THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF MATHEMATICS

WIND-GENERATED SURFACE CURRENT AND WAVES AT AN AIR-WATER INTERFACE

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SPRING 2019

A thesis
submitted in partial fulfillment
of the requirements
for baccalaureate degree
in Mathematics
with honors in Mathematics

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Abstract

At an air-water interface, when wind begins to blow above still water, it sets up an interfacial current as well as motion in the water. We analyze a model for this situation and then look at its stability to perturbations, which correspond to the interfacial waves. This thesis considers a model for the wind and the resulting water motion; we then perturb it to try to understand how waves result at the interface. The first goal is to predict the surface current as a function of wind speed and compare these predictions with measurements taken in the wind-wave facility in the William G. Pritchard Laboratory in the Department of Mathematics at Penn State. The second goal is to predict the growth rate of a wave with a given wavelength and for a given wind speed.

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Acknowledgements

First and foremost, I would like to thank my thesis adviser, Dr. Diane Henderson, for providing me with her expertise, guidance, and assistance throughout the process of writing this thesis. I would not have been able to complete this paper without her guidance.

I would also like to thank Dr. Victoria Sadovskaya and the Women in Mathematics (WIM) for providing me with the Women in Mathematics Research Scholarship during the summer of 2018 to support my research project. Their contribution helped me a great deal to complete this thesis.

This work was supported in part by NSF-DMS-1716159.

Chapter 1

Introduction

Knowledge about wind-generated waves can prove to be incredibly important; for instance, wave forecasts are heavily sought by companies who have commercial interest in countries overseas. In particular, the shipping industry finds it crucial due to the fact that they require guidance for seakeeping needs. Providing a model for wind waves can help with measuring the state of the sea and determine whether sea travel is safe during certain conditions. A summary of models for and measurements of wind-generated waves is presented by Janssen, [4]. For these reasons, it proves to be useful to be able to model wind waves.

We consider a model for the set up of flow in in water by wind and the resulting waves at an air-water interface. Csanady, [2], discusses the base flow model that we will use in this thesis. Much earlier, Miles, [3], presented a different model that allows for turbulence; most models for generation of waves on wind are based on his model.

The model presented by Csanady, [2], is similar to Stokes' First Problem discussed by Kundu [5], in which there is an incompressible Navier-Stokes fluid at rest atop on infinitely long plate. Then, at $t = 0$, the plate moves with a constant speed, and thus, the fluid above it moves. In Csanady's problem, and in the one we use herein, we have two Navier-Stokes fluids at rest, separated by a flat interface. At $t = 0$, we allow the top fluid to move horizontally at a given speed with a uniform profile in the vertical. The water is initially at rest, but reacts to the air flow; thus, both the wind and the current in the water evolve in the vertical and in time.

Then, waves appear as perturbations to this base flow. The main differences between my base flow model and that of Miles [3] is that the base flow used herein is allowed to evolve in time and allowed to interact with the water to set up a current at the interface that then penetrates the water column.

An outline of the thesis is as follows. In Chapter 2, we present the full boundary-value problem for a viscous, two-fluid system. Chapter 3 uses Csanaday's model as our model for the wind and the resulting flow in the water. Comparisons of this models' predictions and Miles' model's predictions with measurements of the air speed and surface current obtained from experiments conducted in the William G. Pritchard Laboratory in the Department of Mathematics at Penn State are shown in Chapter 4. We allow for interfacial waves in Chapter 5, where we analyze the stability of the base flow. The result of this analysis is a linear system of 13 equations in 13 unknowns. We analyze this system in Chapter 6. Here we are not concerned with solving the system; instead, the goal is to determine the growth rate of waves of a given wavelength, so we analyze the properties of the coefficient matrix, which has this quantity embedded in it. In Chapter 7, we discuss the results. The main results are that the predictions of the surface current capture the qualitative, but not quantitative, behavior of the observations. Using our model, we are unable to find solutions for growth rates of waves. Therefore, we conclude that the model is missing some critical feature of the generation of waves. Determining that missing piece will be done in future work.

Chapter 2

The Complete Boundary Value Problem

Here, the fully nonlinear boundary value problem for total flow will be defined, including the governing equations and boundary conditions.

2.1 Governing equations

Consider a system of two fluids separated by an interface. We let position and velocity be given by $\mathbf{x} = \{x, z\}$ and $\mathbf{V}_j^T = \{U_j^T, W_j^T\}$ respectively. The superscript, T, represents the total flow, which will include a base flow (Chapter 3) and a perturbation to the base flow (Chapter 5). The governing equations for two-dimensional flow are the Navier-Stokes equations,

$$\frac{\partial U_j^T}{\partial t} + U_j^T \frac{\partial U_j^T}{\partial x} + W_j^T \frac{\partial U_j^T}{\partial z} = -\frac{1}{\rho_j}\frac{\partial P_j^T}{\partial x} + \nu_j\Big(\frac{\partial^2 U_j^T}{\partial x^2} + \frac{\partial^2 U_j^T}{\partial z^2}\Big), \tag{2.1a}$$

$$\frac{\partial W_j^T}{\partial t} + U_j^T \frac{\partial W_j^T}{\partial x} + W_j^T \frac{\partial W_j^T}{\partial z} = -\frac{1}{\rho_j}\frac{\partial P_j^T}{\partial z} + \nu_j\Big(\frac{\partial^2 U_j^T}{\partial x^2} + \frac{\partial^2 U_j^T}{\partial z^2}\Big) - g. \tag{2.1b}$$

where $j = 1$ represents the flow in the air for which $\eta^T < z < \infty$ and $x \in \mathbb{R}$, and $j = 2$ represents the flow in the water for which $-\infty < z < \eta^T$ and $x \in \mathbb{R}$, where $z = \eta^T(x, t)$ locates the interface. Equation (2.1a) expresses the conservation of the horizontal component of momentum and equation (2.1b) expresses the conservation of the vertical component of momentum. The left-hand side of the equations represents inertia in the flow and the right-hand side represents the forces from gradients in the pressure, $P_j^T(x, z, t)$, and from friction within the flow, where g is gravity, ν_j is a measure of the kinematic viscosity of the jth fluid and ρ_j is the density of the jth fluid. These properties of air and water at 22° C are given in Table 2.1.

j	ρ_j (kg/m^3)	ν_j (cm^2/s)
1	1.196	1.5×10^{-5}
2	997.76	9.55×10^{-7}

Table 2.1: Fluid properties used for the base flow.

We consider flows slow enough that incompressibility is an adequate approximate, so that the flow is divergence-free. Thus, by conservation of mass,

$$\frac{\partial U_j^T}{\partial x} + \frac{\partial W_j^T}{\partial z} = 0. \tag{2.2}$$

in the domains defined for $j = 1$ and $j = 2$. A schematic of the problem is shown in Figure 2.1.

2.2 Boundary conditions at the interface

There are five boundary conditions at the material boundary, $z = \eta^T(x, t)$. They are as follows. The location of the interface is given by $z = \eta^T(x, t)$. Thus, the total derivative of $z - \eta(x, t)$ is zero and provides the kinematic boundary condition,

$$\frac{\partial \eta^T}{\partial t} = W_j^T - U_j^T \frac{\partial \eta^T}{\partial x}. \tag{2.3}$$

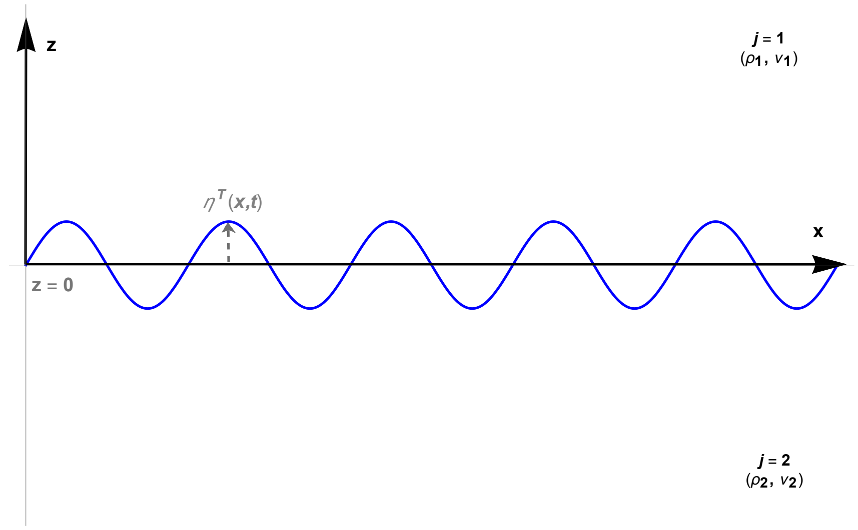

Figure 2.1: Schematic of the problem with height $z > 0$ representing air, $z < 0$ representing water, and $z = 0$ the interface. The values for ρ_j and ν_j can be found in Table 2.1.

Since this condition is satisfied in the limit of $z \to \eta^T(x,t)$ from above and below, it suffices to use either $j = 1$ or $j = 2$; I have used $j = 1$. Then, the statement that the condition has to be satisfied for both fluids gives another boundary condition, which is the continuity of normal velocities,

$$\frac{W_1^T - U_1^T \frac{\partial \eta^T}{\partial x}}{\sqrt{1 + \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2}} = \frac{W_2^T - U_2^T \frac{\partial \eta^T}{\partial x}}{\sqrt{1 + \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2}}. \tag{2.4}$$

Following Batchelor, [1] (pages), we require continuity of tangential velocities in order to avoid infinite shear stresses. Therefore,

$$\frac{U_1^T + W_1^T \frac{\partial \eta^T}{\partial x}}{\sqrt{1 + \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2}} = \frac{U_2^T + W_2^T \frac{\partial \eta^T}{\partial x}}{\sqrt{1 + \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2}}. \tag{2.5}$$

Continuity of normal stress is

$$\begin{aligned} P_1^T - \frac{2\rho_1\nu_1}{\left[1 + \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2\right]} \left[\frac{\partial W_1^T}{\partial z}\left(1 - \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2\right) - \frac{\partial \eta^T}{\partial x}\left(\frac{\partial U_1^T}{\partial z} + \frac{\partial W_1^T}{\partial z}\right)\right] &= \\ P_2^T - \frac{2\rho_2\nu_2}{\left[1 + \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2\right]} \left[\frac{\partial W_2^T}{\partial z}\left(1 - \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2\right) - \frac{\partial \eta^T}{\partial x}\left(\frac{\partial U_2^T}{\partial z} + \frac{\partial W_2^T}{\partial z}\right)\right]&. \end{aligned} \tag{2.6}$$

Continuity of tangential stress is

$$\begin{aligned} \rho_1\nu_1\left[4\frac{\partial \eta^T}{\partial x}\frac{\partial W_1^T}{\partial z} + \left(1 - \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2\right)\left(\frac{\partial U_1^T}{\partial z} + \frac{\partial W_1^T}{\partial x}\right)\right] &= \\ \rho_2\nu_2\left[4\frac{\partial \eta^T}{\partial x}\frac{\partial W_2^T}{\partial z} + \left(1 - \left(\frac{\partial^2 \eta^T}{\partial x^2}\right)^2\right)\left(\frac{\partial U_2^T}{\partial z} + \frac{\partial W_2^T}{\partial x}\right)\right]&. \end{aligned} \tag{2.7}$$

2.3 Boundary conditions as $z \to \pm\infty$

We require that the perturbation motions and pressure decay at infinity. Thus, as $z \to \pm\infty$, the motions are given by the base flow. This problem statement is generic to any two-fluid system. In the remainder of this thesis, we consider the situation of air, $j = 1$, over water, $j = 2$.

Chapter 3

Base Flow Model for Wind

In this chapter, we develop a model for the base flow of the wind in the air, flow within the water, and the resulting current at the interface. For the base flow, we do not allow movement at the interface; thus, in this chapter, $\eta^T = 0$.

3.1 Governing equations

Let $\{U_j, W_j, P_j\}$ be the horizontal velocity, the vertical velocity, and the pressure of the base-flow, respectively, where velocity is represented as $\mathbf{V}_j = \{U_j, W_j\}$. Here, we look for a base flow in the air, ($j = 1, 0 < z < \infty, x \in \mathbb{R}$), and water, ($j = 2, -\infty < z < 0, x \in \mathbb{R}$), where the location of the flat interface between them is at $z = 0$. For $t \leq 0$, both fluids are at rest. Then, at $t > 0$, the top fluid begins to flow only horizontally. Thus, the vertical component of velocity $W_j = 0$. Then, from the incompressibility condition, (2.2), it follows that $\frac{\partial U_j}{\partial x} = 0$. Thus, we can conclude that U_j is independent of x. Therefore, $\mathbf{V}_j = \{U_j(z,t), 0\}$. It follows that the Navier-Stokes equations, (2.1), reduce to

$$\frac{\partial U_j}{\partial t} = -\frac{1}{\rho_j}\frac{\partial P_j}{\partial x} + \nu_j \frac{\partial^2 U_j}{\partial z^2}, \tag{3.1a}$$

$$0 = -\frac{1}{\rho_j}\frac{\partial P_j}{\partial z} - g. \tag{3.1b}$$

From (3.1a), we see that P_j could depend linearly on x, since U_j depends only on z and t. However, we require that P_j be bounded in x; thus, it is independent of x. Then, P_j depends on z as required by (3.1b), so that the pressure is hydrostatic and becomes

$$P_j(z) = -\rho_j g z + P_j^0, \tag{3.2}$$

where P_j^0 is a reference pressure in the limit of approaching the interface from the jth side and lacking any temporal forcing have set the arbitrary time dependence to zero without loss of generality.

Since U_j only depends on z and t, equation (3.1a) cannot contain any x terms. Therefore, it follows that the $\frac{\partial P_j}{\partial z}$ term from equation (3.1a) must equal zero and P_j only depends on z. There also cannot be an x term of the 1st degree because as x increases linearly, the pressure, P_j will continue to increase without bound, which is unphysical. Therefore, the horizontal linear Navier-Stokes equation, (3.1a), reduces further to the heat equation

$$\frac{\partial U_j}{\partial t} = \nu_j \frac{\partial^2 U_j}{\partial z^2}. \tag{3.3}$$

3.2 Boundary conditions

Before we can find the solutions for U_1 and U_2, we have to establish the boundary conditions. We have boundary conditions at ∞, $-\infty$, and at the interface, $z = 0$. In air, we expect the wind will approach a constant value as $z \to \infty$, and we expect the water to have no flow as $z \to -\infty$. So, the following boundary conditions must be fulfilled:

$$U_1 \to U^\infty \text{ as } z \to \infty \tag{3.4a}$$

$$U_2 \to 0 \text{ as } z \to -\infty, \tag{3.4b}$$

where U^{∞} is the speed of the uniform wind.

Then, the boundary conditions at $z = 0$ (for the base flow, $\eta^T(x,t) = 0$) reduce to the following:

- The kinematic condition provides confirmation that the vertical velocity is zero for $j = 1, 2$. So,
$$W_1 = W_2 \equiv 0 \tag{3.5}$$

- Since the interface is flat, the normal velocities are W_j, which have to be continuous across the interface. So again,
$$W_1 = W_2 \equiv 0. \tag{3.6}$$

- Since the interface is flat, the tangential velocities are U_j, which have to be continuous across the surface. So,
$$U_1(0,t) = U_2(0,t) \equiv U_0, \tag{3.7}$$
where U_0 denotes the speed of the surface current and is one quantity we seek to predict.

- Continuity of tangential stress condition reduces to
$$\rho_1 \nu_1 \frac{\partial U_1}{\partial z} = \rho_2 \nu_2 \frac{\partial U_2}{\partial z}. \tag{3.8}$$

- Continuity of normal stress gives the reference pressure at the interface, P_0,
$$P_1(0) = P_1^0 = P_2(0) = P_2^0 \equiv P_0. \tag{3.9}$$

3.3 Solution

We find the solutions, $U_1(z,t)$ and $U_2(z,t)$, using the similarity variables, $\xi_j = \frac{z}{\sqrt{4\nu_j t}}$ where $j = 1, 2$. We look for solutions of the forms

$$U_1(z,t) = f(\xi_1), \tag{3.10a}$$
$$U_2(z,t) = h(\xi_2). \tag{3.10b}$$

It follows that the heat equations, (3.3), for $j = 1, 2$ become

$$f''(\xi_1) + 2\xi_1 f'(\xi_1) = 0, \tag{3.11a}$$
$$h''(\xi_2) + 2\xi_2 h'(\xi_2) = 0. \tag{3.11b}$$

The solutions of (3.11) that satisfies the boundary conditions from section 3.2 are

$$U_1(z,t) = C_0 + \frac{\sqrt{\pi}}{2} C_1 \operatorname{Erf}\Big(\frac{z}{\sqrt{4\nu_1 t}}\Big)$$
$$= C_0 + C_1 \int_0^{\frac{z}{\sqrt{4\nu_1 t}}} e^{-s^2} ds \qquad \text{for } \ 0 < z < \infty,\ t \geq 0, \tag{3.12a}$$
$$U_2(z,t) = C_0 - \frac{\sqrt{\pi}}{2} C_2 \operatorname{Erf}\Big(\frac{z}{\sqrt{4\nu_2 t}}\Big)$$
$$= C_0 + C_2 \int_{\frac{z}{\sqrt{4\nu_2 t}}}^{0} e^{-s^2} ds \qquad \text{for } -\infty < z < 0,\ t \geq 0, \tag{3.12b}$$
$$W_j(z,t) = 0 \qquad \text{for } -\infty < z < \infty,\ t \geq 0, \tag{3.12c}$$
$$P_j(z) = -\rho_j g z + P_0 \qquad \text{for } -\infty < z < \infty,\ t \geq 0. \tag{3.12d}$$

Let $R = \frac{\rho_1}{\rho_2}$ and $V = \frac{\nu_1}{\nu_2}$. Then, the constants C_0, C_1, and C_2 in equations (3.12a)-(3.12d) are

$$C_0 = U^{\infty}\Big(\frac{R\sqrt{V}}{1+R\sqrt{V}}\Big), \tag{3.13a}$$
$$C_1 = \frac{2}{\sqrt{\pi}} U^{\infty}\Big(\frac{1}{1+R\sqrt{V}}\Big), \tag{3.13b}$$
$$C_2 = -\frac{2}{\sqrt{\pi}} U^{\infty}\Big(\frac{R\sqrt{V}}{1+R\sqrt{V}}\Big). \tag{3.13c}$$

The surface current is represented by the constant C_0, which we will denote as U_0. So,

$$U_0 = U_1(0,t) = U_2(0,t) = C_0. \tag{3.14}$$

Furthermore, the surface current is

$$U_0 = U^{\infty}\Big(\frac{R\sqrt{V}}{1+R\sqrt{V}}\Big), \tag{3.15}$$

so that it depends on the fluid properties and the wind speed at infinity, but is fixed in time.

Figure 3.1 shows the graphical depiction of our solution for the base flow, given in (3.12). The different colored lines on the graph depict the various times tested for the equations. Here, we observe that as z grows positively, the horizontal speed, $U_1(z,t)$, increases fairly steadily until it reaches the uniform wind speed, U^{∞}, while as z grows negatively, the $U_2(z,t)$ drops almost immediately to 0, as desired. When $z = 0$, both U_1 and U_2 meet at U_0.

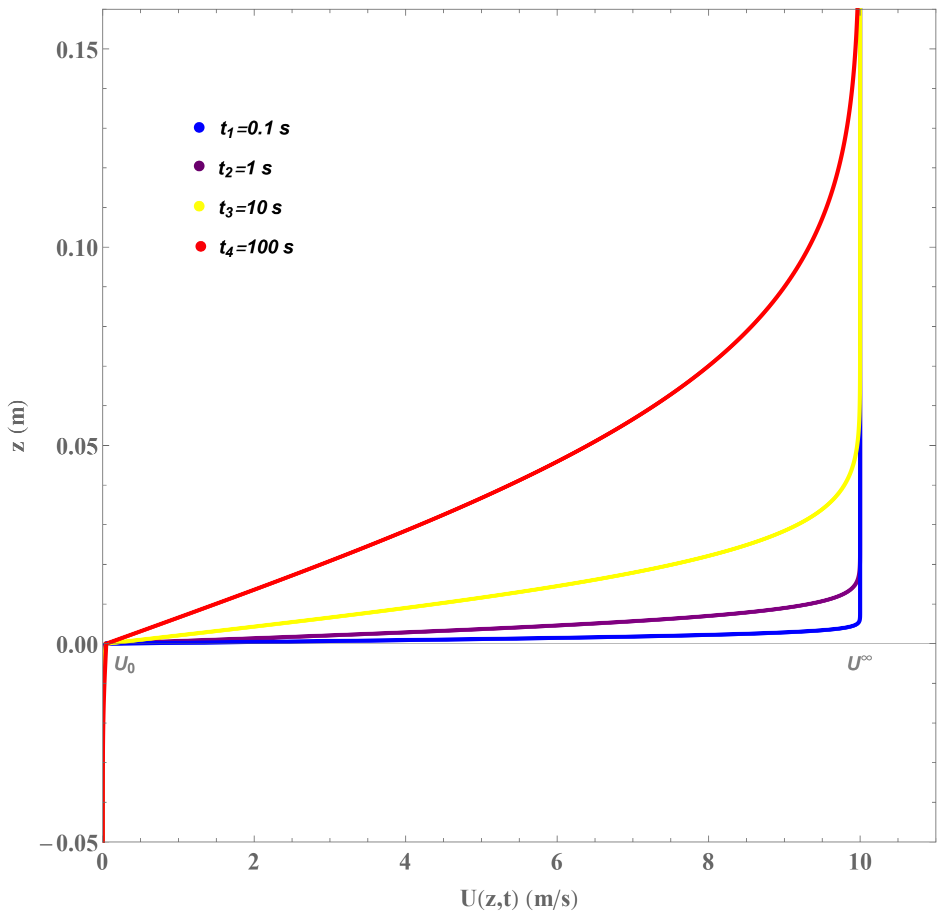

Figure 3.1: Predicted horizontal speed in air ($z > 0$), water ($z < 0$), and at the interface ($z = 0$) when $U^\infty = 10$ m/s.

3.4 Approximate Solution for the Base Flow

The velocities of the base flow depend both on z and t. However, we are interested in the perturbations of this flow for early times, just after the wind has started to blow. So, for these early times, we will approximate the base flow by considering a steady flow that has time, t, as a parameter. Figure 3.1 suggests that at early times, the base flow is linear in z between the interface and the critical heights, $z = z_1$ in the air and $z = z_2$ in the water. Outside of these values, the wind ($z > z_1$) has roughly a uniform in z profile, while the water ($z < z_2$) has zero velocity. Thus, the boundary conditions become

$$\begin{aligned} U_1(z_1) &= U^\infty \\ U_1(0) &= U_0 \\ U_2(0) &= U_0 \\ U_2(z_2) &= 0 \end{aligned} \tag{3.16}$$

where $z_1 > 0$ and $z_2 < 0$ are the viscous boundary layer depths which will be chosen or given by measurement. The boundary conditions on continuity of velocities and stresses still hold, so we require continuity of tangential velocity, (3.7),

$$U_1(0) = U_2(0) \equiv U_0, \tag{3.17}$$

and continuity of tangential stress, (3.8),

$$\rho_1 \nu_1 \frac{\partial U_1}{\partial z} = \rho_2 \nu_2 \frac{\partial U_2}{\partial z}. \tag{3.18}$$

Since we are saying that U_j is independent of time, (3.3) becomes

$$\frac{\partial^2 U_j}{\partial z^2} = 0 \tag{3.19}$$

and it follows that the solutions of these differential equations are

$$U_1(z) = C_1 \, z + U_0, \tag{3.20a}$$

$$U_2(z) = C_2 \, z + U_0. \tag{3.20b}$$

Therefore, the constants C_1 and C_2 for the approximate solution become

$$C_1 = \frac{U^\infty - U_0}{z_1}, \tag{3.21a}$$

$$C_2 = \frac{RV}{z_1}(U^\infty - U_0), \tag{3.21b}$$

where z_1, U^∞, and U_0 are given values. Then, by setting $U_2(z_2) = 0$, we obtain

$$z_2 = -\frac{U_0 \, z_1}{RV(U^\infty - U_0)}. \tag{3.22}$$

Chapter 4

Comparison of Predicted Surface Current with Measurements

In this chapter, I take the predicted surface current model for base flow, $U_0 = U^{\infty}\left(\frac{R\sqrt{V}}{1+R\sqrt{V}}\right)$, and compare it to results from experiments conducted in a wind-wave tank to test the accuracy of the model.

4.1 Graphical Depiction of Base Flow Model for Surface Current

From the previous sections, recall that ρ_j is the density and ν_j is the kinematic viscosity of the jth fluid. ρ_j and ν_j of air and water are given in Table 2.1. Then, the density and viscosity ratios are $R \approx .0012$ and $V \approx 15$. Using these values in (3.15), we obtain the following prediction for surface current speed,

$$U_0 = 0.0047U^{\infty}. \tag{4.1}$$

Equation (4.1) represents our prediction for the speed of the surface current, U_0, using the speed of the wind assuming absence of water waves, U^{∞}. The surface current depends linearly on the wind speed, and once the two fluids are chosen, the constant of proportionality between surface current and wind-speed is determined.

4.2 Experiments

I set up an experiment to see if the predicted model for U_0 from Section 4 is accurate. I took a small piece of circular, flat plastic, colored it black, and used it as a tracker to measure how long it took for it to travel on the water surface between checkpoints in a large tank of water. The wind was generated using a fan and allowed to blow over the water, creating a surface current. The tracker was then added to the water and a stopwatch was used to measure the time it took for it to travel between checkpoints. I did this three times each for four different speeds and averaged the times between each checkpoint in order to minimize error. Figure 4.1 depicts the results of these trials.

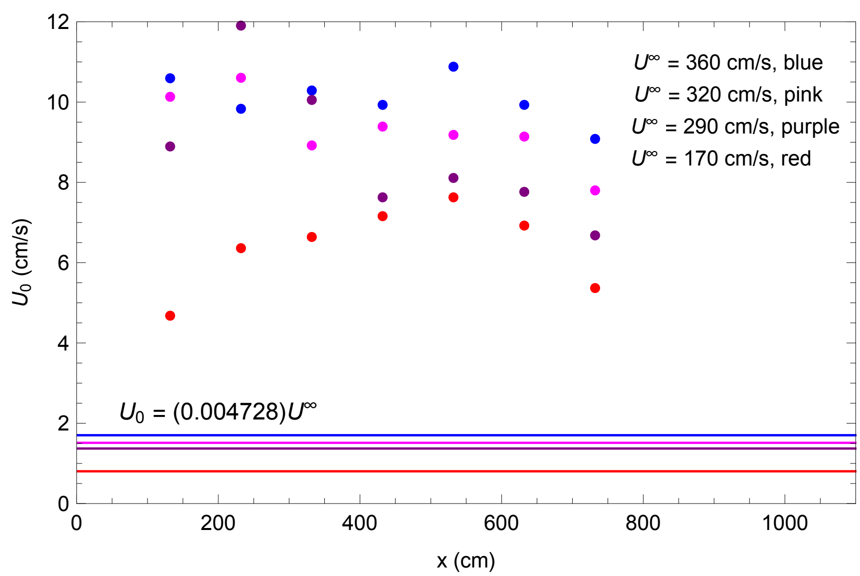

Figure 4.1: Speed of surface current relative to the tracker’s distance from the fan

In Figure 4.1, the colored lines on the bottom of the graph represent the predicted values of U_0 for the various U^{∞} used in the trials. It is clear that in the trials, there was a lot of discrepancy in the first few checkpoints, though this begins to steady by the last few checkpoints. I observed that the time between each checkpoint is not always the same, in contrast to predictions.

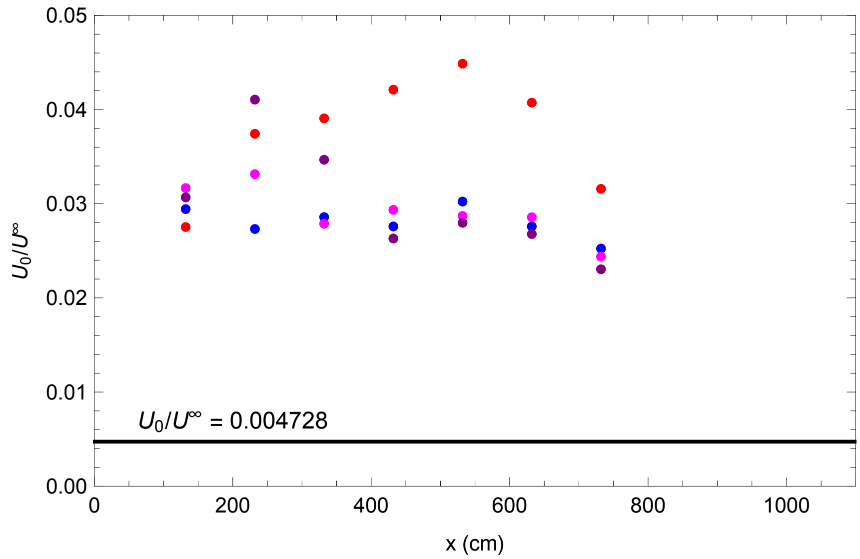

Figure 4.2: Ratio of surface current and wind speed relative to distance from the wind machine

Figure 4.2 represents a graph of $\frac{U_0}{U^{\infty}}$ vs. x. Equation (4.1) predicts that this ratio, $\frac{U_0}{U^{\infty}} = 0.004728$. The solid black line on the bottom of the graph represents this predicted value.

As can be seen in Figure 4.2, the experimental data is also qualitatively horizontal. From both Figure 4.1 and Figure 4.2, it seems as though the model does not work as well for slower speeds as most of the outliers in both graphs are from the trial where $U^{\infty} = 170$ cm/s.

Various explanations for the discrepancies are that the model does not allow for observed turbulent fluctuations in the wind, for observed two-dimensionality of the wind and water surface, and for the waves that had already grown when the measurements were obtained. In future work, the effects of these processes can be investigated.

Chapter 5

Model for Wind Waves

In this chapter, we examine the stability of the base flow to perturbations that are periodic in space and that are allowed to both oscillate and grow/decay in time. There is no constraint on the size of the perturbation wave length.

5.1 The Perturbed Flow Field

Here, we look for waves through perturbation of the base flow. This can be done through Fourier decomposition. Let the phase of the perturbations be

$$\theta = kx - \omega t, \tag{5.1}$$

where $k \in \mathbb{R}$ represents the wave number and $\omega \in \mathbb{C}$ is the frequency. The real part corresponds to the oscillation frequency and the imaginary part corresponds to the growth rate. In the remainder of this thesis, we will fix a value of k and attempt to solve for the corresponding omega to determine the growth rate of waves and the effect of the wind on their frequencies. The total velocity and pressure fields with the perturbations are

$$U_j^T(x,z,t) \;=\; U_j(z) + \epsilon(u_j(z;k)e^{i\theta} + u_j^*(z;k)e^{-i\theta}) + O(\epsilon^2), \tag{5.2a}$$

$$W_j^T(x,z,t) \;=\; 0 + \epsilon(w_j(z;k)e^{i\theta} + w_j^*(z;k)e^{-i\theta}) + O(\epsilon^2), \tag{5.2b}$$

$$P_j^T(x,z,t) \;=\; P_j(z) + \epsilon(p_j(z;k)e^{i\theta} + p_j^*(z;k)e^{-i\theta}) + O(\epsilon^2), \tag{5.2c}$$

$$\eta_j^T(x,t) = 0 + \epsilon(\eta_j(k)e^{i\theta} + \eta_j^*(k)e^{-i\theta}) + O(\epsilon^2), \tag{5.2d}$$

where $j = 1, 2$ and $\epsilon \ll 1$ is a small parameter.

5.2 The Governing Equations

We substitute (5.2) into the governing Navier-Stokes equations, (2.1). We find that the $O(1)$ terms cancel because they correspond to the base flow terms, which exactly satisfy the equations. We are then interested in just the $O(\epsilon)$ terms, so we have

$$e^{i\theta}\Big[-i\omega\, u_j + ik\, u_j\, U_j(z) + C_j\, w_j + \frac{ik}{\rho_j}\, p_j + \nu_j(k^2 u_j - u_j'')\Big] +$$
$$e^{-i\theta}\Big[i\omega\, u_j^* - ik\, u_j^*\, U_j(z) + C_j\, w_j^* - \frac{ik}{\rho_j}\, p_j^* + \nu_j(k^2 u_j^* - u_j''^*)\Big] = O(\epsilon), \tag{5.3a}$$

$$e^{i\theta}\Big[-i\omega\, w_j + ik\, w_j\, U_j(z) + \frac{1}{\rho_j} p_j' + \nu_j(k^2 w_j - w_j'')\Big] +$$
$$e^{-i\theta}\Big[i\omega\, w_j^* - ik\, w_j^*\, U_j(z) - \frac{1}{\rho_j} p_j'^* + \nu_j(k^2 w_j^* - w_j''^*)\Big] = O(\epsilon). \tag{5.3b}$$

Similarly, we substitute (5.2) into the equation for conservation of mass, (2.2), and by gathering the $O(\epsilon)$ terms, we obtain

$$e^{i\theta}\Big[ik\, u_j + w_j'\Big] + e^{-i\theta}\Big[-ik\, u_j^* + w_j'^*\Big] = 0. \tag{5.4}$$

The coefficients in front of the $e^{i\theta}$ and $e^{-i\theta}$ give the same information; therefore, it is sufficient to look at the coefficient of $e^{i\theta}$ only. Thus, (5.3)-(5.4) tell us

$$-i\omega\, u_j + ik\, u_j\, U_j(z) + c_j\, w_j + \frac{ik}{\rho_j}\, p_j + \nu_j(k^2 u_j - u_j'') = 0, \tag{5.5a}$$

$$-i\omega\, w_j + ik\, w_j\, U_j(z) + \frac{1}{\rho_j} p_j' + \nu_j(k^2 w_j - w_j'') = 0, \tag{5.5b}$$

and

$$ik\, u_j + w_j' = 0. \tag{5.6}$$

Equation (5.6) can be rewritten as

$$u_j = \frac{i}{k} w_j', \tag{5.7}$$

and allow us to replace the perturbation horizontal velocities by (derivatives of) the vertical velocities.

5.3 Boundary Conditions for the Perturbed Problem

The boundary conditions at $z = 0$ for the perturbed flow are as follows:

- The kinematic boundary condition becomes

$$w_1(0) = i\eta(k)(kU_0 - \omega). \tag{5.8}$$

- The continuity of normal velocities becomes

$$w_1(0) = w_2(0). \tag{5.9}$$

- The continuity of tangential velocities becomes

$$u_1(0) = u_2(0),$$

  which from (5.7) becomes

$$w_1'(0) = w_2'(0). \tag{5.10}$$

- The continuity of tangential stress becomes

$$\rho_1\nu_1[w_1''(0) + k^2 w_1(0)] = \rho_2\nu_2[w_2''(0) + k^2 w_2(0)]. \tag{5.11}$$

- The continuity of normal stress becomes

$$p_1(0) - 2\rho_1\nu_1[w_1'(0) - ikC_1\eta(k)] = p_2(0) - 2\rho_2\nu_2[w_2'(0) - ikC_2\eta(k)]. \tag{5.12}$$

Additionally, at $z = z_1$,

$$p_1(z_1) = 0, \tag{5.13a}$$

$$w_1(z_1) = 0, \tag{5.13b}$$

$$u_1(z_1) = w_1'(z_1) = 0, \tag{5.13c}$$

and at $z = z_2$,

$$p_2(z_2) = 0, \tag{5.14a}$$

$$w_2(z_2) = 0, \tag{5.14b}$$

$$u_2(z_2) = w_2'(z_2) = 0. \tag{5.14c}$$

5.4 Equations for Pressure and Velocity

Taking the derivative with respect to x of (5.5a) and the derivative with respect to z of (5.5b) as well as the fact that (5.7), we add the two partial derivatives together and we get

$$p_j'' - k^2 p_j = 2ik\rho_j C_j\, w_j, \tag{5.15}$$

which is the differential equation for pressure in air and water. In addition, the right hand-side contains the wind-forcing term, C_j, as well as the vertical velocity term from the perturbation, w_j.

Now, subtracting the x-derivative of (5.5b) from the z-derivative of (5.5a) and substituting (5.7), we get

$$-\nu_j(u_j''' - ik\, w_j'') + (-i\omega + ikU_j(z) + \nu_j k^2)(u_j' - ik\, w_j) = 0. \tag{5.16}$$

Using the equation for vorticity,

$$\Omega_j = u_j' - ik\, w_j, \tag{5.17}$$

(5.16) becomes

$$\Omega_j'' - (a_j z + b_j)\, \Omega_j = 0, \tag{5.18}$$

where

$$a_j = \frac{ik\, C_j}{\nu_j}, \tag{5.19a}$$

$$b_j = \frac{ik\, U_0}{\nu_j} - \frac{i\omega}{\nu_j} + k^2. \tag{5.19b}$$

Let

$$\xi_j(z) = a_j\, z + b_j, \tag{5.20}$$

$$\zeta_j(\xi_j) = \frac{1}{a_j^{2/3}}\,\xi_j, \tag{5.21}$$

$$\Omega_j(z) = f(\zeta_j). \tag{5.22}$$

Therefore, (5.18) becomes

$$f''(\zeta_j) - \zeta_j\, f(\zeta_j) = 0, \tag{5.23}$$

which is in the form of the Airy equation. The solution to this equation is

$$f(\zeta_j) = D_j\,\mathrm{Ai}(\zeta_j) + E_j\,\mathrm{Bi}(\zeta_j). \tag{5.24}$$

where Ai is the Airy function of the first kind and Bi is the Airy function of the second kind.

Therefore, using the fact that

$$\Omega_j(z) = D_j\,\mathrm{Ai}(\zeta_j) + E_j\,\mathrm{Bi}(\zeta_j),$$
$$-\,ik\,\Omega_j(z) = w_j'' - k^2 w_j,$$

are both true, we obtain a differential equation for the perturbed vertical velocity, w_j,

$$w_j'' - k^2 w_j = -ik[\,(D_j\,\mathrm{Ai}(\zeta_j) + E_j\,\mathrm{Bi}(\zeta_j)\,]. \tag{5.26}$$

5.5 Solutions to Pressure and Vertical Velocity

Using the method of variation of parameters to solve differential equations, we obtain the solutions, $p_1(z)$ and $p_2(z)$, of (5.15) for air and water, which are

$$\begin{aligned} p_1(z) = F_1 e^{kz} + G_1 e^{-kz} + i\rho_1 C_1 e^{kz}\Big(\int_0^z e^{-ks} w_1(s)\,ds + \int_z^{z_1} e^{-ks} w_1(s)\,ds\Big) \\ -\,i\rho_1 C_1 e^{-kz}\Big(\int_0^z e^{ks} w_1(s)\,ds + \int_z^{z_1} e^{ks} w_1(s)\,ds\Big) \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} p_2(z) = F_2 e^{kz} + G_2 e^{-kz} + i\rho_2 C_2 e^{kz}\Big(\int_{z_2}^z e^{-ks} w_2(s)\,ds + \int_z^{0} e^{-ks} w_2(s)\,ds\Big) \\ -\,i\rho_2 C_2 e^{-kz}\Big(\int_{z_2}^z e^{ks} w_2(s)\,ds + \int_z^{0} e^{ks} w_2(s)\,ds\Big). \end{aligned} \tag{5.28}$$

Both (5.27) and (5.28) require the solutions of (5.26) for air and water, which are also obtained using the method of variation of parameters and are

$$w_1(z) = H_1 e^{kz} + J_1 e^{-kz} - \frac{i}{2} e^{kz} \Big(\int_0^z I_1(s)\, ds + \int_z^{z_1} I_1(s)\, ds \Big)$$
$$+ \frac{i}{2} e^{-kz} \Big(\int_0^z I_2(s)\, ds + \int_z^{z_1} I_2(s)\, ds \Big) \tag{5.29}$$

and

$$w_2(z) = H_2 e^{kz} + J_2 e^{-kz} - \frac{i}{2} e^{kz} \Big(\int_{z_2}^z I_3(s)\, ds + \int_z^{0} I_3(s)\, ds \Big)$$
$$+ \frac{i}{2} e^{-kz} \Big(\int_{z_2}^z I_4(s)\, ds + \int_z^{0} I_4(s)\, ds \Big), \tag{5.30}$$

where

$$I_1(z) = e^{-kz} [\, D_1 \operatorname{Ai}(\zeta(z)) + E_1 \operatorname{Bi}(\zeta(z))\,] \tag{5.31a}$$
$$I_2(z) = e^{kz} [\, D_1 \operatorname{Ai}(\zeta(z)) + E_1 \operatorname{Bi}(\zeta(z))\,] \tag{5.31b}$$
$$I_3(z) = e^{-kz} [\, D_2 \operatorname{Ai}(\zeta(z)) + E_2 \operatorname{Bi}(\zeta(z))\,] \tag{5.31c}$$
$$I_4(z) = e^{kz} [\, D_2 \operatorname{Ai}(\zeta(z)) + E_2 \operatorname{Bi}(\zeta(z))\,]. \tag{5.31d}$$

5.6 Boundary Conditions with solutions $w_j(z)$ and $p_j(z)$

Here, we rewrite the boundary conditions with (5.27)-(5.30). The kinematic boundary condition from (5.8) becomes

$$H_1 + J_1 + \frac{i}{2} \int_0^{z_1} (I_2(s) - I_1(s))\, ds = i\eta(k)(kU_0 - \omega). \tag{5.32}$$

The continuity of normal velocities, (5.9), is

$$H_1 + J_1 + \frac{i}{2} \int_0^{z_1} (I_2(s) - I_1(s))\, ds = H_2 + J_2 + \frac{i}{2} \int_{z_2}^{0} (I_4(s) - I_3(s))\, ds. \tag{5.33}$$

The continuity of tangential velocities from (5.10) becomes

$$H_1 - J_1 - \frac{i}{2} \int_0^{z_1} (I_1(s) + I_2(s))\, ds = H_2 - J_2 - \frac{i}{2} \int_{z_2}^{0} (I_3(s) + I_4(s))\, ds. \tag{5.34}$$

The continuity of tangential stress, (5.11), is

$$\rho_1 \nu_1 \Bigg[2kH_1 + 2kJ_1 - i\Big(D_1 \operatorname{Ai}(\frac{b_1}{a_1^{2/3}}) + E_1 \operatorname{Bi}(\frac{b_1}{a_1^{2/3}}) \Big) + ik \int_0^{z_1} (I_2(s) - I_1(s))\, ds \Bigg] =$$
$$\rho_2 \nu_2 \Bigg[2kH_2 + 2kJ_2 - i\Big(D_2 \operatorname{Ai}(\frac{b_2}{a_2^{2/3}}) + E_2 \operatorname{Bi}(\frac{b_2}{a_2^{2/3}}) \Big) + ik \int_{z_2}^{0} (I_4(s) - I_3(s))\, ds \Bigg]. \tag{5.35}$$

From (5.12), the continuity of normal stress becomes

$$F_1+G_1+iC_1\rho_1\int_0^{z_1}(e^{-ks}-e^{ks})w_1(s)\,ds-2k\rho_1\nu_1\Big[H_1-J_1-iC_1\eta(k)-\frac{i}{2}\int_0^{z_1}(I_1(s)+I_2(s))\,ds\Big]=$$

$$F_2+G_2+iC_2\rho_2\int_{z_2}^{0}(e^{-ks}-e^{ks})w_2(s)\,ds-2k\rho_2\nu_2\Big[H_2-J_2-iC_2\eta(k)-\frac{i}{2}\int_{z_2}^{0}(I_3(s)+I_4(s))\,ds\Big]. \tag{5.36}$$

In addition to the five conditions at $z=0$, we have conditions at $z=z_1$ and $z=z_2$. At $z=z_1$, (5.13) become

$$p_1(z_1)=F_1e^{kz_1}+G_1e^{-kz_1}+iC_1\rho_1\Big(e^{kz_1}\int_0^{z_1}e^{-ks}w_1(s)\,ds-e^{-kz_1}\int_0^{z_1}e^{ks}w_1(s)\,ds\Big)=0, \tag{5.37a}$$

$$w_1(z_1)=H_1e^{kz_1}+J_1e^{-kz_1}+\frac{i}{2}\Big(e^{-kz_1}\int_0^{z_1}I_2(s)\,ds-e^{kz_1}\int_0^{z_1}I_1(s)\,ds\Big)=0, \tag{5.37b}$$

$$w_1'(z_1)=H_1e^{kz_1}-J_1e^{-kz_1}-\frac{i}{2}\Big(e^{kz_2}\int_0^{z_1}I_1(s)\,ds+e^{-kz_1}\int_0^{z_1}I_2(s)\,ds\Big)=0. \tag{5.37c}$$

And, at $z=z_2$, (5.14) become

$$p_2(z_2)=F_2e^{kz_2}+G_2e^{-kz_2}+iC_2\rho_2\Big(e^{kz_2}\int_{z_2}^{0}e^{-ks}w_2(s)\,ds-e^{-kz_2}\int_{z_2}^{0}e^{ks}w_2(s)\,ds\Big)=0, \tag{5.38a}$$

$$w_2(z_2)=H_2e^{kz_2}+J_1e^{-kz_2}+\frac{i}{2}\Big(e^{-kz_2}\int_{z_2}^{0}I_4(s)\,ds-e^{kz_2}\int_{z_2}^{0}I_3(s)\,ds\Big)=0, \tag{5.38b}$$

$$w_2'(z_2)=H_2e^{kz_2}-J_2e^{-kz_2}-\frac{i}{2}\Big(e^{kz_2}\int_{z_2}^{0}I_3(s)\,ds+e^{-kz_2}\int_{z_2}^{0}I_4(s)\,ds\Big)=0. \tag{5.38c}$$

Given that there are a total of 13 constants ($D_j, E_j, F_j, G_j, H_j, J_j$, and η for $j=1,2$) that need to be found, we use the vertical Navier-Stokes equation, (5.5b), at $z=z_1$ for $j=1$ and at $z=z_2$ for $j=2$ in order to add two equations. Thus, we have

$$w_1''(z_1)-\frac{1}{\rho_1\nu_1}p_1'(z_1)=0, \tag{5.39a}$$

and

$$w_2''(z_2)-\frac{1}{\rho_2\nu_2}p_2'(z_2)=0. \tag{5.39b}$$

We have a total 13 unknown variables and 13 equations, which, in the next chapter, we will attempt to analyze.

Chapter 6

Analyzing the Perturbed Boundary-Value Problem

In this chapter, I use Mathematica to analyze the system of 13 linear linear equations and 13 unknowns, written as $\mathbf{M} \cdot \mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \mathbf{0}$. We are not interested in solving the system for the unknowns; instead, we want to determine the frequency, ω, of the waves, which has a real part as well as an imaginary part. We are interested in solving for the imaginary part in order to find the growth rate of the wave.

6.1 Power Series Representation of the Airy Equation of the First and Second Kind

In order to find the coefficients in front of each unknown variable, we must be able to solve the integral of the Airy functions. Due to the fact that we are unable to find the integral of the Airy functions of the first and second kind directly, we will find the power series form of both functions and then perform the integration.

Given (5.23), the power series form of (5.24) is

$$\mathrm{Ai}(\zeta_j) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdots (3n-2)}{(3n)!} \; \zeta_j^{3n}, \tag{6.1a}$$

and

$$\mathrm{Bi}(\zeta_j) = \zeta_j + \sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdots (3n-1)}{(3n+1)!} \; \zeta_j^{3n+1}, \tag{6.1b}$$

both of which converge for all values. We will approximate these series using the first two terms, so that

$$\mathrm{Ai}(\zeta_j) \approx 1 + \frac{\zeta_j^3}{3 \cdot 2} + O(\zeta_j^9), \tag{6.2a}$$

and

$$\mathrm{Bi}(\zeta_j) \approx \zeta_j + \frac{\zeta_j^4}{4 \cdot 3} + O(\zeta_j^{10}). \tag{6.2b}$$

Substituting (6.2) for Ai and Bi in (5.32)-(5.39b), we are able to evaluate an approximation of the integrals.

6.2 Setting up the Matrix

The matrix $\mathbf{M}$ is a square matrix with the entries as follows,

$$\mathbf{M} = \begin{array}{c|ccccccccccccc}
 & D_1 & E_1 & F_1 & G_1 & H_1 & J_1 & \eta & D_2 & E_2 & F_2 & G_2 & H_2 & J_2 \\
\hline
5.32 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.33 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.34 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.35 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.36 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.37a & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.37b & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.37c & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.38a & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.38b & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.38c & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.39a & . & . & . & . & . & . & . & . & . & . & . & . & . \\
5.39b & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

where the row labels indicate the corresponding equation being addressed and the column labels correspond to the 13 unknowns; each entry in the matrix corresponds to a coefficient of the one of the 13 unknowns (listed at the top of the matrix) and one of the 13 equations (listed at the left of the matrix) . Since these values are all large, I have not included them in the matrix itself and instead chose to show how the matrix is set up. I used Mathematica to set up the full matrix.

It follows that

$$\mathbf{x} = \begin{pmatrix} D_1 \\ E_1 \\ F_1 \\ G_1 \\ H_1 \\ J_1 \\ \eta \\ D_2 \\ E_2 \\ F_2 \\ G_2 \\ H_2 \\ J_2 \end{pmatrix}$$

and $\mathbf{b} = \mathbf{0}$ of 13 rows.

6.3 Attempting to solve the matrix

Our goal is to find the values of ω that satisfy $\mathbf{M} \cdot \mathbf{x} = \mathbf{0}$. One option is that we look for values of ω that correspond to $\det(\mathbf{M}) = 0$, but that allows for non-physical solutions. For this system, there can only be one free variable. For example, if the surface displacement is zero, then there cannot be any perturbation velocities or pressures. Therefore, rather than find ω that result in $\det(\mathbf{M}) = 0$, we seek values of ω that result in $\text{rank}(\mathbf{M}) = 12$, guaranteeing that there is only one free variable. We will test values of parameters ω (both real and imaginary parts) for different values of k so that $\mathbf{M}$ has rank 12.

Before I tested values of ω and k, I need to fix z_1 at an appropriate value. I fixed $z_1 = 0.01$ meters to correspond to measurements of the base flow profile.

Then, after fixing z_1, C_1 and C_2 follow from equation set (3.21). Similarly, z_2 also follows from (3.22).

We then fix the uniform wind speed at $U^\infty = 3.6$ m/s due to the fact that this speed was used in experiments in the Pritchard Lab. As a result, using (3.15) and the values for ρ_1, ρ_2, ν_1, and ν_2 from Table 2.1, $U_0 = 0.0167391$ m/s.

To separate out the real and imaginary parts of omega, we write it as,

$$\omega = \omega_R + i\omega_I, \tag{6.3}$$

where ω_R is frequency of oscillations and in the absence of wind, corresponds to

$$\omega_R = \sqrt{\frac{k\, g\, (\rho_2 - \rho_1) + k^3\, T_0}{\rho_1 + \rho_2}}. \tag{6.4}$$

Here, g is the gravitational constant, $g = 9.81$ m/s^2 and $T_0 = 72$ dynes/cm is the surface tension of a clean air-water interface. The two parameters that we must test values for are k and ω_I.

We proceed by choosing a value of k and varying ω_I, then changing k, and so on. I first choose values for k between 50 $\frac{1}{\text{cm}}$ and 250 $\frac{1}{\text{cm}}$ in increments of 50. Once a value for k is chosen, I used a for-loop to test values for ω_I between 0 $\frac{1}{\text{s}^2}$ and 10 $\frac{1}{\text{s}^2}$ in increments of 1. I also tested values for ω_I between 0 $\frac{1}{\text{s}^2}$ and 1 $\frac{1}{\text{s}^2}$ in increments of 0.1. I constructed the for-loop such that if the rank of the matrix $\mathbf{M}$ is 12, then I would store the values of k and ω_I in an array.

After trying all of these values, I did not find any values for k and ω_I that resulted in a rank of 12 for matrix $\mathbf{M}$; for all of the values tested, the rank is 9. This means that this problem does not have any physically relevent solutions; this will be further addressed in the discussion chapter.

Chapter 7

Discussion

Unfortunately, I was unable to find a frequency ω and wave number k that would allow matrix $\mathbf{M}$ to have rank of 12 and thus, resulting in $\eta(k)$ being the only free variable. There may be a couple explanations for this.

Since there are two parameters, the frequency ω, which has a real part and imaginary part, and the wave number k, it is difficult to test all possible values and combinations for these parameters. This is a major disadvantage to this problem and may possibly be the reason why we did not obtain desirable results for ω and k that would work to complete this model. Another aspect of the model that could possibly be causing discrepancy is the power series approximation for the Airy functions of the first and second kind; since we were unable to integrate the functions themselves and had to use an approximation, there is a possibility that the error is too great.

A more likely reason we did not find solutions is that the model could lack the key ingredients that causes the growth of the wind waves. We used a simplified model for wind waves and then approximated it. For future work, we can investigate the boundary conditions we chose for the approximate model and we can investigate the behavior of perturbations to the full base flow, rather than to our approximated version of it. If these considerations do not yield solutions, then it is likely that the turbulence, which we did not allow for this model, is a required ingredient for the growth of wind waves.

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