estimated conversion gain quickly diverges away from the actual conversion gain as \( \Delta R \) is increased. From the figure, for \( \Delta R/R \) of approximately 0.5%, \( \hat{G}_c \) is at 90% of \( G_c \).
3.2 Solver and Integration Parameter Selection

In order to determine the appropriate solver, the concept of stiffness needs to be understood. The idea of stiffness originated fifty years ago when computers became available for widespread use in solving ODEs. It is not well known what makes a problem stiff. In fact, there exist many definitions of stiffness, such as:
• Systems containing very fast components as well as very slow components [19].

• Systems that represent coupled physical systems having components varying with very different timescales: that is they are systems having some components varying much more rapidly than the others [20].

• Stiff equations are equations where certain implicit methods perform better, usually tremendous better, than explicit ones [21].

Brugnano proposed a unified definition in which systems are stiff when the stiffness ratio,

\[ \sigma_c = \frac{k_c}{\gamma_c} = \max_{t \in [0,T]} \frac{|y(t)|}{\frac{1}{T} \int_0^T |y(t)|dt}, \]  

is much greater than unity [22]. This definition is applicable to many scenarios as discussed by Brugnano. For oscillatory systems, such as the marginal oscillator, Brugnano showed that \( \sigma_c \) can be approximated as

\[ \sigma_c = 2\pi \frac{T}{T^*}, \]  

where \( T \) is the length of the simulation and \( T^* \) is the period of oscillation. In a marginal oscillator, the time constant for change in amplitude is given by Viswanathan as [12]

\[ \tau = 2RCG_c. \]  

For a typical application where \( R = 60 \, \text{k}\Omega, \, C = 100 \, \text{pF}, \) and \( G_c = 10, \, \tau = 1.2 \times 10^{-4} \, \text{s}. \) To observe changes in amplitude, set \( T = 5\tau. \) For oscillation at 3.3 MHz,

\[ T^* = \frac{1}{3.3 \, \text{MHz}} = 3.0303 \times 10^{-7} \, \text{s}. \]  

It follows that

\[ \sigma_c = \frac{T}{T^*} = \frac{5\tau}{T^*} = 1980 \gg 1. \]  

The marginal oscillator is stiff because \( \sigma_c \gg 1. \)

The marginal oscillator system is solved using the MATLAB ODE suite, a collection of codes for solving initial value problems given by first-order systems of ODEs. Functions in the ODE suite are categorized by how they choose the step-size and how they solve for the next value of the response. Fixed-step solvers provide a solution at points in time separated by a fixed interval, while variable-step solvers adjust the interval between solution points. Explicit solvers directly compute the next solution using known quantities, while implicit solvers find the next solution by solving a set of equations using an iterative technique.

Both fixed-step and variable-step solvers calculate the next simulation time by adding the current simulation time and the step size. For fixed-step solvers, the step size is constant and user-specified. For variable-step solvers, the step size is chosen by the solver to meet specified relative
and absolute error tolerances. The relative tolerance specifies the allowable error with respect

to each state variable. The absolute tolerance represents the allowable error for state variable

values close to zero. A variable-step solver is preferable, because it can significantly shorten the

simulation time. This is because, for a given level of accuracy, the solver can dynamically adjusts

the step size as necessary and thus reduce the number of steps.

Explicit solver calculates the values of the next state as a function of previous states and state

derivatives. In contrast, implicit solvers find the next state by solving a coupled set of algebraic

equations dependent on the next state derivative using a Newton-like method. The result of

this approach is that implicit solvers provide greater stability for oscillatory behavior, like that

found in stiff systems. However, they do so by sacrificing accuracy and are more computationally

expensive. Therefore, implicit solvers are generally used for stiff systems, while explicit solvers

are used for nonstiff systems.


It is desirable to use a variable-step solver to reduce simulation time and memory usage.

Out of the available variable-step solvers, this study chooses **ode15s** for two reasons. First, the

ODE describing the marginal oscillator is very stiff. This means an implicit solver other than

**ode23t**, which is only effective for moderately stiff systems [23], must be used. Second, a high

level of accuracy is desired. This rules out **ode23s** and **ode23tb**, which are more effective at

crude tolerances [24, 25].

The **ode15s** solver has several parameters. This thesis adjusts the relative and absolute toler-

ces to strike a balance between accuracy and simulation time. To determine the appropriate
tolerances, simulations were conducted for several different tolerances, with a nominal amplitude

$A$ of 0.25 V, a change in losses $\Delta R$ of 47.389 $\Omega$, and a desired conversion gain of 10. This study

chooses the appropriate tolerances by comparing simulated and theoretical values of the esti-

mated conversion gain $\hat{G}_c$, the nominal amplitude $A$, the new amplitude $A$ from a change $\Delta R$ in

the losses, the natural frequencies $f_n$ at both amplitudes, and the time constants $\tau$ transitioning

between the two amplitudes. In addition, the total simulation time is taken into account.

The simulated values are obtained using Simulink according to the following procedure. The

Simulink model was first run for a total of 9.3 ms. With $R = \overline{R}$, the simulated voltage was given

2 ms to reach steady-state and is then observed for 1 ms. Then, the resistance is changed to

$R = \overline{R} - \Delta R$ over 0.1 ms. At the new $R$, the voltage is again given 2 ms to reach steady-state and

observed for 1 ms. Finally, the resistance is changed back to $R = \overline{R}$ over 0.1 ms, and the voltage

is observed in the same manner as before. Then, the positive and negative peaks of the voltage

are found. From this, it is possible to determine the envelope voltage as well as the oscillation

frequency. The conversion gain can be estimated from the envelope voltage using [Eq. (3.11)]. For

the two transition regions, the corresponding rise or fall time is determined as the time it takes

for the envelope voltage to transition between 10% and 90% of the difference between the two

maximums and minimums.
steady-state values. From the rise and fall times, it is possible to calculate the time constants using $\tau = t_r / \ln 9$ and $\tau = t_f / \ln 9$, respectively. Table 3.1 compares theory with the simulation results, obtained using 64-bit Matlab R2010a on a Windows 7 computer with an Intel Core 2 Duo T9300 processor at 2.5 GHz and 4 GB of RAM. The simulation results are mostly consistent with theory except for the time constants $\tau$. Therefore, the solver tolerances are chosen based on the simulated time constants and the simulation time. This study finds the best compromise between accuracy and speed is achieved with a relative tolerance of $10^{-10}$ and an absolute tolerance of $10^{-30}$. This set of tolerances results in more accurate time constants compared with other sets with similar or shorter simulation times, while a more accurate set of tolerances needs nearly double the simulation time.

It is not always possible to use a variable-step solver. As described in Section 3.3, a fixed-step solver is sometimes necessary. Therefore, this study also determines the preferred fixed-step solver. While ode14x is the only implicit fixed-step solver, simulations show that the explicit solvers have comparable accuracy and much shorter simulation times. Therefore, all of the fixed-step solvers need to be considered. To determine the appropriate solver and step size, simulations were performed with the same procedure that produced Table 3.1 except that the simulations were done for different fixed-step solvers and step sizes. Table 3.2 shows the results of the simulations. As with the previous set of simulations, the simulation results are mostly consistent with theory except for the time constants $\tau$. Therefore, the solver and step size are chosen based on the simulated time constants and the simulation time. Note that the accuracy of the simulations is largely determined by the step size for solvers of sufficiently high orders. Using a higher order solver than ode4 does not result in greater accuracy, while using a lower order solver reduces accuracy. The simulated time constants for step sizes of 0.25 ns and 0.5 ns are both very close to theory, while those for larger step sizes are less accurate. Therefore, this study finds the best compromise between accuracy and speed is achieved with the ode4 solver with a step size of 0.5 ns.

### 3.3 Thermal Noise

Thermal noise is of importance to marginal oscillator performance, because it sets the SNR of measurements. To better understand the effect of thermal noise, this section has three goals. The first goal is to incorporate thermal noise into the simulation model for the marginal oscillator. The second goal is to use simulation to verify the relationship between the noise factor (NF) and the conversion gain given in Eq. (2.52). For the third goal, it is well known that the SNR in pulsed quadrupole resonance spectroscopy varies as the square root of the coil Q-factor \([26]\). The coil Q-factor is usually on the order of 100. However, recent attempts at Penn State by Pusateri have resulted in high-temperature superconductor coils with Q-factors on the order of 500,000. The third goal is to verify using simulation whether this relationship still exists for very high Q-factors.

This thesis assumes the thermal noise is Gaussian white noise. However, simulation of ideal-
### Table 3.1. Comparison of ode15s solver parameters.

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<th>Tolerance</th>
<th>$\hat{T}_\epsilon$</th>
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<th>$A$ [V]</th>
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<th>$\tau$ [µs]</th>
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### Table 3.2. Comparison of fixed-step solvers.

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<th>$A$ [V]</th>
<th>$f_n$ [Hz]</th>
<th>$\tau$ [µs]</th>
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<tr>
<td>ode3</td>
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<td>0.2477719</td>
<td>3308000</td>
<td>3308000</td>
<td>121.003</td>
<td>121.003</td>
</tr>
<tr>
<td>ode3</td>
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<td>0.2496966</td>
<td>0.2479778</td>
<td>3308000</td>
<td>3308000</td>
<td>120.109</td>
<td>120.590</td>
</tr>
<tr>
<td>ode3</td>
<td>0.25</td>
<td>0.2499961</td>
<td>0.2480040</td>
<td>3308000</td>
<td>3308000</td>
<td>119.971</td>
<td>120.315</td>
</tr>
<tr>
<td>ode4</td>
<td>2</td>
<td>0.2499815</td>
<td>0.2479895</td>
<td>330803</td>
<td>3307996</td>
<td>125.819</td>
<td>130.015</td>
</tr>
<tr>
<td>ode4</td>
<td>1</td>
<td>0.2499955</td>
<td>0.2480034</td>
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<td>3308003</td>
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<td>122.035</td>
</tr>
<tr>
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<td>0.2499989</td>
<td>0.2480068</td>
<td>3308000</td>
<td>3308000</td>
<td>120.109</td>
<td>120.590</td>
</tr>
<tr>
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<td>0.2480076</td>
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<td>120.315</td>
</tr>
<tr>
<td>ode5</td>
<td>2</td>
<td>0.2499820</td>
<td>0.2479900</td>
<td>330803</td>
<td>3307996</td>
<td>125.337</td>
<td>129.189</td>
</tr>
<tr>
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<td>0.2480034</td>
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<td>3308003</td>
<td>121.003</td>
<td>122.516</td>
</tr>
<tr>
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<td>0.2499989</td>
<td>0.2480068</td>
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<td>3308000</td>
<td>120.109</td>
<td>120.590</td>
</tr>
<tr>
<td>ode5</td>
<td>0.25</td>
<td>0.2499997</td>
<td>0.2480076</td>
<td>3308000</td>
<td>3308000</td>
<td>119.971</td>
<td>120.315</td>
</tr>
<tr>
<td>ode8</td>
<td>10</td>
<td>0.2495502</td>
<td>0.2475617</td>
<td>3307996</td>
<td>3307996</td>
<td>909.966</td>
<td>909.966</td>
</tr>
<tr>
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<td>0.2498875</td>
<td>0.2478963</td>
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<td>3307996</td>
<td>168.126</td>
<td>907.074</td>
</tr>
<tr>
<td>ode8</td>
<td>2</td>
<td>0.2499820</td>
<td>0.2479900</td>
<td>330803</td>
<td>3307996</td>
<td>125.337</td>
<td>129.189</td>
</tr>
<tr>
<td>ode8</td>
<td>1</td>
<td>0.2499955</td>
<td>0.2480034</td>
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<td>3308003</td>
<td>121.003</td>
<td>122.516</td>
</tr>
<tr>
<td>ode8</td>
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<td>0.2499989</td>
<td>0.2480068</td>
<td>3308000</td>
<td>3308000</td>
<td>120.109</td>
<td>120.590</td>
</tr>
<tr>
<td>ode14x</td>
<td>10</td>
<td>0.2495798</td>
<td>0.2475910</td>
<td>3307996</td>
<td>3307996</td>
<td>909.966</td>
<td>909.893</td>
</tr>
<tr>
<td>ode14x</td>
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<td>0.2498885</td>
<td>0.2478972</td>
<td>3307996</td>
<td>3307996</td>
<td>169.020</td>
<td>905.010</td>
</tr>
<tr>
<td>Theoretical</td>
<td></td>
<td>0.25</td>
<td>0.2480079</td>
<td>3308000</td>
<td>120.105</td>
<td>120.280</td>
<td>—</td>
</tr>
</tbody>
</table>
ized white noise with a constant PSD is impossible, so band-limited white noise is used instead. To produce band-limited white noise, a fixed-step solver must be used. To see why, consider the PSD of band-limited white noise shown in Figure 3.4, which is nonzero for $-B < f < B$. This PSD leads to an autocorrelation described by a sinc function with zero crossings at $t = kT$, where $T = 1/2B$ and $k$ is a nonzero integer. At a fixed sample rate of $T$, sampled values of band-limited Gaussian white noise are uncorrelated and independent in the Gaussian case. Hence, band-limited Gaussian white noise can be simulated using an independent and identically-distributed sequence of Gaussian random variables if the sample rate is fixed, as is the case with a fixed-step solver. Therefore, this thesis uses the fixed-step solver ode4 with a step size of $T = 0.5$ ns for the reasons described in Section 3.2.

The band-limited Gaussian white noise is simulated using a Gaussian random number generator with a mean of zero and a variance of $\sigma^2 = \frac{4kTB}{R}$, where $k$ is Boltzmann’s constant, $T$ is the temperature in Kelvin, $B$ is the bandwidth of the system, and $R$ is the losses of the resonant circuit. The standard noise temperature of $T = 290$ K is used. The bandwidth is $B = 1/2T = 1$ GHz. Figure 3.5 shows the addition of the Gaussian random number generator that produces band-limited Gaussian white noise, represented by GWN, into the block diagram of the marginal oscillator.

This study considers the effect of thermal noise on the envelope of the voltage. To simplify simulations, only the steady-state envelope is considered. To do so, set the initial amplitude $a = A$. The simulations were run for Q-factors ranging from order of magnitudes of zero to six and conversion gains of 1, 2, 5, 10, 25, 50, 100, 500, 1000, and $10^6$. For each Q-factor and conversion gain $G_c$, it is possible to calculate the resistance $R$ using Eq. (2.8) and the angle $\theta$ using Eq. (2.35). As mentioned before, the nominal value of $g_A R$ is 1.0344. It follows that $g_A = 1.0344/R$. Knowledge of $G_c$, $R$, $g_A$, and $\theta$ allows $v_T$ and $g_B$ to be found using Eqs. (2.32) and (2.36) respectively. For each combination of Q-factor and conversion gain, the
system is simulated twice, once with noise and once without, each time for 10 μs. The two envelope voltages are then calculated. The rms envelope voltage is then found by taking the standard deviation of the difference between the two envelope voltages. The simulation results are shown in Table 3.3. Viswanathan predicts that the rms envelope noise given by Eq. (2.47) is proportional to the square root of $G_c$ and independent of the Q-factor. However, simulation shows that the amount of noise is also affected by the Q-factor. To investigate the effect of the Q-factor on the rms envelope noise, it is useful to consider the plot of rms envelope noise versus Q-factor in Figure 3.6. In particular, consider the curve corresponding to a conversion gain of $10^6$. It can be seen that the marginal oscillator limits the maximum rms envelope noise, and this limit is inversely proportional to the square root of the Q-factor. This makes sense, because the Q-factor is inversely proportional to the bandwidth of the resonant circuit. Thus, increasing the Q-factor decreases the bandwidth, resulting in decreased rms envelope noise. For significantly small Q-factor and conversion gain, for which the rms envelope noise is not limited, theory and simulation are in agreement.
It is useful to compare the NF for theory and simulation. Theory predicts that the NF given by Eq. (2.52) is inversely proportional to the square root of $G_c$. Using Eq. (2.51), the NF from simulation can be computed as

$$\text{NF} = \frac{\text{SNR}_{\text{Q-meter}}}{\text{SNR}_{\text{m.o.}}} = \frac{\Delta A_{\text{rms,Q-meter}}/v_{\text{rms,Q-meter}}}{\Delta A_{\text{rms,m.o.}}/v_{\text{rms,m.o.}}},$$

where $\Delta A_{\text{rms}}$ is the change in rms amplitude from a change $\Delta R$ in the losses and $v_{\text{rms}}$ is the rms envelope noise. Using the conversion gain approximation in Eq. (3.11) and the fact that the conversion gain of a Q-meter is unity, it follows that

$$\text{NF} = \frac{(A/\sqrt{2})(\Delta R/R)}{G_c(A/\sqrt{2})(\Delta R/R)} \frac{v_{\text{rms,m.o.}}}{v_{\text{rms,Q-meter}}} = \frac{v_{\text{rms,m.o.}}}{G_c v_{\text{rms,Q-meter}}}.$$  

It has already been shown that for sufficiently small Q-factors, the rms envelope noise for the marginal oscillator is directly proportional to the square root of $G_c$. Because the rms envelope noise for the Q-meter is a constant, it follows that the NF is inversely proportional to the square root of $G_c$, as predicted by theory.

It is also useful to determine the effect of the Q-factor on the SNR of the marginal oscillator using simulation. The SNR of the marginal oscillator as defined Section 2.5 is

$$\text{SNR}_{\text{m.o.}} = \frac{\Delta A_{\text{rms}}}{v_{\text{rms}}},$$

where $\Delta A_{\text{rms}}$ is the change in rms amplitude from a change $\Delta R$ in the losses, independent of the Q-factor, and $v_{\text{rms}}$ is the rms envelope noise. As determined above, for sufficiently large
Table 3.4. Effect of a sampled-time NRC.

<table>
<thead>
<tr>
<th>$f_s$ [MHz]</th>
<th>$\bar{G}_c$</th>
<th>$\bar{A}$ [V]</th>
<th>$A$ [V]</th>
<th>$f_n$ [Hz]</th>
<th>$\tau$ [µs]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq 46$</td>
<td>Does Not Oscillate</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>25.670</td>
<td>0.1724287</td>
<td>0.1688877</td>
<td>3305029</td>
<td>3305026</td>
</tr>
<tr>
<td>50</td>
<td>13.838</td>
<td>0.1834444</td>
<td>0.1814137</td>
<td>3305213</td>
<td>3305210</td>
</tr>
<tr>
<td>75</td>
<td>10.177</td>
<td>0.2197864</td>
<td>0.2179970</td>
<td>3306155</td>
<td>3306154</td>
</tr>
<tr>
<td>100</td>
<td>9.990</td>
<td>0.2326172</td>
<td>0.2307581</td>
<td>3306621</td>
<td>3306618</td>
</tr>
<tr>
<td>200</td>
<td>9.950</td>
<td>0.245382</td>
<td>0.2435836</td>
<td>3307311</td>
<td>3307311</td>
</tr>
<tr>
<td>500</td>
<td>9.958</td>
<td>0.2492712</td>
<td>0.2472855</td>
<td>3307726</td>
<td>3307725</td>
</tr>
<tr>
<td>1000</td>
<td>9.981</td>
<td>0.2498045</td>
<td>0.2478142</td>
<td>3307863</td>
<td>3307863</td>
</tr>
<tr>
<td>Continuous</td>
<td>9.944</td>
<td>0.2499947</td>
<td>0.2480059</td>
<td>3308001</td>
<td>3308000</td>
</tr>
</tbody>
</table>

Q-factors, the rms envelope voltage is inversely proportional to the square root of the Q-factor. It follows that the SNR of the marginal oscillator varies with the square root of the Q-factor.

3.4 Sampled-Time Implementation

Realizing a NRC with even a simple $i$-$v$ characteristic as shown in Figure 2.3b is not an easy task using analog circuits. To simplify the hardware representation of the NRC, it is useful to implement the NRC using a sampled-data system. This thesis seeks to ultimately show the feasibility of implementing the NRC using a sampled-data system. As a first step, this study investigates implementing the NRC using a sampled-time system, where time is quantized but amplitude is not. To more closely match the sampled-data implementation given in Section 2.3.2, the NRC in the simulation is implemented as the current

$$i(t) = \frac{\bar{G}(v(kT)) - v(t)}{R_f} \quad \text{for} \quad kT \leq t < (k+1)T,$$

(3.24)

where $T = 1/f_s$, the sample period, is equal to the inverse of the sample rate $f_s$. This implementation is similar to the implementation in Figure 2.5a except that $\bar{G}(v)$ is now quantized in time. Figure 3.7 shows the marginal oscillator block diagram with this sampled-time implementation of the NRC. The simulations were performed for different sample rates using the same procedure that generated Table 3.1. The feedback resistance $R_f$ is 100 Ω for the simulations. Table 3.4 shows the effect of different sample rates $f_s$ on marginal oscillator performance. Simulation reveals that a minimum sample rate of 47 MHz is necessary to achieve oscillation. As the sample rate increases, the results more closely match the continuous results. Namely, as the sample rate increases, the conversion gain decreases, the oscillation amplitudes increase, the oscillation frequencies increase, and the time constants decrease. It is interesting to note that the conversion gain increases when the NRC is sampled. This reveals the possibility of using quantization of the NRC in time in order to increase the conversion gain.
3.5 Sampled-Data Implementation

Having determined the sample rate needed for the marginal oscillator to function, this study investigates the effect of implementing the NRC using a sampled-data system, where amplitude is quantized in addition to time. To do so, a quantizer is added before the input to $G(v)$ in Figure 3.7. The resulting marginal oscillator with a sampled-data implementation of the NRC is shown in Figure 3.8. As before, the resistance $R'$ is fixed at 100 Ω. The quantizer calculates the output based on the input such that

$$y = q \times \text{round} \left( \frac{u}{q} \right), \quad (3.25)$$

where $y$ is the output, $u$ is the input, and $q$ is the quantization interval. It is convenient to relate the quantization interval $q$ to the number of quantization bits $N$ by

$$q = \frac{2v_{\text{max}}}{2^N}, \quad (3.26)$$

where the $v_{\text{max}}$ is the maximum output of the quantizer and $-v_{\text{max}}$ is the minimum. This study chooses $v_{\text{max}} = 5$ V. The simulations use with a sample rate of 100 MHz, which produces good results and is achievable with current technology. The same procedure as in Section 3.4 is used for the simulations. Table 3.5 shows the simulation results of the effect of quantization in both time and amplitude on the performance of a marginal oscillator. The results show that a quantizer resolution of 10 bits and a sample rate of 100 MHz is sufficient to achieve the desired conversion gain. However, a higher resolution of 12 bits lowers the time constant, which is useful to reduce the necessary measurement time. Any additional increase in resolution has minimal effect on the performance of the marginal oscillator.

In a physical implementation of the NRC by a discrete-time DSP system, such as shown in Figure 2.9, the output of the NRC would be delayed. Delays of 100 ns can be expected.
with current technology. Keeping in mind that the nominal oscillation period $T_n = 1/f_n$ is 302.3 ns, a delay that is a third of the oscillation period could negatively impact the oscillation of the marginal oscillator. Therefore, the effect of a delayed NRC output on marginal oscillator performance is investigated. Consider the marginal oscillator block diagram in Figure 3.9 where the output sampled-data NRC is delayed by a time $d$. The marginal oscillator was simulated with a sample rate of 100 MHz and 14 bits of quantization for several different delay values. The simulation results are shown in Table 3.6. Small delays less than or equal to 6 ns allow the marginal oscillator to still oscillate. The delay of 6 ns corresponds to approximately 7° of phase delay at the nominal frequency. A delay of 100 ns, which corresponds to 119° of phase delay, would cause the marginal oscillator to not function. However, delays close to a multiple of the oscillation period also allow oscillation without too much impact on the conversion gain. This result reveals the feasibility of a physical implementation of a marginal oscillator with a sampled-data NRC by increasing the delay to a multiple of the oscillation period.

Another method of allowing the marginal oscillator to still oscillate is to implement larger gains in the feedback path to counter the loss of amplitude that results from a large NRC output delay. This thesis implements the larger gains by determining the gains $g_A$ and $g_B$ as above, and then, increasing them by a gain factor $\alpha$ slightly larger than unity. The new gains are given by $\alpha g_A$ and $\alpha g_B$, respectively. Tables 3.7 and 3.8 show the simulation results for various gain factors $\alpha$ for delays of 10 ns and 20 ns. The results show that increasing the gain in the feedback path can cause the marginal oscillator to function for larger delays, but the resulting oscillation frequency is lowered. Therefore, the effect of large delays can be mitigated by choosing appropriate gain factors.
Table 3.5. Effect of a sampled-data NRC with a sample rate of 100 MHz.

<table>
<thead>
<tr>
<th>Quant. [bits]</th>
<th>$G_c$</th>
<th>$A$ [V]</th>
<th>$f_n$ [Hz]</th>
<th>$\tau$ [µs]</th>
<th>$A$ to $A$</th>
<th>$\overline{A}$ to $\overline{A}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td>Does Not Oscillate</td>
</tr>
<tr>
<td>6</td>
<td>0.989</td>
<td>0.1222574</td>
<td>0.1221607</td>
<td>3306613</td>
<td>3306612</td>
<td>910.002</td>
</tr>
<tr>
<td>7</td>
<td>2.826</td>
<td>0.2200098</td>
<td>0.2195124</td>
<td>3306611</td>
<td>3306615</td>
<td>40.535</td>
</tr>
<tr>
<td>8</td>
<td>4.599</td>
<td>0.2325667</td>
<td>0.2317113</td>
<td>3306618</td>
<td>3306613</td>
<td>38.195</td>
</tr>
<tr>
<td>10</td>
<td>10.142</td>
<td>0.232189</td>
<td>0.2304322</td>
<td>3306620</td>
<td>3306621</td>
<td>462.537</td>
</tr>
<tr>
<td>12</td>
<td>10.047</td>
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<td>0.2307748</td>
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<td>3306618</td>
<td>122.775</td>
</tr>
<tr>
<td>14</td>
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<td>0.232676</td>
<td>0.2307588</td>
<td>3306620</td>
<td>3306618</td>
<td>119.747</td>
</tr>
<tr>
<td>16</td>
<td>9.989</td>
<td>0.2326176</td>
<td>0.2307558</td>
<td>3306620</td>
<td>3306618</td>
<td>120.986</td>
</tr>
</tbody>
</table>

No Quant.     | 9.990       | 0.2326172 | 0.2307581   | 3306621    | 3306618   | 120.986          | 121.674           |

Figure 3.9. Block diagram of marginal oscillator with delayed sampled-data NRC.

Table 3.6. Effect of a sampled-data NRC with a sample rate of 100 MHz and 14 bits of quantization.

<table>
<thead>
<tr>
<th>Delay [ns]</th>
<th>$G_c$</th>
<th>$A$ [V]</th>
<th>$f_n$ [Hz]</th>
<th>$\tau$ [µs]</th>
<th>$A$ to $A$</th>
<th>$\overline{A}$ to $\overline{A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.988</td>
<td>0.2326200</td>
<td>0.2307613</td>
<td>3306620</td>
<td>3306618</td>
<td>119.747</td>
</tr>
<tr>
<td>1</td>
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<td>0.2271184</td>
<td>0.2249508</td>
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<td>3306341</td>
<td>885.645</td>
</tr>
<tr>
<td>2</td>
<td>10.202</td>
<td>0.2203419</td>
<td>0.2185433</td>
<td>3306063</td>
<td>3306062</td>
<td>888.542</td>
</tr>
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<td>5</td>
<td>11.092</td>
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<td>0.1937776</td>
<td>3316213</td>
<td>3316213</td>
<td>872.041</td>
</tr>
<tr>
<td>6</td>
<td>13.587</td>
<td>0.1852133</td>
<td>0.1832001</td>
<td>3307927</td>
<td>3307925</td>
<td>884.373</td>
</tr>
<tr>
<td>$\geq 7$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Does Not Oscillate</td>
</tr>
</tbody>
</table>

$T_n = 302.3$ | 7.688       | 0.2328011 | 0.2313692   | 3312655    | 3306654   | 126.421          | 137.360           |

$2T_n$       | 9.882       | 0.2337906 | 0.2319357   | 3306686    | 3310685   | 125.044          | 124.768           |

$3T_n$       | 9.945       | 0.2343103 | 0.2324458   | 3308717    | 3309717   | 128.552          | 129.172           |
Table 3.7. Effect of a sampled-data NRC with a sample rate of 100 MHz, 14 bits of quantization, 10 ns delay, and gain factor $\alpha$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$G_c$</th>
<th>$\bar{A}$ [V]</th>
<th>$A$ [V]</th>
<th>$f_n$ [Hz]</th>
<th>$\tau$ [$\mu$s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.03</td>
<td>6.61</td>
<td>0.2001323</td>
<td>0.1990731</td>
<td>3303743</td>
<td>3303740</td>
</tr>
<tr>
<td>1.05</td>
<td>9.873</td>
<td>0.2444268</td>
<td>0.2424963</td>
<td>3303743</td>
<td>3303740</td>
</tr>
<tr>
<td>1.0525</td>
<td>9.926</td>
<td>0.2503142</td>
<td>0.2483274</td>
<td>3303745</td>
<td>3303740</td>
</tr>
<tr>
<td>1.055</td>
<td>9.785</td>
<td>0.2562818</td>
<td>0.2542758</td>
<td>3303743</td>
<td>3303741</td>
</tr>
<tr>
<td>1.06</td>
<td>9.632</td>
<td>0.2684544</td>
<td>0.2663856</td>
<td>3303745</td>
<td>3303741</td>
</tr>
<tr>
<td>1.07</td>
<td>9.572</td>
<td>0.2957769</td>
<td>0.2935121</td>
<td>3303744</td>
<td>3303742</td>
</tr>
</tbody>
</table>

Table 3.8. Effect of a sampled-data NRC with a sample rate of 100 MHz, 14 bits of quantization, 20 ns delay, and gain factor $\alpha$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$G_c$</th>
<th>$\bar{A}$ [V]</th>
<th>$A$ [V]</th>
<th>$f_n$ [Hz]</th>
<th>$\tau$ [$\mu$s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.15</td>
<td>9.926</td>
<td>0.2427437</td>
<td>0.2408161</td>
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<tr>
<td>1.1525</td>
<td>9.976</td>
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Chapter 4

Discussion

4.1 Summary

This thesis accomplished its aims. First, it was verified that Viswanathan’s theoretical predictions are correct through simulation. This was accomplished by choosing an appropriate solver and solver parameters, along with establishing a protocol for estimating conversion gain from simulation results. In addition, the theoretical predictions were also verified through experiment.

Second, it was shown through simulation that a sampled-data implementation of the NRC is feasible. This thesis was able to determine the minimum sample rate and quantization level that allows the marginal oscillator to properly function. It was also shown that the negative effect of a delayed NRC output due to the sampled-data implementation can be mitigated using larger gains on the feedback path or by increasing the delay to a multiple of the oscillation period.

Third, the effect of thermal noise on the marginal oscillator was taken into account. This thesis shows that Viswanathan’s prediction of the rms envelope noise is inadequate and only correct for sufficiently small Q-factors.

4.2 Future Work

This thesis was able to predict several properties of the marginal oscillator using simulation. Further work in this area would need to experimentally verify the results. In addition, theory needs to be improved to describe the results seen.


The equivalent bandwidth of a system is the bandwidth of an ideal brick-wall filter that gives the same noise power as the actual system. Assuming white noise and a system with a transfer function $H(j\omega)$, Stremler gives the equivalent bandwidth as

$$B_{eq} = \frac{\int_{0}^{\infty} |H(j\omega)|^2 d\omega}{|H(j\omega_0)|^2},$$

(A.1)

where $\omega_0$ is the midband frequency that maximizes $|H(j\omega)|$. This definition is illustrated in Figure A.1.

This thesis wants to find the equivalent bandwidth of a parallel RLC circuit. From Eq. (2.9),

![Figure A.1. Graphic definition of equivalent bandwidth.](image)
the transfer function that relates the noise current $i_n(t)$ to the voltage $v(t)$ is

$$H(j\omega) = \frac{1}{C} \frac{j\omega}{(j\omega)^2 + \frac{\omega_n^2}{Q} j\omega + \omega_n^4}. \quad (A.2)$$

It follows that

$$|H(j\omega)|^2 = \frac{\omega^2/C^2}{(\omega^2 - \omega_n^2)^2 + \frac{\omega_n^2}{Q^2} \omega^2}. \quad (A.3)$$

Using $\omega_0 = \omega_n$ for a parallel RLC circuit, it follows from Eq. (A.3) that

$$|H(j\omega_0)|^2 = \frac{1}{C^2} \frac{Q^2}{\omega_n^2}. \quad (A.4)$$

Using the equivalent bandwidth definition in Eq. (A.1), it follows from Eqs. (A.3) and (A.4) that

$$B_{eq} = \frac{\omega_n^2}{Q^2} \int_0^\infty \frac{\omega^2}{\omega^4 + \left(\frac{\omega_n^2}{Q^2} - 2\omega_n^2\right) \omega^2 + \omega_n^4} d\omega. \quad (A.5)$$

Because the integrand in Eq. (A.5) is an even function of $\omega$, the integral can be expressed equivalently as

$$B_{eq} = \frac{\omega_n^2}{2Q^2} \int_{-\infty}^\infty \frac{\omega^2}{\omega^4 + \left(\frac{\omega_n^2}{Q^2} - 2\omega_n^2\right) \omega^2 + \omega_n^4} d\omega. \quad (A.6)$$

The integral in Eq. (A.6) is difficult to evaluate using elementary calculus but is easily evaluated using contour integration. To see this, consider integrating the complex function

$$f(z) = \frac{\omega_n^2}{2Q^2} \frac{z^2}{z^4 + \left(\frac{\omega_n^2}{Q^2} - 2\omega_n^2\right) z^2 + \omega_n^4} \quad (A.7)$$

along the contour $A$ shown in Figure A.2, which consists of the line segment along the real axis

![Figure A.2](image-url)
from $-a$ to $a$ and the semicircle in the upper half-plane centered at the origin. Choose $a$ large enough such that both poles of $f(z)$ in the upper half-plane are contained in $A$. This integral can be found by noting that $f(z)$ has two poles in the upper half-plane, and the sum of their residues is

$$\sum \text{Res} = \frac{\omega_n}{4Qj}. \quad (A.8)$$

It follows from the residue theorem that

$$\oint_A f(z) \, dz = 2\pi j \sum \text{Res} = \frac{\pi \omega_n}{2Q} \quad (A.9)$$

for sufficiently large $a$. All that remains is to show that

$$B_{eq} = \oint_A f(z) \, dz. \quad (A.10)$$

Note that

$$\oint_A f(z) \, dz = \int_{-a}^{a} f(z) \, dz + \int_{\text{Arc}} f(z) \, dz. \quad (A.11)$$

Using the estimation lemma, it can be shown that

$$\left| \int_{\text{Arc}} f(z) \, dz \right| \leq \frac{\omega_n^2}{2Q^2} a^4 + \frac{\pi a^3}{Q^2} - 2\omega_n^2 \left| a^2 + \omega_n^4 \right| \to 0 \text{ as } a \to \infty, \quad (A.12)$$

which means

$$\int_{\text{Arc}} f(z) \, dz \to 0 \text{ as } a \to \infty \quad (A.13)$$

by the sandwich theorem. Using Eqs. (A.11) and (A.13) it follows that

$$\oint_A f(z) \, dz = \int_{-a}^{a} f(z) \, dz \text{ as } a \to \infty. \quad (A.14)$$

Using Eqs. (A.9) and (A.14) this thesis finds the equivalent bandwidth to be

$$B_{eq} = \int_{-\infty}^{\infty} f(z) \, dz = \lim_{a \to \infty} \int_{-a}^{a} f(z) \, dz = \oint_A f(z) \, dz = \frac{\pi \omega_n}{2Q} \text{ rad/s.} \quad (A.15)$$

Using Eq. (2.8) it can be equivalently expressed as

$$B_{eq} = \frac{\pi}{2RC} \text{ rad/s.} \quad (A.17)$$

Note that the equivalent bandwidth is inversely proportional to the Q-factor. Therefore, the thermal noise seen across the resonant circuit decreases as the Q-factor increases. Also, the
equivalent bandwidth is related to the 3 dB bandwidth $\beta$ by

$$B_{eq} = \frac{\pi}{2} \beta.$$  \hfill (A.18)

Interestingly, this is the same relationship found between the equivalent and 3 dB bandwidths of a first-order Butterworth filter.
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