THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF SCIENCE, ENGINEERING, AND TECHNOLOGY

THE RESULTING SHAPE OF A HANGING CHAIN WITH A FUNCTION BOUNDARY CONDITION

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Abstract

Mathematics has built the foundation for many areas of study. The creation of some of the concepts of mathematics are rooted in the need to solve physics problems. A common physics problem that is used as an example in calculus is the catenary problem. The catenary is the shape formed by a chain connected to two poles. This problem has been heavily studied and analyzed using the Newtonian approach of forces and a system being in equilibrium. We consider the calculus of variations approach to the hanging chain problem. The calculus of variations was created in parallel to calculus; however, they differ in how one would approach a problem such as the catenary. The calculus of variations focuses on the fact that a chain connected between two poles tends toward a shape that minimizes the system's potential energy, thus forming a catenary. We derive the equation for the catenary then explore the consequences of two hanging chain problems using methods of the calculus of variations. The first problem having boundary conditions related to fixing the left endpoint to a specific point while constraining the right endpoint to freely move along the line, $mx_1 + b$. The second problem allowing the left endpoint to move freely up and down a pole say, $y = x_0$ with the right endpoint the same as the first problem. Both problems yield a general solution of,

$$y(x) = c_1 \cosh\left(\frac{x_1}{c_1} + c_2\right),$$

with c_1 and c_2 being constants dependent on the initial boundary conditions which we analyze.

Specifically, the results of this research will need to be explored further in other research projects to have greater worldly applications. However, this research will offer assistance to other young mathematicians in learning and understanding a problem in the calculus of variations.

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Acknowledgements

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Above all I give thanks to God for the physical, mental, and spiritual strength to complete this work.

Chapter 1

Introduction

1.1 Motivation

The applications of mathematics have played a major role in the advancement of various fields of scientific study. In fact, the motivation for the creation of calculus was credited to the need to solve physics problems in a more structured and systematic way. The calculus of variations was developed in parallel to calculus, around the mid seventeenth century, thus they share basic notation and tools but differ primarily in their methodological approaches [2, 4]. A classic problem that is still widely studied today, in both calculus and the calculus of variations, is the catenary problem of determining the exact form of a hanging chain pinned at both end points. The solution has been derived using the Newtonian Calculus and the Calculus of Variations approach. The basic problem has since been revisited and altered.

1.2 Calculus background

If the function is known then calculus allows us to find the lowest point of the curve. If the function is unknown we focus on the use of the Calculus of Variations approach to derive the function that depicts the shape of a chain hanging between two poles. If we use Newtonian mechanics we can determine an equation that contains information about the slope of the hanging cable: a differential equation [1]. Integration can now be used to find the curve that satisfies the equation just mentioned. The calculus of variations approach to determine the shape of a hanging cable does not involve using say, Newton's second law; it involves a recognition that systems tend towards configurations that minimize their potential energy. The problem of the shape of a hanging cable when fixed between two poles can now be stated as: amongst all the curves hanging between two poles, which minimizes the potential energy of the cable? The function of interest, in this case, is the catenary: represented mathematically by the hyperbolic cosine function, denoted as cosh [3].

Seen in Figure 1.1, the catenary depicts one form of connection between two fixed points. The mathematical ideas behind the calculus of variation's techniques will be explored more extensively.

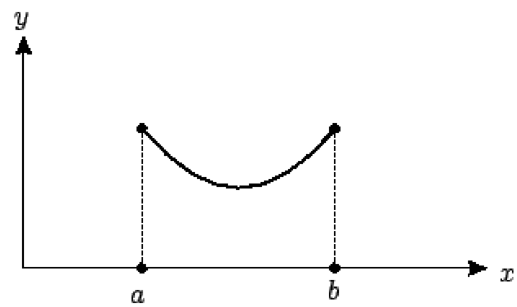

Figure 1.1: A Catenary [5]

Chapter 2

Elements of the theory of the calculus of variations

2.1 The first variation

A fundamental problem in the calculus of variations is to find a function $y(x)$ that is an extremum, say a minimizer, of a functional, a function of functions,

$$J\big[y(x)\big] = \int_{x_0}^{x_1} F\big(x, y, y'\big), \tag{2.1}$$

where $y(x)$ satisfies the boundary conditions $y(x_0) = y_0,\ y(x_1) = y_1$. The functions y and F are assumed to be sufficiently smooth.

Recall that, a function of a single variable, $h(x)$, achieves a local minimum at $x = x_0$ if for all nearby x values, say within ϵ units of x_0, $h(x) \geq h(x_0)$ [4].

Similarly, a functional, $J[y]$ achieves a local minimum at $y = \hat{y}(x)$ if for all functions nearby of the form $\hat{y}(x) + \epsilon h(x),\ J[y] \geq J[\hat{y}]$, where $\hat{y}(x)$ satisfies the above boundary conditions and $h(x_0) = h(x_1) = 0$ [8, 2, 3].

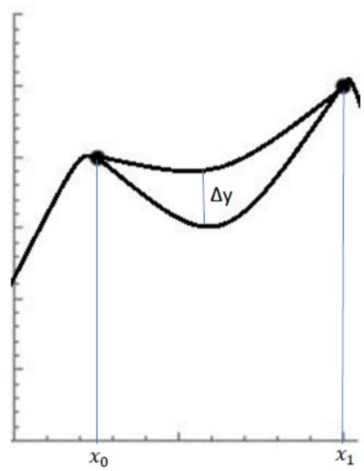

Figure 2.1: A variation of y

A necessary property for a local minimum of a single variable function, $h(x)$, is to be located at a critical point, $x = x_0$, where $h'(x_0) = 0$.

Likewise, $J[y]$ has a local minimum at $y = \hat{y}(x)$, where,

$$\frac{d}{d\epsilon}J[\hat{y} + \epsilon h]|_{\epsilon=0} = 0.$$

This type of derivative for a functional is known as a Gateaux derivative and is an analogous to a directional derivative from multi-variable calculus.

Now,

$$\frac{d}{d\epsilon}J[\hat{y} + \epsilon h]|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{J[\hat{y} + \epsilon h] - J[\hat{y}]}{\epsilon}.$$

The above is denoted $\delta J(\hat{y}, h)$ and is called the first variation of J at $\hat{y}$.

The numerator of this difference quotient,

$$\Delta J = J[\hat{y} + \epsilon h] - J[\hat{y}]$$

is simply a variation, or change in J as seen in figure (2.1).

Note $h \in C^2[x_0, x_1]$ refers to functions that are twice differentiable with each derivative continuous within the interval $[x_0, x_1]$. For concreteness, we define two sets, S and H, which define conditions for the smoothness of $\hat{y}$ and h

$$S = \{y \in C^2[x_0, x_1] : y(x_0) = y_0, y(x_1) = y_1\}$$

$$H = \{h \in C^2[x_0, x_1] : h(x_0) = 0, h(x_1) = 0\}$$

and boundary constraints for $\hat{y}, h$ that guarantee that functions of the form $\hat{y}(x) + \epsilon h(x)$ satisfy the original boundary conditions (2.1).

Now, using Leibniz's integration rule

$$\frac{d}{d\epsilon}\left(\int_{x_0}^{x_1} f(x, \epsilon)dx\right) = \int_{x_0}^{x_1} \frac{d}{d\epsilon} f(x, \epsilon)dx$$

and Taylor's Theorem for a function of two variables

$$f(x, y) = f(x_0, x_1) + f_x(x - x_0) + f_y(y - x_1) + \cdots$$

applied to ΔJ centered around $\epsilon = 0$ one can show

$$\delta J(y, h) = \int_{x_0}^{x_1} F_y(x, y, y')h + F_{y'}(x, y, y')h'\, dx$$

at $y \in S$ and for any $h \in H$.

Since we established that the variation of J must vanish, for J to be stationary, neither increasing or decreasing,

$$\delta J = \int_{x_0}^{x_1} F_y h + F_{y'} h'\, dx = 0.$$

Using integration by parts one can show

$$\delta J = h(x)F_{y'}|_{x_0}^{x_1} - \int_{x_0}^{x_1} h(x)(F_y - \frac{d}{dx}F_{y'})dx = 0. \tag{2.2}$$

Lemma 2.1.1 *If $f(x)$ is continuous on $[x_0, x_1]$ and if*

$$\int_{x_0}^{x_1} f(x)h(x)dx = 0$$

for all $h \in C[x_0, x_1]$ such that $h(x_0) = h(x_1) = 0$ then $f(x) = 0$ for all $x \in [x_0, x_1]$ [3].

Here $h \in C[x_0, x_1]$ means functions, h, that are continuous within the interval $[x_0, x_1]$. Using lemma (2.1.1), also known as the Fundamental lemma of the calculus of variations, we can deduce

$$F_y - \frac{d}{dx}F_{y'} = 0, \tag{2.3}$$

since $h(x_0) = h(x_1) = 0$.

Equation 2.3 is known as the Euler-Lagrange Equation and must hold for the functional to have an extremum.

Theorem 2.1.2 *Let $J[y]$ be a functional of the form*

$$\int_{x_0}^{x_1} F(x, y, y')\, dx,$$

for all $y \in S$. Then a necessary condition for $J[y]$ to have an extremum for a given function $y(x)$ is that $y(x)$ satisfy the Euler-Lagrange equation

$$F_y - \frac{d}{dx}F_{y'} = 0\ [3].$$

The Euler-Lagrange equation is a second order partial differential equation known as a Lagrangian, which characterizes the state of a physical system. Mathematically, the Lagrangian is kinetic energy minus potential energy, most often used to examine how a system changes with time. However, our system is not dependant on time and thus our function only represents the potential energy of the chain.

For special forms of $F(x, y, y')$ we can obtain a first order differential equation by integrating the Euler-Lagrange equation. Specifically, if the integrand, $F\big(x, y, y'\big)$, does not depend explicitly on x then the functional will be of the form

$$J\big[y\big] = \int_{x_0}^{x_1} F\big(y, y'\big)\, dx,$$

and the Euler-Lagrange equation becomes

$$F_y - \frac{d}{dx}F_{y'} = F_y - F_{y'y}y' - F_{y'y'}y''.$$

Multiplying both sides of the above equation by y' yields

$$F_y y' - F_{y'y}{y'}^2 - F_{y'y'}y'y'' = \frac{d}{dx}(F - y'F_{y'}),$$

using the multi-variable chain rule for differentiation. Since the right side is equal to 0, from (2.3) we obtain a first integral

$$F - y'F_{y'} = C,$$

where C is an arbitrary constant.

2.2 Classical catenary problem

Here we will be looking for the stable, or stationary, function $y\big(x\big)$ that minimizes the length of a hanging cable between $[x_0, x_1]$ by minimizing the potential energy, mgy, where m is the mass per unit length of the cable; g denotes gravitational acceleration; and $y(x)$ is the height of the cable as seen in (2.2). Mass and gravity in this problem are constant. It is sufficient to analyze y at each infinitesimal piece of chain, ds, between $[x_0, x_1]$ where $ds = \sqrt{1 + y'^2}$, the arc length of an element of the cable, by the functional:

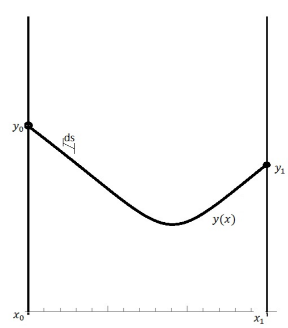

Figure 2.2: Classical catenary problem

$$J\big[y(x)\big] = \int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx. \tag{2.4}$$

that satisfies the boundary conditions or rather fixed end point conditions since y is fixed at x_0, x_1,

$$y\big(x_0\big) = y_0,\ y\big(x_1\big) = y_1.$$

Note that there are no restrictions being imposed on the length of the chain here. Hence we call this boundary value problem (BVP) the unconstrained or variable length catenary problem.

Recall the special case of the Euler-Lagrange equation for the integrand $F(x, y, y')$ from Section 2.1. Since our integand does not depend directly on x,

$$F - y'F_{y'} = C. \tag{2.5}$$

By substituting $F = y\sqrt{1+(y')^2}$ into (2.5), and computing the partial derivative of F with respect to y' we obtain $F_{y'} = \frac{yy'}{\sqrt{1+(y')^2}}$ [6].

$$y\sqrt{1+(y')^2} - \frac{y(y')^2}{\sqrt{1+(y')^2}} = C.$$

Solving for y' results in a first order differential equation that can be solved using the technique of separation of variables as follows:

$$y' = \frac{\sqrt{y^2+C^2}}{C},$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2+C^2}}{C},$$

$$dx = \frac{C\,dy}{\sqrt{y^2+C^2}}.$$

Integrating both sides yields:

$$x = C\cosh^{-1}\left(\frac{y}{C}\right) + C_2$$

or

$$y = c_1\cosh\left(\frac{x}{c_1} + c_2\right). \tag{2.6}$$

2.3 Variable endpoint BVP's

In this work we consider two types of variable endpoint conditions.

First, suppose we do not require that the values of y are fixed at the given endpoints x_0, x_1, but allow them both to vary freely on curves, $\varphi(x),\ \psi(x)$.

(See pages 59-61 in Gelfand and Fomin's work for full derivation [3]).

Suppose,

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\, dx,$$

where the endpoints $(x_0, y_0), (x_1, y_1)$ are required to lie on the curves $y = \varphi(x),\ y = \psi(x)$, respectively. Then J is stationary when the following are satisfied,

$$F - y'F_{y'} = C, \tag{2.5}$$

$$[F + (\varphi' - y')F_{y'}]|_{x=x_0} = 0, \tag{2.7}$$

$$[F + (\psi' - y')F_{y'}]|_{x=x_1} = 0, \tag{2.8}$$

(2.7),(2.8) are known as the transversality conditions. The transversality conditions are derived in a similar manner as the Euler-Lagrange equation, by computing $\delta J(y, h)$ where now variations in y and x are considered as shown in figure (2.3).

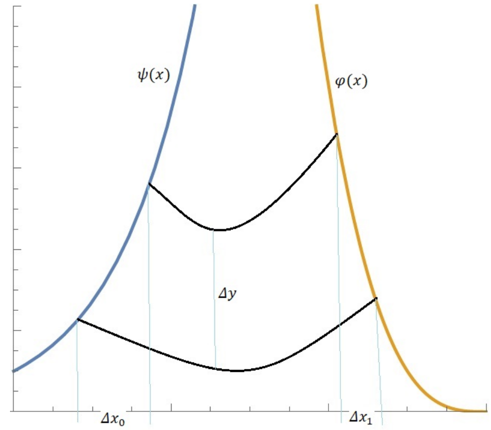

Figure 2.3: Variations for x and y

When we consider the functional (2.4), the transversality conditions simplify as follows.

$$J[y] = \int_{x_0}^{x_1} y\sqrt{1+(y')^2}\,dx,$$

where $F = y\sqrt{1+(y')^2}$. Substituting F into the transversality conditions results in:

$$F + (\varphi' - y')F_{y'} = \frac{(1+y'\varphi')F}{1+y'^2} = 0,$$

$$F + (\psi' - y')F_{y'} = \frac{(1+y'\psi')F}{1+y'^2} = 0.$$

It then follows that

$$y' = -\frac{1}{\varphi'}$$

for the left hand side and

$$y' = -\frac{1}{\psi'}$$

for the right hand side. Thus with functionals of our form, transversality reduces to orthogonality between $y(x)$ and the corresponding curve each endpoint lies on.

2.3.1 Natural boundary conditions

Suppose the value of y is not specified at either x_0 or x_1. Then $h(x)$ is no longer required to vanish at x_0, x_1. In that case for the term

$$h(x)F_{y'}|_{x_0}^{x_1}$$

from equation (2.2), to vanish we need

$$F_{y'}|_{x_0} = F_{y'}|_{x_1} = 0$$

for any h regardless of the value of $h(x_0)$ and $h(x_1)$.

Observe that

$$F_{y'} = \frac{yy'}{\sqrt{1+(y')^2}}.$$

Since we have no constraints on $y(x_0), y(x_1)$, $y'(x_0) = y'(x_1) = 0$ is a necessary condition for $F_{y'}|_{x_0} = F_{y'}|_{x_1} = 0$

These boundary conditions are called natural boundary conditions. We note that for the natural boundary condition $y' = 0$, orthogonality between y and the poles each endpoint lies on, is a special case of the transversality conditions that arise when you allow an end point to slide freely along the line $x = x_0$.

Chapter 3

Variable length catenary problem with a fixed left end, variable right end

We now consider the following problem: Minimize,

$$J[y(x)] = \int_{x_0}^{x_1} y\sqrt{1+(y')^2}\,dx \tag{2.4}$$

subject to the boundary condition $y(0) = 1$, $y(x_1) = mx_1 + b = y_1$ as seen in figure (3.1).

From (2.6) we know that that $y = c_1 \cosh\left(\frac{x}{c_1} + c_2\right)$ is our candidate for a local minimum.

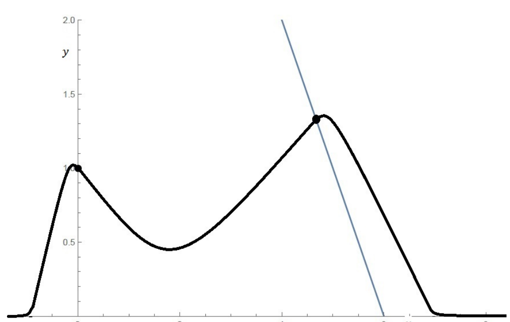

Figure 3.1: Infinite length catenary with fixed left end, variable right end

Note that so far $c_1 \neq 0$ is the only restriction required with respect to the unknown constants/-parameters, c_1, c_2, x_1, y_1, m, b.

3.1 Consequences of the boundary conditions

Now x_0 and y_0 are specified while the right end point, (x_1, y_1), is constrained to lie on the line $mx + b$; m denotes the non-zero slope of the line and b is the y-intercept, as seen in figure (3.1). Recall:

$$y(0) = 1 \tag{3.1}$$

$$y(x_1) = mx_1 + b = y_1 \tag{3.2}$$

The transversality condition associated with the free right endpoint becomes,

$$y'(x_1) = -\frac{1}{m} \tag{3.3}$$

The boundary conditions and the transversality condition produce three equations in terms of the unknowns, c_1, c_2, x_1, y_1, m, b, as follows:

From boundary condition (3.1) and (2.6) we deduce that

$$c_1 \cosh c_2 = 1. \tag{3.4}$$

Using (3.2) and (2.6) shows,

$$y(x_1) = c_1 \cosh\left(\frac{x_1}{c_1} + c_2\right) = mx_1 + b = y_1 \tag{3.5}$$

Lastly, from the transversality condition (3.3) and (3.5) we obtain

$$\sinh\left(\frac{x_1}{c_1} + c_2\right) = -\frac{1}{m} \tag{3.6}$$

or

$$\sinh^{-1}\left(-\frac{1}{m}\right) = \frac{x_1}{c_1} + c_2 \tag{3.7}$$

3.1.1 Existence of Solutions

At this point we can divide (3.5) by (3.6) to form a single transcendental equation:

$$\coth\left(\frac{x_1}{c_1} + c_2\right) = \frac{-my_1}{c_1}.$$

Solutions of the above determine whether or not solutions of the BVP exist.

Multiplying the right hand side by $\frac{x_1+c_1c_2}{x_1+c_1c_2}$ yields,

$$\coth\left(\frac{x_1}{c_1} + c_2\right) = \frac{-my_1(x_1 + c_1c_2)}{c_1(x_1 + c_1c_2)}.$$

Let $u = \frac{x_1}{c_1} + c_2 = \frac{x_1+c_1c_2}{c_1}$ and $\alpha = \frac{-my_1}{x_1+c_1c_2}$. Thus we have,

$$\coth u = \alpha u \tag{3.8}$$

or

$$\frac{\coth u}{u} = \alpha. \tag{3.9}$$

Note here that $\frac{\coth u}{u}$ is even since both u and $\coth u$ are odd functions. Thus $\frac{\coth(-u)}{(-u)} = \frac{\coth u}{u}$. First note that $\frac{\coth u}{u}$ has no roots since $\coth u \neq 0$, thus does not intersect the u-axis. Now applying the first derivative test for (3.8) we have that the only critical point of,

$$\frac{d}{du}\frac{\coth u}{u} = \frac{u(1 - \coth^2 u) - \coth u}{u^2}$$

is $u = 0$ where the derivative is undefined. Thus we can look at the value of the derivative when evaluated at -1 and 1.

$$\frac{d}{du}\frac{\coth u}{u}\bigg|_{u=1} = -2.037$$

$$\frac{d}{du}\frac{\coth u}{u}\bigg|_{u=-1} = 2.037$$

Hence (3.9) is increasing when $u < 0$ and decreasing when $u > 0$ by the first derivative test.

Now we determine the end behavior of $\frac{\coth u}{u}$ as $u \to \infty$ and as $u \to 0^+$, since (3.8) is even the above extreme conditions are the same for $u \to -\infty$ and $u \to 0^-$.

$$\lim_{u\to\infty} \frac{\coth u}{u} = \lim_{u\to\infty} \coth u \times \frac{1}{u} = 0,$$

since

$$\lim_{u\to\infty} \coth u = 1.$$

Then

$$\lim_{u\to 0^+} \frac{\coth u}{u} = 1 - \coth^2 u = \infty.$$

using L'Hopital's limit rule.

Thus the left side of (3.9) is always greater than 0 and so we only get solutions to (3.9) if $\alpha > 0$ as well.

3.1.2 Signs of parameters $x_1,\ c_1,\ c_2,\ m,\ b,\ y_1$

Now we can conclude that

$$sgn(-m) = sgn(y_1) = sgn(x_1 + c_1 c_2).$$

Note for each $\alpha > 0$ we get two u values, a positive and negative, but since the left hand side of (3.9) is an even function the resulting u values are negatives of each other thus we can restrict our admissible field to only $u > 0$. Restricting the domain to $(0, \infty)$, the function on the left side of (3.9) becomes one to one and thus only has one u for each $\alpha > 0$. Hence,

$$sgn(x_1 + c_1 c_2) = sgn(c_1) > 0,$$

from (3.10).

Therefore, $m < 0$ and $y_1 > 0$. Then by our construction of α, since there are no restrictions on c_2, $x_1 \neq 0$. In fact since the domain of y is $[x_0, x_1]$, where $x_0 = 0$, x_1 must be positive. Here note that we have

$$y_1 = mx_1 + b,$$

or

$$y_1 - b = mx_1 < 0,$$

thus

$$y_1 - b < 0$$

therefore

$$b > 0.$$

We can also say that $x_1 < \frac{-b}{m}$ and $b > -mx_1$ since,

$$y_1 = mx_1 + b > 0.$$

Rewriting equation (3.4) we can solve c_1 in terms of c_2.

$$c_1 = \frac{1}{\cosh c_2} \tag{3.10}$$

Now, equation (3.10) restricts $c_1 \in (0, 1]$ while $c_2 \in (-\infty, \infty)$.

Note that $\cosh(\sinh^{-1} x) = \sqrt{1 + x^2},\ \forall x \in \mathbb{R}$.

Now we solve for $\frac{x_1}{c_1} + c_2$ in equation (3.3) and substitute the result into (3.5) giving,

$$c_1 \cosh\left(\sinh^{-1}\left(-\frac{1}{m}\right)\right) = c_1\sqrt{1+\frac{1}{m^2}} = mx_1 + b. \tag{3.11}$$

Then we solve for x_1 and obtain

$$x_1 = \frac{c_1\sqrt{1+\frac{1}{m^2}} - b}{m} = \frac{\sqrt{1+\frac{1}{m^2}}}{m\cosh c_2} - \frac{b}{m}. \tag{3.12}$$

Now, since (3.11) holds and $c_1 \in (0,1]$, $y_1 \in \left(0, \sqrt{1+\frac{1}{m^2}}\right]$ with $x_1 \in (0, \frac{-b}{m})$.

We see in the y_1 verses c_1 plot, figure (3.2) that we have three scenarios. We could have two solutions. For $m = -9$ and $b = 10$ we get two c_1 values for the y_1 values near 0.5. We get a unique solution at the minimum of y_1 and for values above, in this case about 0.7. Then we have no solutions for c_1 when y_1 is less then the minimum.

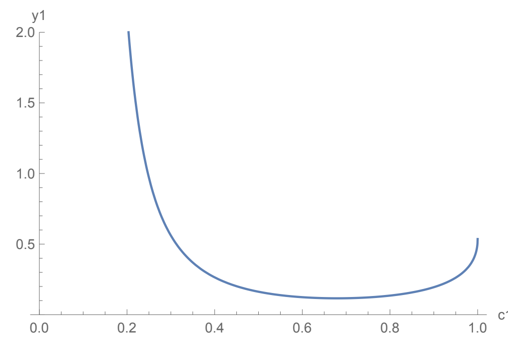

Figure 3.2: y_1 versus c_1 with $m = -9,\ b = 10$

3.2 Analysis

Now we determine the behavior of y_1 and the length of the cable as c_1 approaches 0 from the left and 1 from the right with m, b held constant.

One can show,

$$\lim_{c_1\to 0^-} x_1 = \frac{c_1\sqrt{1+\frac{1}{m^2}} - b}{m} = \frac{-b}{m}$$

and

$$\lim_{c_1\to 1^+} x_1 = \frac{c_1\sqrt{1+\frac{1}{m^2}} - b}{m} = \frac{\sqrt{1+\frac{1}{m^2}} - b}{m}.$$

Now recall,

$$L = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx$$

Thus for our function (2.6) we get,

$$\begin{aligned} L &= \int_0^{x_1} \sqrt{1 + \sinh^2\left(\frac{x}{c_1} + c_2\right)}\, dx \\ &= \int_0^{x_1} \sqrt{\cosh^2\left(\frac{x}{c_1} + c_2\right)}\, dx \\ &= \int_0^{x_1} \cosh\left(\frac{x}{c_1} + c_2\right) dx \end{aligned}$$

where we have used the hyperbolic trigonometric identity, $\cosh^2 x - \sinh^2 x = 1$. Thus,

$$\begin{aligned} L &= c_1 \sinh\left(\frac{x_1}{c_1} + c_2\right) - c_1 \sinh c_2 \\ &= \frac{c_1}{-m} - \tanh c_2, \end{aligned} \tag{3.13}$$

since $\sinh\left(\frac{x_1}{c_1} + c_2\right) = -\frac{1}{m}$ from (3.6) and $c_1 = \frac{1}{\cosh c_2}$ in (3.10).

Note we want the length to be positive thus we add the constraint that $L > 0$. This results in

$$-\frac{c_1}{m} - \tanh c_2 > 0,$$

$$-\frac{c_1}{m} > \tanh c_2,$$

thus

$$0 < c_1 < -m,$$

also to ensure that the length is positive we need $c_2 < 0$.

Note,

$$\lim_{c_1 \to 0^+} c_2 = -\infty.$$

Now evaluating the end behavior of the length,

$$\begin{aligned}\lim_{c_1 \to 0^+} L &= \lim_{c_1 \to 0^+} -\frac{c_1}{m} - \tanh c_2 \\ &= \lim_{c_1 \to 0^+} -\tanh c_2 \\ &= 1.\end{aligned}$$

Now note that

$$\lim_{c_1 \to 1^-} c_2 = 0.$$

Then

$$\begin{aligned}\lim_{c_1 \to 1^-} L &= \lim_{c_1 \to 1^-} -\frac{c_1}{m} - c_1 \sinh c_2 \\ &= -\frac{1}{m} - \sinh 0, \\ &= -\frac{1}{m},\end{aligned}$$

since if $c_1 = 1$ then $c_2 = 0$ from (3.10). Note that $-\frac{1}{m} > 0$ since m is less than zero. Here we plot the length of the catenary between two poles using the length formula (3.13) along with (3.12) and (3.10) which all depend on m and b. This is shown in figure (3.3) where $m = -1$ and for any b.

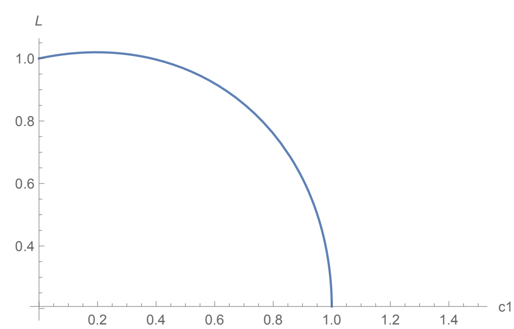

Figure 3.3: Length function with $m = -5$

Lastly, recalling (3.11), we have

$$c_1 = \frac{y_1}{\sqrt{1 + \frac{1}{m^2}}},$$

thus as $c_1 \to 0$ we get,

$$\frac{y_1}{\sqrt{1+\frac{1}{m^2}}} = 0,$$

which implies,

$$\lim_{c_1 \to 0^+} y_1 = 0.$$

Then it follows from (3.11) that

$$\lim_{c_1 \to 1^-} y_1 = \sqrt{1+\frac{1}{m^2}}\,.$$

3.3 Several graphs of $y(x)$

Here we show the resulting catenary with several different values for c_1 to show how $y(x)$ changes as c_1 varies. Record the changes of y with respect to c_1. For the following graphs let $m = -5$ and $b = 5$ unless labeled otherwise.

Figure (3.4) shows $y(x)$ if we let $c_1 = 1$ then $c_2 = -\cosh^{-1} 1$ from (3.10).It follows that,

$$x_1 = \frac{c_1\sqrt{1+\frac{1}{m^2}} - b}{m} = \frac{\sqrt{1+\frac{1}{(-1)^2}} - 5}{(-1)} \approx 0.7960,\ y_1 \approx 1.3339$$

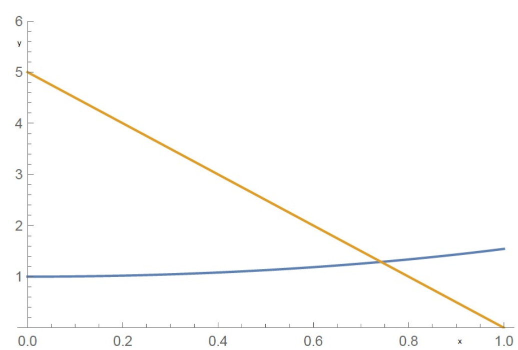

Figure 3.4: $y(x)$ with $c_1 = 1, m = -5, b = 5$

Note figure (3.4) graphically shows the results of the limits of y_1 and x_1 as c_1 approaches 1^+.

Now let $c_1 = 0.5$ then $c_2 = -\cosh^{-1} 2$. Notice in figure (3.5) as c_1 is smaller than in figure

(3.4) that y_1 is closer to 0. For comparison figure (3.6) has the same c_1 value now with $m = -6$ and $b = 4$.

We can show that for (3.5),

$$x_1 \approx 0.8980,\ y_1 \approx 0.5190$$

and for (3.6).

$$x_1 \approx 0.5822,\ y_1 \approx 0.5013$$

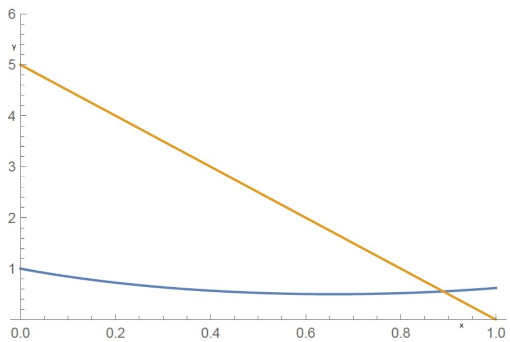

Figure 3.5: $y(x)$ with $c_1 = 0.5, m = -5, b = 5$

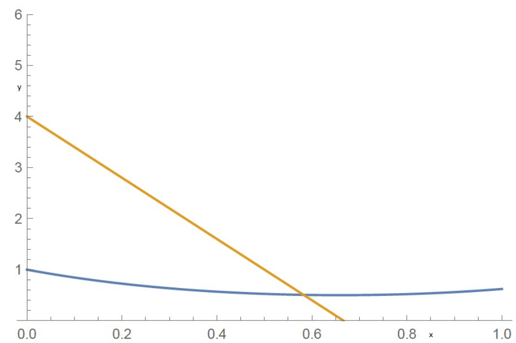

Figure 3.6: $y(x)$ with $c_1 = 0.5$,$m = -6, b = 4$

Notice that the catenary, $y(x)$ is the same in both figures (3.5) and (3.6). Hence the catenary is not directly dependent on m and b. The only changes are in the values for x_1 and y_1.

Lastly, figure (3.7) shows $y(x)$ if we let $c_1 = 0.25$ then $c_2 = -\cosh^{-1} 4$.

Then,

$$x_1 \approx 0.9490,\ y_1 \approx 0.4241$$

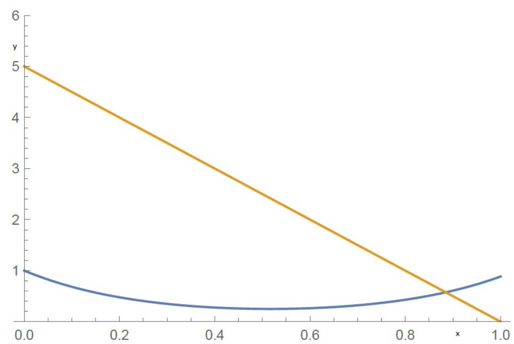

Figure 3.7: $y(x)$ with $c_1 = 0.25, m = -5, b = 5$

Notice as c_1 decreases so does y_1, as is expected based on the limit calculations. Thus the catenary shape becomes more evident as c_1 approaches zero. We would expect physically that the right endpoint would slide down toward zero but one would think it would be sudden versus the gradual tendency towards zero. One would also expect that the larger m with smaller b the faster y_1 would approach zero. The stability of the catenary is evident.

Chapter 4

Unconstrained length catenary problem with an unspecified left end, variable right end

We now consider the problem where y_0 is allowed to slide freely up and down the line $x = x_0$, as seen in figure (4.1) while the right end point, (x_1, y_1) is constrained to lie on the line $mx + b$.

Hence the problem is: Minimize

$$J[y(x)] = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx \tag{2.4}$$

subject to the boundary condition $y(x_1) = mx_1 + b$ and that y is unspecified at $x_0 = 0$.

From (2.6) we know that that $y = c_1 \cosh\left(\frac{x}{c_1} + c_2\right)$ is our local minimizing candidate. At this point the only constraint on our unknowns is that $c_1 \neq 0$ from (2.6).

Recall from section (2.3) that

$$F_{y'}|_{x=x_0} = \frac{y(x_0)y'(x_0)}{\sqrt{1 + (y'(x_0))^2}} = 0 \;\Rightarrow\; y'(x_0) = 0.$$

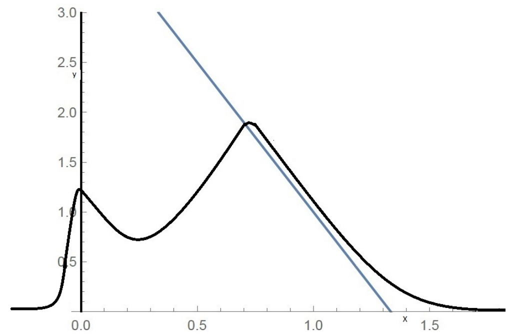

Figure 4.1: Unspecified left end, variable right end with infinite length

4.1 Consequences of the boundary conditions

For this problem we now have the boundary condition,

$$y(x_1) = mx_1 + b, \tag{4.1}$$

one natural boundary condition on the left hand side,

$$y'(0) = 0, \tag{4.2}$$

and one transversality condition,

$$y'(x_1) = -\frac{1}{m} \tag{4.3}$$

on the right hand side.

Immediately from the natural boundary condition (4.2) and properties of hyperbolic sine, we get

$$\sinh c_2 = 0 \quad \Rightarrow \quad c_2 = 0$$

Now from boundary condition (4.1) and the transversality condition (4.3) we can deduce,

$$c_1 \cosh\left(\frac{x_1}{c_1}\right) = mx_1 + b = y_1 \tag{4.4}$$

$$\sinh\left(\frac{x_1}{c_1}\right) = -\frac{1}{m} \tag{4.5}$$

or

$$\sinh^{-1}\left(-\frac{1}{m}\right) = \frac{x_1}{c_1}. \tag{4.6}$$

4.1.1 Existence of Solutions

Now if we divide equation (4.4) by (4.5) and divide both side of this single equation by c_1 we get,

$$\frac{c_1 \cosh \frac{x_1}{c_1}}{\sinh \frac{x_1}{c_1}} = y_1(-m)$$

or

$$\coth \frac{x_1}{c_1} = \frac{y_1(-m)}{c_1}. \tag{4.7}$$

Multiply both sides of (4.7) by $\frac{x_1}{x_1}$ yields,

$$\coth \frac{x_1}{c_1} = \frac{y_1(-m)x_1}{c_1 x_1}$$

Now call $\frac{x_1}{c_1} = u$ and $\frac{y_1(-m)}{x_1} = \alpha$ so we have,

$$\coth u = \alpha u \text{ or } \frac{\coth u}{u} = \alpha. \tag{4.8}$$

Using the analysis of (3.8) to determine conditions upon which we have solutions to the BVP, we can argue from (4.7) that we again get solutions to equation (4.8) when $\alpha > 0$.

4.1.2 Signs of parameters $x_1,\ c_1,\ y_1,\ m,\ b$

Thus

$$sgn(-my_1) = sgn(x_1) > 0$$

since $u > 0$, we get that $sgn(x_1) = sgn(c_1) > 0$. Thus $m < 0, y_1 > 0, x_1 > 0$ and $c_1 > 0$. Here note that we have

$$y_1 = mx_1 + b,$$

or

$$y_1 - b = mx_1 \leq 0,$$

thus

$$y_1 - b < 0,$$

therefore

$$b > 0,$$

or rather $b > -mx_1$.

4.2 Analysis

Now we obtain expressions for c_1 and x_1 which highlight their dependence on $m,\ b$. If we substitute (4.6) into (4.4) we get

$$c_1 = \frac{mx_1 + b}{\sqrt{1 + \frac{1}{m^2}}}$$

from the hyperbolic trigonometric identity used in (3.11) and

$$x_1 = \frac{b}{\frac{\sqrt{1+\frac{1}{m^2}}}{\sinh^{-1}\left(-\frac{1}{m}\right)} - m} = \frac{b\sinh^{-1}\left(-\frac{1}{m}\right)}{\sqrt{1 + \frac{1}{m^2}} - m\sinh^{-1}\left(-\frac{1}{m}\right)}.$$

Here if we look at the y_1 versus c_1 graph, figure (4.2), we can show there are two values for c_1 with the same resulting y_1. Note we get two values for c_1 for all y_1 expect at the minimum y_1 dependant on the m and b we get a unique solution. Also below the y_1 minimum you get no solutions.

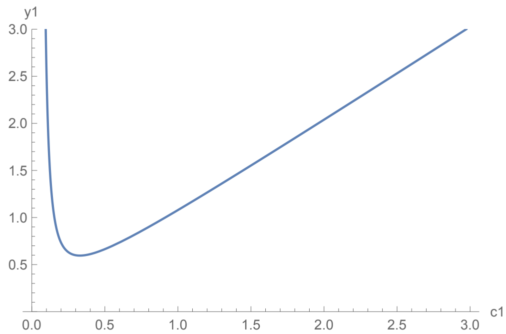

Figure 4.2: y_1 versus c_1 with $m = -5,\ b = 4$

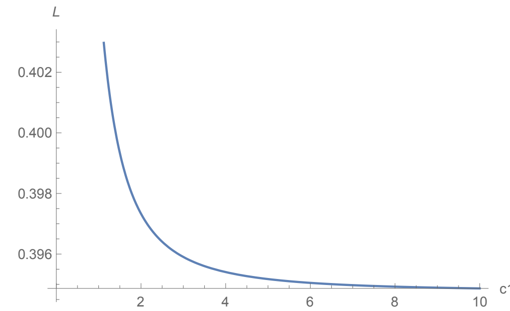

Figure 4.3: Length versus c_1, $m = -5,\ b = 4$

However we are looking for the catenary that minimizes (2.4) and we notice in figure (4.3) that the length is minimized as c gets larger so now our candidate for a minimizing function is

$$y(x) = c_1 \cosh \frac{x}{c_1},$$

where c_1 and x_1 are given by the formulas above.

Here we can explore the end behavior of $x_1, c_1, y_1,$ and the arc length are catenary produced by our minimizing candidate $y(x)$ as m approaches 0 from the left hand side and as m approaches $-\infty$ with a fixed b value.

As $m \to -\infty$, we are approaching the BVP where the left hand endpoint is allowed to move freely along the line $x = x_0$ and the right hand endpoint lies on a nearly vertical line.

Recall that,

$$\lim_{x \to -\infty} \frac{1}{x} = 0. \tag{4.9}$$

Thus,

$$\lim_{m\to-\infty} x_1 = \lim_{m\to-\infty} \frac{b\sinh^{-1}\left(-\frac{1}{m}\right)}{\sqrt{1+\frac{1}{m^2}} - m\sinh^{-1}\left(-\frac{1}{m}\right)} \approx \frac{b\sinh^{-1}0}{1} - \frac{b}{m} = 0.$$

Since we set $x_0 = 0$, as m gets large x_1 approaches x_0 i.e. the distance between x_0 and x_1 is shrinking towards 0. Thus both endpoints are tending towards being constrained on the line $x = x_0 = x_1$.

Next we show the end behavior of c_1. Note that,

$$\lim_{m\to-\infty} c_1 = \lim_{m\to-\infty} \frac{mx_1 + b}{\sqrt{1+\frac{1}{m^2}}},$$
$$= \frac{\lim_{m\to-\infty} mx_1 + \lim_{m\to-\infty} b}{\lim_{m\to-\infty} \sqrt{1+\frac{1}{m^2}}}.$$

Thus

$$\lim_{m\to-\infty} mx_1 = \lim_{m\to-\infty} \frac{mb\sinh^{-1}\left(-\frac{1}{m}\right)}{\sqrt{1+\frac{1}{m^2}} - m\sinh^{-1}\left(-\frac{1}{m}\right)}.$$

Applying L'Hopital's limit rule twice,

$$\lim_{m\to-\infty} mx_1 = \lim_{m\to-\infty} \frac{\frac{b}{(\frac{1}{m^2}+1)^{\frac{3}{2}} m^4}}{\frac{2m^2+2}{(\frac{1}{m^2}+1)^{\frac{3}{2}} m^6}}.$$

Multiplying by $\frac{m^4}{m^4}$,

$$= \lim_{m\to-\infty} \frac{\frac{-b}{(\frac{1}{m^2}+1)^{\frac{3}{2}}}}{\frac{2}{(\frac{1}{m^2}+1)^{\frac{3}{2}}} + \frac{2}{(\frac{1}{m^2}+1)^{\frac{3}{2}} m^2}} = \frac{-b}{2}.$$

Then

$$\lim_{m\to-\infty} c_1 = \frac{-b}{2} + b = \frac{b}{2}.$$

Now as m approaches $-\infty$, y_1 behaves as follows; using the definition found in equation (2.6) for y_1:

$$\begin{aligned}
\lim_{m\to-\infty} y_1 &= \lim_{m\to-\infty} c_1 \cosh \frac{x_1}{c_1} \\
&= \lim_{m\to-\infty} c_1 \cosh \frac{\lim_{m\to-\infty} x_1}{\lim_{m\to-\infty} c_1} \\
&= \frac{b}{2} \cosh 0 = \frac{b}{2}.
\end{aligned}$$

Finally, recall the length is determined by

$$L = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx$$

Thus for our function (2.6) we get,

$$\begin{aligned}
L &= \int_0^{x_1} \sqrt{1 + \sinh^2\left(\frac{x_1}{c_1}\right)} dx \\
&= \int_0^{x_1} \sqrt{\cosh^2\left(\frac{x_1}{c_1}\right)} dx \\
&= \int_0^{x_1} \cosh\left(\frac{x_1}{c_1}\right) dx
\end{aligned}$$

using the hyperbolic trigonometric identity, $\cosh^2 x - \sinh^2 x = 1$. Thus,

$$L = c_1 \sinh \frac{x_1}{c_1}$$

As m gets large,

$$\begin{aligned}
\lim_{m\to-\infty} L &= \lim_{m\to-\infty} c_1 \sinh \frac{x_1}{c_1}, \\
&= \lim_{m\to-\infty} c_1 \sinh \frac{\lim_{m\to-\infty} x_1}{\lim_{m\to-\infty} c_1}, \\
&= \frac{b}{2} \sinh 0 = 0,
\end{aligned}$$

since $x_1 \to 0$ and $c_1 \to \frac{b}{2}$.

From the form of our potential minimizer (2.6), we deduce that $y(0) = c_1 = y_0$ since we have shown that the length approaches 0 and $y_0,\ y_1$ both tend to $\frac{b}{2}$ as $m \to -\infty$, we can conclude that $y_1 - y_0 \to 0$.

Figure (4.4) depicts the catenary as $m \to -\infty$.

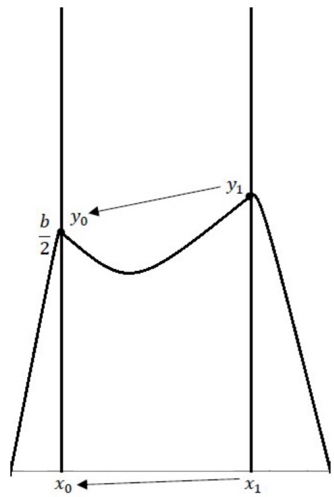

Figure 4.4: Visual as $m \to -\infty$

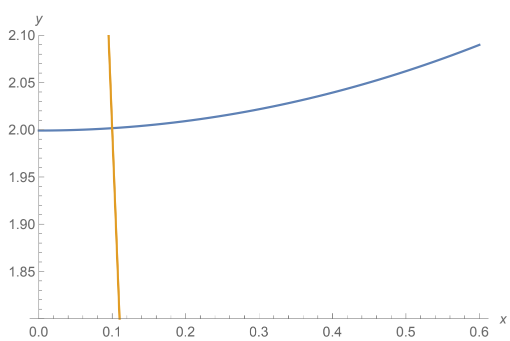

Figure 4.5: x_1 near 0 with $m = -20,\ b = 4$

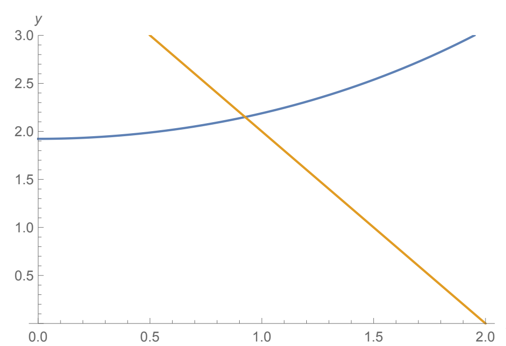

Figure 4.6: x_1 away from 0 with $m = -2,\ b = 4$

We can see in figures (4.4), (4.5), and (4.6) that $x_1 - x_0 \to 0$ as does $y_1 - y_0 \to 0$, as the poles are getting infinitely close to each other and consequently $y_1 \to y_0 = c_1 \to \frac{b}{2}$. We notice that the catenary shape is minimal as $m \to -\infty$. For the fixed length problem one would expect the chain to fold in on itself but with the variable length we are seeing that the chain is retracting causing the length to diminish.

Now we look at how x_1, c_1, y_1, L behave as $m \to 0$ from the left hand side. We first take the limit

$$\lim_{m\to 0^-} x_1 = \lim_{m\to 0^-} \frac{b\sinh^{-1}\left(-\frac{1}{m}\right)}{\sqrt{1+\frac{1}{m^2}} - m\sinh^{-1}\left(-\frac{1}{m}\right)}$$
$$= \lim_{m\to 0^-} \frac{b}{-m^2 - 1 + m^3\sinh^{-1}\left(-\frac{1}{m}\right)\sqrt{\frac{1}{m^2}+1}} = 0,$$

by L'Hopital's limit rule, since the derivative of a constant, b, is 0. Similarly as when m got large, as $m \to 0^-$, $x_1 \to x_0 = 0$.

Now for c_1,

$$\lim_{m\to 0^-} c_1 = \lim_{m\to 0^-} \frac{mx_1 + b}{\sqrt{1+\frac{1}{m^2}}} = 0,$$

since $\sqrt{1+\frac{1}{m^2}} \to \infty$ as $m \to 0^-$. Here we can also note that $y(0) = c_1 = y_0$ thus as m approaches 0 from the left, $y_0 \to 0$.

Next we show the behavior of y_1 as m approaches 0 from the left. Note we are using equation (4.1) for y_1.

Now,

$$\lim_{m\to 0^-} y_1 = \lim_{m\to 0^-} mx_1 + b = b.$$

Finally, as $m \to 0^-$,

$$\lim_{m\to 0^-} L = \lim_{m\to 0^-} c_1 \sinh\frac{x_1}{c_1}.$$

Using equation (4.5) we can rewrite the limit as,

$$\lim_{m\to 0^-} \frac{c_1}{-m}$$

Applying L'Hopital's rule to the indeterminate form $\frac{0}{0}$ we have

$$\lim_{m\to 0^-} -\frac{b\left(1+m^2-\sqrt{1+\frac{1}{m^2}}\,m^3\,\sinh^{-1}\left(\frac{1}{m}\right)\right)}{\sqrt{1+\frac{1}{m^2}}\,m^3\left(\sqrt{1+\frac{1}{m^2}}+m\,\sinh^{-1}\left(\frac{1}{m}\right)\right)^2} = b$$

Figures (4.7) and (4.8) support the above length limits.

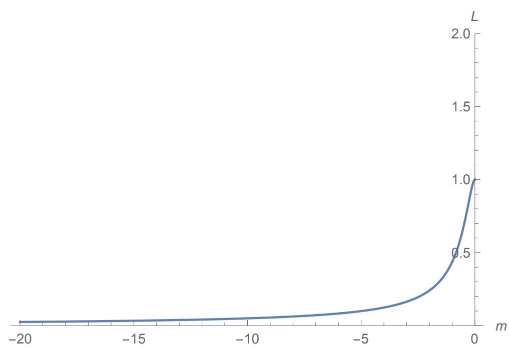

Figure 4.7: Length verses slope: b = 1

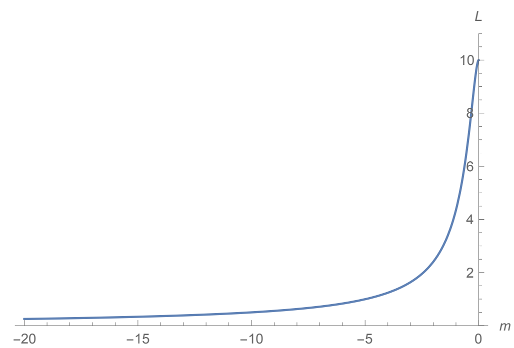

Figure 4.8: Length verses slope: b = 10

The extreme situation where m approaches 0 from the left is shown in figures (4.9).

We see that as the right hand pole becomes nearly horizontal and the right endpoint, y_1 slides towards b. Also notice that the left endpoint is sliding down towards the point, $(x_0, 0)$ or $(0, 0)$ in our case. Thus our catenary is approaching the vertical line, $x = 0$ on the interval $[y_0, y_1] = [0, b]$ for some fixed b as seen in figure (4.9).

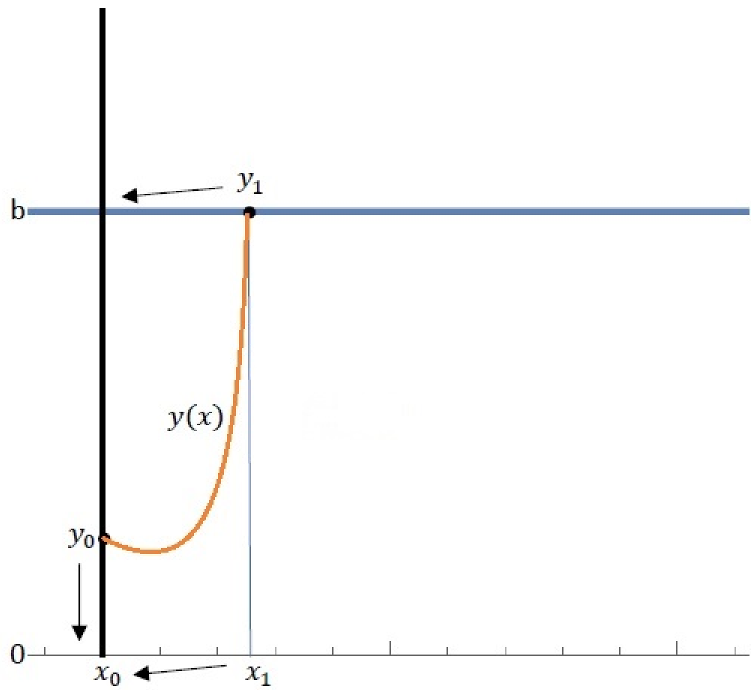

Figure 4.9: Plot at m $\approx 0^-$

4.3 Several graphs of $y(x)$

Here we graph multiple catenaries to display how $y(x)$ varies as m and b vary.

Figure (4.10) depicts our solution if we let $m = -1$ and $b = 1$. Then

$$x_1 = \frac{1 \sinh^{-1}\left(-\frac{1}{(-1)}\right)}{\sqrt{1+\frac{1}{(-1)^2}} - (-1)\sinh^{-1}\left(-\frac{1}{(-1)}\right)} = \frac{\sinh^{-1} 1}{\sqrt{2} + \sinh^{-1} 1} \approx 0.3841,$$

$$c_1 = \frac{mx_1 + b}{\sqrt{1+\frac{1}{m^2}}} = \frac{(1-)(0.3841)+1}{\sqrt{1+\frac{1}{(-1)^2}}} = \frac{1}{\sqrt{2}} - \frac{\sinh^{-1} 1}{2+\sqrt{2}\sinh^{-1} 1} \approx 0.4356,$$

$$y_1 = mx_1 + b = (-1)(0.3841) + 1 \approx 0.6161,$$

$$L = c_1 \sinh\frac{x_1}{c_1} = 0.4356 \sinh\frac{0.3841}{0.4356} \approx 0.3964.$$

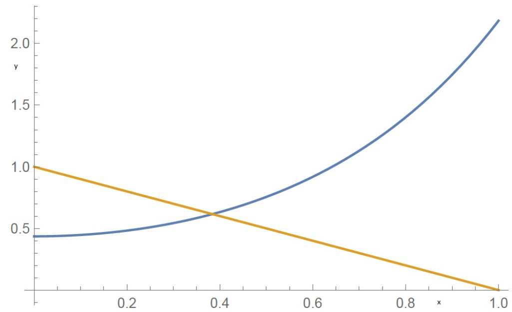

Figure 4.10: $y(x)$ *with* $m = -1$ *and* $b = 1$

Now let $m = -5$ and $b = 5$. Figure (4.11) shows the resulting catenary. With

$$x_1 = \frac{5\sinh^{-1}\left(-\frac{1}{(-5)}\right)}{\sqrt{1+\frac{1}{(-5)^2}} - (-1)\sinh^{-1}\left(-\frac{1}{(-5)}\right)} = \frac{5\sinh^{-1}\frac{1}{5}}{\sqrt{26}+5\sinh^{-1}\frac{1}{5}} \approx 0.1631,$$

$$c_1 = \frac{(-5)(0.1631)+5}{\sqrt{1+\frac{1}{(-5)^2}}} \approx 2.4835,$$

$$y_1 = mx_1 + b = (-5)(0.1631) + 5 \approx 4.1847,$$

$$L = c_1 \sinh\frac{x_1}{c_1} = 2.4835\sinh\frac{0.1631}{2.4835} \approx 0.1632.$$

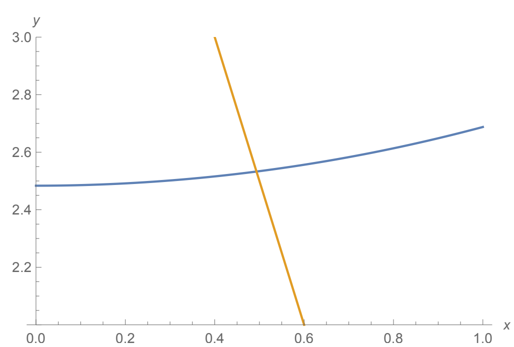

Figure 4.11: $y(x)$ *with* $m = -5$ *and* $b = 5$

Notice how as m decreases the catenary tends toward a straight line.

Lastly let $m = -1$ and $b = 5$. Figure (4.12) shows the resulting catenary. With

$$x_1 = \frac{5\sinh^{-1}\left(-\frac{1}{(-1)}\right)}{\sqrt{1+\frac{1}{(-1)^2}} - (-10)\sinh^{-1}\left(-\frac{1}{(-1)}\right)} = \frac{5\sinh^{-1} 1}{\sqrt{2}+\sinh^{-1} 1} \approx 0.4309,$$

$$c_1 = \frac{(-1)(0.4309)+5}{\sqrt{1+\frac{1}{(-1)^2}}} \approx 3.2309,$$

$$y_1 = mx_1 + b = (-1)(0.4309) + 5 \approx 4.5691,$$

$$L = c_1 \sinh\frac{x_1}{c_1} = 3.2309 \sinh\frac{0.4309}{3.2309} \approx 0.43214.$$

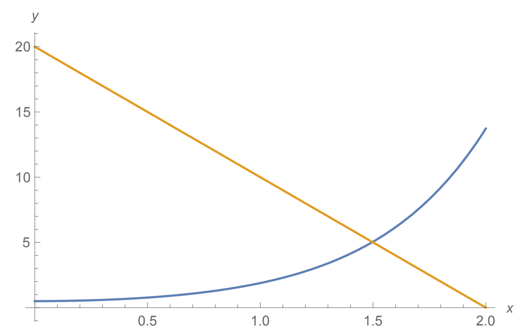

Figure 4.12: $y(x)$ *with* $m = -10$ *and* $b = 20$

The idea of having a left endpoint allowed to freely slide up and down may go against intuition. One would expect the left endpoint to immediately drop to zero but instead steadily approaches zero as m gets large. Also notice in figures (4.10), (4.11), and (4.12) that as b increases the curve of the catenary becomes more evident.

Chapter 5

Conclusion

We have thoroughly derived and analyzed minimizing candidates for two calculus of variation variable length problems.

First, if given boundary conditions of $y\big(0\big) = 1$, $y\big(x_1\big) = mx_1 + b = y_1$ and asked to minimize a functional say,

$$J[y(x)] = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx$$

we have a minimizing candidate of (2.6) with $c_1 > 0, c_2 < 0$ thus resulting in $y_1 > 0, x_1 > 0$, and $L > 0$ given when $m < 0$ and $b > 0$ for the right hand endpoint condition.

Then we considered the same functional subject to $y\big(x_1\big) = mx_1 + b$ where y is unspecified at $x_0 = 0$.We still have the same candidate, (2.6) except here we have $c_2 = 0, c_1 > 0$. This change results from the transversality conditions. To ensure that $L > 0$ we require that $m < 0$ and $b > 0$.

Chapter 3	**Chapter 4**
$m < 0$	$m < 0$
$b > -mx_1$	$b > -mx_1$
$0 < x_1 < \frac{-b}{m}$	$0 < x_1 < \frac{-b}{m}$
$0 < c_1 < 1$ and $c_1 < -m$	$c_1 > 0$
$c_2 < 0$	$c_2 = 0$
$0 < y_1 < \sqrt{1 + \frac{1}{m^2}}$	$y_1 > 0$
$0 < L < -\frac{1}{m}$	$0 < L < b$

Table 5.1: Ranges for parameters for Chapter 3 and Chapter 4 problems

5.1 Necessary and sufficient conditions for a local minimizer

The one question we did not answer in either problem is, is our minimizing candidate a local minimizer? The process of computing δJ, the first variation, to determine the Euler-Lagrange equation and its subsequent solution is analogous to computing the derivative of a function, setting it equal to 0 and finding critical points. Both techniques yield only potential extrema, meaning critical points and the Euler-Lagrange equation form the centerpiece of the necessary condition for a function or functional to have an extremum. We need something like the second derivative test from calculus to assert that the solution to the Euler-Lagrange equation produces a local minimum. We need sufficient conditions to classify a local extremum.

For $J[y] = \int_{x_0}^{x_1} f(x, y, y')\, dx$ subject to $y(x_0) = y_0, y(x_1) = y_1$, for any $h \in H, y \in S$, from section (2.1), one can define the second variation of J:

$$\delta^2 J[h, y] = \int_{x_0}^{x_1} (h^2 f_{yy} + 2hh' f_{yy'} + h'^2 f_{y'y'})dx.$$

Theorem 5.1.1 *Suppose that J has a local extremum in S at y. If y is a local minimum, then*

$$\delta^2 J[h, y] \geq 0, \ \forall\, h \in H \ [7].$$

This result is hard to use in practice. Instead we will use,

Theorem 5.1.2 *Let $J[y]$ be as in Theorem (5.1.1). Suppose J has a local minimum in S at y. Then*

$$f_{y'y'} \geq 0, \; \forall \, x \in [x_0, x_1].$$

Note the proof of theorem (5.1.2) uses theorem (5.1.1) and $\delta^2 J$.

For our catenary problem,

$$F(x, y, y') = y\sqrt{1 + y'^2},$$

then for any y

$$F_{y'y'} = \frac{y}{\sqrt{1 + y'^2}},$$

which is greater than or equal to zero when $y \geq 0$. For both problems we note that $y > 0$ so this condition is satisfied.

Then theorem (5.1.3) gives sufficient conditions for J to have a local minimum at an extremal, y.

Theorem 5.1.3 *Let F be a smooth function of x, y, y' and let y be a smooth extremal for J, as defined in theorem (5.1.1), such that $F_{y'y'} > 0, \; \forall \, x \in [x_0, x_1]$. If there are no points in the interval $(x_0, x_1]$ conjugate to x_0 then J has a local minimum at y [3].*

A Conjugate point refers to the existence of non trivial solutions, u, to the Jacobi accessory equation, a Ricatti second order linear differential equation, that satisfies $u(x_0) = u(k) = 0$ [8, 2]. For a BVP involving transversality conditions, the theory is quite complicated and not readily applicable.

5.2 Further research

Here we studied two BVP's in the calculus of variations with non-standard, variable boundary conditions and determined the shape of a hanging chain. Several extensions to this work are possible.

Mathematically, one could now analyze how the inclusion of a variable force changes the shape of the hanging cable; it should deviate from the hyperbolic cosine. With the addition of a force one could apply Theorem (5.1.3) to determine if the minimizer candidate is in fact a local minimum. Without the introduction of a force, one could study problems where the endpoints lie on nonlinear curves such as, $y = \ln x$ or $y = e^x$. In addition, one could try to verify the results of this work experimentally.

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