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A FAST DIFFUSION EQUATION

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Abstract

The fast diffusion equation is $u_t = -\Delta u^\beta$ for some $0 < \beta < 1$. There is a conformally invariant analogue of the fast diffusion equation which has an interesting relationship to sharp Gagliardo–Nirenberg inequalities. This project involves carrying out a detailed study of this flow on the round spheres as a first step towards understanding the behavior of the flow on general manifolds.

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Chapter 1

Introduction and Preliminaries

Given a compact manifold, the Hardy-Littlewood-Sobolev (HLS) inequality [1] and the Gagliardo-Nirenberg-Sobolev (GNS) inequality [3] have been shown to be conformally invariant. Moreover, Carlen, Carrillo, and Loss have proven that there is an unique relationship between the sharp HLS inequality of the Laplacian, the sharp GNS inequality, and the fast diffusion equation (FDE) [2]. Our goal is to show that the FDE is also conformally invariant.

The main result of this thesis is that this unique relationship between the HLS, GNS, and FDE extends to the special case of the round sphere. As an application, stability is proven.

This work provides a model for conformally invariant FDE's on compact manifolds. Furthermore, this work is expected to generalize to other FDE's on the sphere. In the following, we review some basic definitions and related work in the literature. Throughout this paper, we adopt the convention that $-\Delta$ is non-negative.

1.1 Sobolev Spaces

A Sobolev space is a special case of a Banach space.

Definition 1.1.1. *A complete metric space is a metric space in which every Cauchy sequence is convergent.*

Definition 1.1.2. *A complex vector space X is said to be a normed linear space if for each $x \in X$ there is associated a non-negative real number $\|x\|$, called the norm of x, such that :*

(a) $\|x + y\| \leq \|x\| + \|y\|$ *for all* x,$y \in X$,

(b) $\|ax\| = |a|\|x\|$ *if* $x \in X$ *and* $a \in \mathbb{C}$,

(c) $\|x\| \geq 0$, *and* $\|x\| = 0$ *if and only if* $x = 0$.

Definition 1.1.3. *A Banach space is a normed linear space which is complete in the metric defined by its norm.*

L^p-norms are incredibly important when studying PDE's. Therefore, the definition of L^P-norms is included below.

Definition 1.1.4. *Let μ be a measure on X. If $1 \leq p < \infty$ and if f is a complex measurable function on X, define:*

$$\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

and let $L^p(\mu)$ consist of all f for which:

$$\|f\|_p < \infty.$$

We call $\|f\|_p$ the L^p norm of f.

Remark 1.1.5. *Throughout this paper, the following notation holds true:* $\|f\|_p = (\int_{\mathbb{R}^d} |f|^p \, \mathrm{d}x)^{\frac{1}{p}}$ *or* $\|f\|_p = (\int_{S^d} |f|^p \, \mathrm{d}x)^{\frac{1}{p}}$ *for* $p \in [1, \infty)$.

Definition 1.1.6. *A (real) Hilbert Space H is a vector space over* $\mathbb{R}$*, equipped with a scalar product*

$$(\cdot, \cdot) : H \times H \to \mathbb{R}$$

that satisfies the following properties:

(a) $(x, y) = (y, x)$ *for all* x *and* $y \in H$.

(b) $(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda(x_1, y) + \lambda_2(x_2, y)$ *for all* $\lambda_1, \lambda_2 \in \mathbb{R}$, $, x_2, y \in H$.

(c) $(x, x) > 0$ *and for all* $x \neq 0$, $x \in H$.

(d) H is complete with respect to the norm $\|x\| := (x, x)^{\frac{1}{2}}$.

Furthermore, in a Hilbert space, the Schwarz Inequality

$$|(x, y)| \leq \|x\| \cdot \|y\|$$

holds. Moreover, equality holds if and only if x and y are linearly dependent.

Remark 1.1.7. *Every Hilbert Space is a Banach Space.*

Definition 1.1.8. *Let* μ *be a measure on* X*. The Sobolev space* $W^{1,2}(X)$ *is defined as the completion of* $C_c^\infty(X)$ *of smooth functions of compact support with respect to the norm* $\|u\|_{W^{1,2}(X)} := (u, u)^{1/2}_{W^{1,2}(X)}$*, where the inner product* $(\cdot, \cdot)$ *is given by*

$$(u, v)_{W^{1,2}(X)} := \int_X uv \, \mathrm{d}\mu + \sum_{i=1}^{d} \int_X D_i u \cdot D_i v \, \mathrm{d}\mu.$$

Again, let μ be a measure on X. The square of The $W^{1,2}$ norm of a C^1-function f is the sum of the square of the L^2-norm and the square of the L^2-norm of its gradient:

$$\|f\|^2_{W^{1,2}} = \int |f|^2 \, \mathrm{d}\mu + \int |\nabla f|^2 \, \mathrm{d}\mu.$$

Lemma 1.1.9. $W^{1,2}(X)$ *is complete with respect to* $\| \cdot \|_{W^{1,2}}$*, and is hence a Hilbert space.*

Therefore, we can define the $W^{1,2}(X)$ space when $X = \mathbb{R}^d$ and $X = S^d$.

Definition 1.1.10. *A functional $\mathscr{F}$ is a real-valued function on a vector space V.*

1.2 Basic definitions and review of related work

The sharp form of the Hardy-Littlewood-Sobolev Inequality states that for all non-negative measurable functions f on $\mathbb{R}^d$, and all $0 < \lambda < d$ it follows that:

$$\frac{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{f(x)f(y)}{|x-y|^\lambda}\,\mathrm{d}x\mathrm{d}y}{\|f\|_p^2} \leq \frac{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{h(x)h(y)}{|x-y|^\lambda}\,\mathrm{d}x\mathrm{d}y}{\|h\|_p^2} \tag{1.1}$$

where

$$h(x) = (\frac{1}{1+|x|^2})^{\frac{2d-\lambda}{2}} \tag{1.2}$$

and where $p = \frac{2d}{(2d-\lambda)}$ [2].

Specifically, Carlen, Carrillo, and Loss [2] looked at the case of $\lambda = d-2$. They referenced from other work that the $\lambda = d-2$ case of the Sharp HLS inequality expresses the L^p-norm smoothing properties of $(-\Delta)^{-1}$ on $\mathbb{R}^d$. This follows from (1.1) and the fact that for $d \geq 3$

$$\int_{\mathbb{R}^d} f(x)[(-\Delta)^{-1}f](x)\mathrm{d}x = \frac{1}{(d-2)|S^{d-1}|}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{f(x)f(y)}{|x-y|^\lambda}\,\mathrm{d}x\mathrm{d}y,$$

where $|S^{d-1}|$ denotes the surface area of the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$. Carlen, Carillo, and Loss [2] also pointed out that for $\lambda = d-2$, equation (1.1) can be rewritten as $\mathscr{F}[f] \geq 0$ for all $f \in L^{\frac{2d}{(d+2)}}(\mathbb{R}^d)$, where:

$$\mathscr{F}[f] := C_s\|f\|^2_{\frac{2d}{d+2}} - \int_{\mathbb{R}^d} f(x)[(-\Delta)^{-1}f](x)\mathrm{d}x \tag{1.3}$$

with

$$C_s := \frac{4}{d(d-2)}|S^d|^{-2/d}.$$

Specifically, Carlen, Carillo, and Loss [2] refer to $\mathscr{F}$ as the HLS functional on $L^{\frac{2d}{(d+2)}}(R^d)$. Furthermore, [2] defines g to be a smooth function of compact support and denotes the inner product:

$$\langle g, f\rangle = \int_{\mathbb{R}^d} g(x)f(x)\mathrm{d}x$$

Definition 1.2.1. *Consider m ∈ (0,1), then the fast diffusion equation is*

$$\frac{\partial}{\partial t}u(x,t) = \Delta u^m(x,t) \tag{1.4}$$

The main result of [2] is that the HLS functional is decreasing along a particular fast diffusion equation, and hence HLS minimizers are attracting steady states for this flow. This is a consequence of the following three results.

Theorem 1.2.2 ([2]). *Let $f \in L^{\frac{2d}{(d+2)}}(\mathbb{R}^d)$ be non-negative,and suppose that f satisfies*

$$\sup_{|x|>R} f(x)|x|^{\frac{2}{1-m}} < \infty$$

for some $R > 0$, and $m = \frac{d}{d+2}$, ensuring that f is integrable. Let us further suppose that

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} h(x)\,\mathrm{d}x = M_*,$$

where h is given by (1.2) *with $\lambda = d-2$. Let u(x,t) be the solution of* (1.4) *with $m = \frac{d}{d+2}$ and $u(x,1) = f(x)$. Then, for all $t > 1$,*

$$\frac{\partial}{\partial t}\mathscr{F}[u(\cdot,t)] = -2\mathscr{D}[u^{\frac{(d-1)}{(d+2)}}(\cdot,t)],$$

where

$$\mathscr{D}[g] := C_s\frac{d(d-2)}{(d-1)}\|g\|_{\frac{2d}{(d-1)}}^{\frac{4}{(d-1)}}\|\nabla g\|_2^2 - \|g\|_{\frac{2(d+1)}{(d-1)}}^{\frac{2(d+1)}{(d-1)}}.$$

Theorem 1.2.3 ([2]). *Let $g \in W^{1,2} \cap L^{\frac{2(d+1)}{(d-1)}}(\mathbb{R}^d)$, be non-negative. Then $\mathscr{D}[g] \geq 0$. The results in*

(1.2.3) *are a consequence of the* $p = \frac{d+1}{d-1}$ *cases of the GNS Inequality [4]. These can be written in the form below ,where* $C_{GNS} = C_s \frac{d(d-2)}{(d-1)}$ *is the best constant for which the inequality is valid for all smooth g with compact support,*

$$C_{GNS} \|g\|_{\frac{2d}{d-1}}^{\frac{4}{d-1}} \|\nabla g\|_2^2 \geq \|g\|_{\frac{2(d+1)}{d-1}}^{\frac{2(d+1)}{d-1}}.$$

Theorem 1.2.4 ([2]). *Let* $f \in L^{\frac{2d}{(d+2)}}, d \geq 3$ *be non-negative. Then* $\mathscr{F}[f] \geq 0$ *, and* $\mathscr{F}[f] = 0$ *if and only if f is a multiple of* $h(x/s - x_0)$ *for some* $s > 0$*, and some* $x_0 \in S^d$*, and where h is given by* (1.2) *with* $\lambda = d - 2$*.*

Note that Theorems (1.2.2) - (1.2.4) imply that the HLS functional $\mathscr{F}$ is monotone decreasing along this flow and hence the HLS minimizers are the attracting steady states for a certain fast diffusion flow. This paper will prove the analogous results on the sphere.

1.3 Stereographic Projection

Stereographic projection is a conformal map from the sphere to the Euclidean Space. One of the main aims of this paper is to generalize the conclusions from the previous results shown in Carlen, Carillo, and Loss [2] in Euclidean spaces, to spherical spaces.

Definition 1.3.1 ([5]). *Stereographic Projection is the map* $\Phi : S^d \setminus (0, ..., 0, 1) \rightarrow \mathbb{R}^d$*,*

$$\Phi(x) = \left(\frac{\bar{x}}{1 - x_{n+1}} \right)$$

where $\bar{x} = (x_1, x_2, ..., x_d)$ *,* $x \in \mathbb{R}^{d+1}$ *, and* $x = (\bar{x}, x_{d+1})$ *. Also,* $x \in S^d$ *implies that*

$$|\bar{x}|^2 + x_{d+1}^2 = 1$$

Lemma 1.3.2. *For* $\mathbb{R}^d$ *consider* $\rho(\vec{x}) = |\vec{x}|$ *and* $\vartheta = \frac{\vec{x}}{|\vec{x}|}$*. Also, for* S^d *let* $cos\phi(\xi) = \xi_{d+1}$*,* $\theta = \frac{\bar{\xi}}{|\bar{\xi}|}$*,* *and* $\xi = (\bar{\xi}, \xi_{d+1})$*. It then follows that the map* Φ *can be written in polar coordinates such that,*

$$\Phi(\rho, \theta) = (\rho(\phi, \theta), \vartheta(\rho, \theta))$$

where $\rho(\Phi(\phi, \theta)) = \cot \frac{\phi}{2}$ *and* $\vartheta(\Phi(\phi, \theta)) = \theta$*.*

Furthermore, we can define the pullback of the map Φ.

Definition 1.3.3. $\Phi^* \colon C^\infty(\mathbb{R}^d) \to C^\infty(S^d)$ *is the map*

$$(\Phi^* u)(\xi) := u\left(\Phi(\xi)\right).$$

Lemma 1.3.4. *Let the metrics on* S^d *and* $\mathbb{R}^d$ *be defined as follows [6] :*

$$g_S := d\phi^2 + \sin^2 \phi \, d\theta^2,$$
$$g_{\mathbb{R}} := d\rho^2 + \rho^2 \, d\vartheta^2,$$

respectively, where $d\rho^2 := (d\rho)^2$ *and* θ *is a variable on the* $(d-1)$ *dimensional sphere. Then it follows that the pullback of* $g_{\mathbb{R}}$ *is:*

$$\Phi^* g_{\mathbb{R}} = (1 - \cos\phi)^{-2} \left(d\phi^2 + \sin^2 \phi \, d\theta^2\right).$$

Proof. By definition,

$$\Phi^* g_R = d\rho(\Phi(\phi, \theta))^2 + \rho^2(\Phi(\phi, \theta)) \, d\vartheta(\Phi(\phi, \theta))^2.$$

Since $\rho(\Phi(\phi, \theta)) = \cot \frac{\phi}{2}$ and $\vartheta(\Phi(\phi, \theta)) = \theta$, we deduce that

$$\begin{aligned}\Phi^* g_R &= \left(d(\cot(\phi/2))\right)^2 + \cot^2 \frac{\phi}{2} \, d\theta^2 \\ &= \left(-\frac{1}{2} \csc^2 \frac{\phi}{2} \, d\phi\right)^2 + \cot^2 \frac{\phi}{2} \, d\theta^2\end{aligned}$$

$$= \left(2\sin^2\frac{\phi}{2}\right)^{-2} d\phi^2 + \cot^2\frac{\phi}{2}\,d\theta^2$$
$$= \left(2\sin^2\frac{\phi}{2}\right)^{-2} \left(d\phi^2 + (2\sin\frac{\phi}{2}\cos\frac{\phi}{2})^2\,d\theta^2\right)$$
$$= (1-\cos\phi)^{-2}\left(d\phi^2 + \sin^2\phi\,d\theta^2\right). \qquad \square$$

Lemma 1.3.5. *The volume element of* $\Phi^* g_{\mathbb{R}}$ *is*

$$dvol_{\Phi^* g_{\mathbb{R}}} = (1-\cos\phi)^{-d} dvol_{g_S}.$$

Proof. This is an immediate consequence of the definition

$$dvol_g = \sqrt{\det g}\,dx^1 \cdots dx^d$$

of the Riemannian volume element. $\square$

Theorem 1.3.6. *Let*

$$L := -\Delta_{g_S} + \frac{d(d-2)}{4},$$

denote the conformal Laplacian on (S^d, g_S)*. Then for all* $f \in C^\infty(S^d)$*, we have*

$$-\Delta_{\Phi^* g_{\mathbb{R}}} f = (1-\cos\phi)^{\frac{d+2}{2}} L\left((1-\cos\phi)^{-\frac{d-2}{2}} f\right).$$

In particular, this gives the change of variables formula

$$\Phi^*\left(-\Delta_{g_{\mathbb{R}}} f\right) = (1-\cos\phi)^{\frac{d+2}{2}} L\left((1-\cos\phi)^{-\frac{d-2}{2}} \Phi^* f\right),$$

for all $f \in C^\infty(\mathbb{R}^d)$.

Proof. Recall that $\Delta g = \frac{1}{\sqrt{\det g}}\partial_i(\sqrt{\det g} g^{ij}\partial_j)$ [6].

Take $a = \frac{d-2}{2}$ and consider f to be independent of θ.

$$
\begin{aligned}
&-\Delta_{\Phi^* g_{\mathbb{R}}}(1-\cos\phi)^a f\\
&= \frac{(1-\cos\phi)^d}{(\sin\phi)^{d-1}} d\phi \left(\frac{(\sin\phi)^{d-1}}{(1-\cos\phi)^{d-2}} d\phi (1-\cos\phi)^a f \right)\\
&= \frac{(1-\cos\phi)^d}{(\sin\phi)^{d-1}} d\phi \left(\frac{(\sin\phi)^{d-1}}{(1-\cos\phi)^{d-2}} [a(\sin\phi)(1-\cos\phi)^{d-1} f + (1-\cos\phi)^a f'] \right)\\
&= \frac{(1-\cos\phi)^d}{(\sin\phi)^{d-1}} d\phi \left(a(\sin\phi)^d (1-\cos\phi)^{a-d+1} f + \sin\phi^{d-1}(1-\cos\phi)^{a-d+2} f' \right)\\
&= \frac{(1-\cos\phi)^d}{(\sin\phi)^{d-1}} \Big(\frac{d(d-2)}{2} (\sin\phi)^{d-1} (1-\cos\phi)^{(\frac{d-2}{2}-d+1)} \cos\phi f\\
&\quad + \frac{d-2}{2} \sin\phi^{d+1} \Big(\frac{d-2}{2} - d + 1 \Big) (1-\cos\phi)^{(\frac{d-2}{2}-d)} f + \frac{d-2}{2} \sin\phi^d (1-\cos\phi)^{(\frac{d-2}{2}-d+1)} f'\\
&\quad + (d-1)\sin\phi^{d-2}\cos\phi(1-\cos\phi)^{(\frac{d-2}{2}-d+2)} f'\\
&\quad + \Big(\frac{d-2}{2} - d + 2 \Big) \sin\phi^{d-1} (1-\cos\phi)^{(\frac{d-2}{2}-d+1)} \sin\phi f' + \sin\phi^{d-1}(1-\cos\phi)^{\frac{d-2}{2}-d+2} f'' \Big)\\
&= \frac{d(d-2)}{2} \cos\phi (1-\cos\phi)^{(\frac{d-2}{2}+1)} f + \frac{-d(d-2)}{4} \sin\phi^2 (1-\cos\phi)^{\frac{d-2}{2}} f\\
&\quad + \frac{d-2}{2} \sin\phi (1-\cos\phi)^{(\frac{d-2}{2}+1)} f' + (d-1)\cot(1-\cos\phi)^{(\frac{d-2}{2}+2)} f'\\
&\quad + \frac{-d+2}{2} (1-\cos\phi)^{(\frac{d-2}{2}+1)} \sin\phi f' + \sin\phi (1-\cos\phi)^{(\frac{d-2}{2}+2)} f''\\
&= \frac{d(d-2)}{2} \cos\phi (1-\cos\phi)^{\frac{d}{2}} f + \frac{-d(d-2)}{4} \sin\phi^2 (1-\cos\phi)^{\frac{d-2}{2}} f\\
&\quad + \frac{d-2}{2} \sin\phi (1-\cos\phi)^{\frac{d}{2}} f' + \frac{-d+2}{2} (1-\cos\phi)^{\frac{d}{2}} \sin\phi f'\\
&= \frac{d(d-2)}{2} \cos\phi (1-\cos\phi)^{\frac{d}{2}} f + \frac{-d(d-2)}{4} \sin\phi^2 (1-\cos\phi)^{\frac{d-2}{2}} f
\end{aligned}
$$

Recall that $g_{S^n} := d\phi^2 + \sin^2\phi g_{S^{n-1}}$. Therefore, if $f(x) = f(\phi)$, then

$$-\Delta_{gs} f = f'' + (d-1)\cot\phi f'$$

Consequently, both $\sin\phi(1-\cos\phi)^{a+2}f''$ and $(n-1)\cot(1-\cos\phi)^{a+2}f'$ are in $-\Delta_{gs}f$. $\square$

Proposition 1.3.7. *Let* $W = (1-\cos\phi)^{\frac{-d-2}{2}}\,\Phi^{*}u$*, then the functional* $\mathscr{F}$ (1.3) *pulled-back to the sphere is given by,*

$$\mathscr{F}[u] = \frac{4}{d(d-2)}|S^{d}|^{\frac{-2}{d}}\,\|W\|^{2}_{L^{\frac{2d}{d+2}}} - \int_{S^{d}} W(L^{-1})W\,\mathrm{d}vol g_{s} \tag{1.5}$$

Proof. Carlen, Carrilo, and Loss [2] stated that,

$$\mathscr{F}[u] = \frac{4}{d(d-2)}|S^{d}|^{\frac{-2}{d}}\,\|u\|^{2}_{L^{\frac{2d}{d+2}}} - \int_{\mathbb{R}^{d}} u(x)[L^{-1}u]x\,\mathrm{d}vol g_{s}$$

First, we can compute the pullback of the $2d/(d+2)$-norm,

$$\begin{aligned}
\|\Phi^{*}u\|^{2}_{L^{\frac{2d}{d+2}}} &= \left(\int_{\mathbb{R}^{d}}|\Phi^{*}u|^{\frac{2d}{d+2}}\,\mathrm{d}vol g_{s}\right)^{\frac{d+2}{d}}\\
&= \left(\int_{S^{d}}|\Phi^{*}u|^{\frac{2d}{d+2}}\,(1-\cos\phi)^{-d}\,\mathrm{d}vol g_{s}\right)^{\frac{d+2}{d}}\\
&= \left(\int_{S^{d}}|\,(1-\cos\phi)^{\frac{-d-2}{2}}\,\Phi^{*}u|^{\frac{2d}{d+2}}\,\mathrm{d}vol g_{s}\right)^{\frac{d+2}{d}}\\
&= \left(\int_{S^{d}}|W|^{\frac{2d}{d+2}}\,\mathrm{d}vol g_{s}\right)^{\frac{d+2}{d}}
\end{aligned}$$

Also, the formula for $\Phi^{*}\Delta$ is given in Theorem 1.3.6. Therefore, the formula for $\Phi^{*}\Delta^{-1}$ is as follows:

$$\Phi^{*}\left((-\Delta_{g_{\mathbb{R}}})^{-1}f\right) = (1-\cos\phi)^{\frac{d-2}{2}}L^{-1}\left((1-\cos\phi)^{-\frac{d+2}{2}}\Phi^{*}f\right).$$

Next, we can compute the pullback of the second integral; this yields

$$\begin{aligned}
&\int_{s^{d}}\Phi^{*}u\,(1-\cos\phi)^{\frac{d-2}{2}}\,L^{-1}[(1-\cos\phi)^{-\frac{(d+2)}{2}}\,\Phi^{*}u]\,(1-\cos\phi)^{-d}\,\mathrm{d}vol g_{s}\\
&= \int_{s^{d}}\Phi^{*}u\,(1-\cos\phi)^{\frac{-d-2}{2}}\,L^{-1}[(1-\cos\phi)^{\frac{-d-2}{2}}\,\Phi^{*}u]\,\mathrm{d}vol g_{s}
\end{aligned}$$

$$= \int_{s^d} W(L^{-1})W \, \mathrm{d}volg_s. \qquad \square$$

Chapter 2

The Fast Diffusion Case on the Sphere

2.1 Monotonicity of $\mathscr{F}$ Along Fast Diffusion

As stated in Chapter 1, the main aim of this paper is to extend the relationship between the FDE, HLS, and GNS to the sphere. For our specific case of the fast diffusion equation, we want to show that $\mathscr{F}$ defined in (1.3.7) is monotone decreasing along this flow. This is a consequence of two results. First we compute the evolution of $\mathscr{F}$ along the FDE.

Theorem 2.1.1. *Let $f \in L^{\frac{2d}{(d+2)}}(\mathbb{R}^d)$ be non-negative, and suppose that f satisfies*

$$\sup_{|x|>R} f(x)|x|^{\frac{2}{1-m}} < \infty$$

for some $R > 0$, and $m = \frac{d}{d+2}$, ensuring that f is integrable. Let us further suppose that

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} h(x)\,\mathrm{d}x = M_*,$$

where h is given by (1.2) *with* $\lambda = d - 2$*. Let u(x,t) be the solution of* (1.4) *with* $m = \frac{d}{d+2}$ *and u(x,1)=f(x). Set* $W = (1-\cos\phi)^{\frac{-d-2}{2}}\Phi^* u$*. Then, for all* $t > 1$*,*

$$\frac{\partial}{\partial t}\mathscr{F}[W(\cdot,t)] = -2\mathscr{D}[W^{\frac{(d-1)}{(d+2)}}(\cdot,t)],$$

where $\mathscr{F}$ *is* (1.5) *and*

$$\mathscr{D}[g] := C_s \frac{d(d-2)}{(d-1)}\|g\|^{\frac{4}{(d-1)}}_{\frac{2d}{(d-1)}}\Big(\int_{s^d}\big(|\nabla g|^2 + \frac{(d+1)(d-1)}{4}g^2\big)(1-\cos\phi) - \frac{d-1}{2}\int_{s^d} g^2\Big) - \|g\|^{\frac{2(d+1)}{(d-1)}}_{\frac{2(d+1)}{(d-1)}}.$$

Additionally, $\frac{\partial}{\partial t}W = LW^{\frac{d}{d+2}}$. Second, we have, by pulling back via stereographic projection, the following GNS inequality.

Theorem 2.1.1. *Let* $u \in L^{\frac{2d}{(d+2)}}(S^d)$*, be non-negative. Then* $\mathscr{D}[u]$ *is non-negative.*

2.2 Stability of Steady State

Theorem 2.2.1 characterizes the steady state solutions for the specific fast diffusion case we are studying.

Theorem 2.2.1. *Let* $f \in L^{\frac{2d}{(d+2)}}(S^d), d \geq 3$ *be non-negative. Then* $\mathscr{F}[f] \geq 0$*, and* $\mathscr{F}[f] = 0$ *if and only if f is a multiple of* $h(\Phi\Psi_{O,\delta}x)$ *for some conformal transformation* $\Psi_{O,\delta}$ *of* S^d*,where O stands for the rotations and* δ *stands for the dilations, and where h is given by* (1.2).

Chapter 3

Proofs of Main Theorems

3.1 Proof of Theorem 2.1.1

Proof. Let $W = (1 - \cos\phi)^{\frac{-d-2}{2}} \Phi^* u$ and take the functional $\mathscr{F}$ from (1.5),

$$\mathscr{F}[u] = \frac{4}{d(d-2)} |S^d|^{\frac{-2}{d}} \|W\|^2_{L^{\frac{2d}{d+2}}} - \int_{S^d} W(L^{-1})W \, \mathrm{d}vol g_s \tag{3.1}$$

Next, we will take the derivative with respect to t. Since S^d and the constant in front is not in terms of t , only the terms with W matter.

$$\begin{aligned}
\frac{\partial}{\partial t}\mathscr{F}[u] &= \frac{4}{d(d-2)} |S^d|^{\frac{-2}{d}} \frac{\partial}{\partial t} \|W\|^2_{L^{\frac{2d}{d+2}}}(S^d) - \frac{\partial}{\partial t} \int_{S^d} W(L^{-1})W \, \mathrm{d}vol g_s \\
&= \frac{4}{d(d-2)} |s^d|^{\frac{-2}{d}} \frac{\partial}{\partial t} \int_{S^d} |\, (1 - \cos\phi)^{\frac{-d-2}{2}} \, \Phi^* u \,|^{\frac{2d}{d+2}} \mathrm{d}vol g_s)^{\frac{(d+2)}{d}} \\
&\quad - \frac{\partial}{\partial t} \int_{S^d} W(L^{-1})W \, \mathrm{d}vol g_s
\end{aligned}$$

The lines below go through the process of taking the derivative by using the chain rule, power rule, and etc.

$$\frac{\partial}{\partial t} f[u] = \frac{4(d+2)}{d^2(d-2)} |S^d|^{\frac{-2}{d}} \Big(\int_{S^d} |\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u\,|^{\frac{2d}{d+2}} \mathrm{d}vol g_s\Big)^{\frac{2}{d}}$$
$$\times \frac{\partial}{\partial t}\Big(\int_{S^d} |\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u|^{\frac{2d}{d+2}}\ \mathrm{d}vol g_s\Big) - \frac{\partial}{\partial t}\int_{S^d} W(L^{-1})W\ \mathrm{d}vol g_s$$

$$= \frac{4(d+2)}{d^2(d-2)} |S^d|^{\frac{-2}{d}} \Big(\int_{S^d} |\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u\,|^{\frac{2d}{d+2}} \mathrm{d}vol g_s\Big)^{\frac{2}{d}}$$
$$\times \Big(\int_{S^d} \frac{2d}{d+2} |\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u|^{\frac{d-2}{d+2}}\ |\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* \Delta u^{\frac{d}{d+2}}\ |\mathrm{d}vol g_s\Big)$$
$$- \frac{\partial}{\partial t}\int_{S^d} [(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u(L^{-1})\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u\,]\mathrm{d}vol g_s$$

$$= \frac{4(d+2)}{d^2(d-2)} |S^d|^{\frac{-2}{d}} \Big(\int_{S^d} |\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u\,|^{\frac{2d}{d+2}} \mathrm{d}vol g_s\Big)^{\frac{2}{d}}$$
$$\times \Big(\int_{S^d} \frac{2d}{d+2} |\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u|^{\frac{d-2}{d+2}}\ |\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* \Delta u^{\frac{d}{d+2}}\ |\mathrm{d}vol g_s\Big)$$
$$- \int_{S^d} [(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* \Delta u^{\frac{d}{d+2}}(L^{-1})\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u$$
$$+ ((1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u\,)(L^{-1})\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* \Delta u^{\frac{d}{d+2}}]\mathrm{d}vol g_s$$

Specifically for ease going forward we will write W, as before, where:

$$W = (1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* u.$$

Also, we can combine the constants and bring them to the front of the first part of the equation.

$$\frac{\partial}{\partial t} f[u] = \frac{8}{d(d-2)} |S^d|^{\frac{-2}{d}} \Big(\int_{S^d} |W|^{\frac{2d}{d+2}} \mathrm{d}vol g_s\Big)^{\frac{2}{d}}$$
$$\times \Big(\int_{S^d} |W|^{\frac{d-2}{d+2}} [(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* \Delta u^{\frac{d}{d+2}}\ ]\mathrm{d}vol g_s\Big)$$
$$- 2\int_{S^d} (W)(L^{-1})\Big(\,(1-\cos\phi)^{\frac{-d-2}{2}}\ \Phi^* \Delta u^{\frac{d}{d+2}}\Big)\mathrm{d}vol g_s$$

Next, recall Section 1.3 result for $\Phi^*\Delta$ on (S^n, g_S) (1.3.6).

$$\frac{\partial}{\partial t} f[u] = \frac{8}{d(d-2)} |S^d|^{\frac{-2}{d}} (\int_{S^d} |W|^{\frac{2d}{d+2}} \mathrm{d}volg_s)^{\frac{2}{d}}$$
$$\times (\int_{S^d} |W|^{\frac{d-2}{d+2}} [L\left((1-\cos\phi)^{\frac{-d-2}{2}} \Phi^* u^{\frac{d}{d+2}}\right)] \mathrm{d}volg_s)$$
$$- 2\int_{S^d} (W)(L^{-1}) L\left((1-\cos\phi)^{\frac{-d-2}{2}} \Phi^* u^{\frac{d}{d+2}}\right) \mathrm{d}volg_s$$

Let $V = (1-\cos\phi)^{\frac{-d-2}{2}} \Phi^* u^{\frac{d}{d+2}}$.

$$\frac{\partial}{\partial t} f[u] = \frac{8}{d(d-2)} |S^d|^{\frac{-2}{d}} (\int_{S^d} |W|^{\frac{2d}{d+2}} \mathrm{d}volg_s)^{\frac{2}{d}}$$
$$\times (\int_{S^d} |W|^{\frac{d-2}{d+2}} [L(V)] \mathrm{d}volg_s)$$
$$- \int_{S^d} 2(V)(W) \mathrm{d}volg_s$$

We can easily rewrite the first integral in terms of the L^p Norm.

$$\frac{\partial}{\partial t} f[u] = \frac{8}{d(d-2)} |S^d|^{\frac{-2}{d}} \|W\|^{\frac{4}{d+2}}_{L^{\frac{2d}{d+2}}} \int_{s^d} |W|^{\frac{d-2}{d+2}} LV \,\mathrm{d}volg_s - \int_{s^d} 2(V)W \,\mathrm{d}volg_s$$

Next, we need to write everything in terms of Y := $W^{\frac{d-1}{d+2}}$. Consider the following:

$$\|W\|^{\frac{4}{d+2}}_{L^{\frac{2d}{d+2}}} = [(\int_{s^d} |W|^{\frac{2d}{d+2}} \,\mathrm{d}volg_s)^{\frac{d+2}{2d}}]^{\frac{4}{d+2}} = \left(\int_{s^d} |W|^{\frac{2d}{d+2}} \,\mathrm{d}volg_s\right)^{\frac{2}{d}}.$$

Since $Y = W^{\frac{d-1}{d+2}}$,

$$\|W\|^{\frac{4}{d+2}}_{L^{\frac{2d}{d+2}}} = \left(\int_{s^d} |Y|^{\frac{2d}{d-1}} \,\mathrm{d}volg_s\right)^{\frac{2}{d}} = \|Y\|^{\frac{4}{d-1}}_{L^{\frac{2d}{d-1}}}.$$

We need to rewrite the following in the correct variables:

$$V = (1-\cos\phi)^{\frac{-d-2}{2}} \Phi^* u^{\frac{d}{d+2}}$$

$$
\begin{aligned}
&= (1-\cos\phi)^{\frac{-d-2}{2}}\,(1-\cos\phi)^{\frac{-(-d-2)d}{2(d+2)}}\,W^{\frac{d}{d+2}} \\
&= (1-\cos\phi)\,W^{\frac{d}{d+2}} \\
&= (1-\cos\phi)\,Y^{\frac{d}{d-1}}.
\end{aligned}
$$

Therefore,

$$
\int_{s^d} |W|^{\frac{d-2}{d+2}} LV \,\mathrm{d}volg_s = \int_{s^d} Y^{\frac{d-2}{d-1}} L\,(1-\cos\phi)\,Y^{\frac{d}{d-1}}\,\mathrm{d}volg_s.
$$

We can also use the previous results to write:

$$
\int_{s^d} 2(V)W \,\mathrm{d}volg_s = \int_{s^d} 2\,(1-\cos\phi)\,Y^{\frac{2d+2}{d-1}})\mathrm{d}volg_s
$$

Combining all the previous parts:

$$
\begin{aligned}
\frac{\partial}{\partial t} f[u] = {} & \frac{8}{d(d-2)} |s^d|^{\frac{-2}{d}} \|Y\|^{\frac{4}{d-1}}_{L^{\frac{2d}{d-1}}} \int_{s^d} |Y|^{\frac{d-2}{d-1}} L\,(1-\cos\phi)\,Y^{\frac{d}{d-1}}\,\mathrm{d}volg_s \\
& - \int_{s^d} 2((1-\cos\phi)\,Y^{\frac{2d+2}{d-1}})\,\mathrm{d}volg_s
\end{aligned}
$$

We need to take the Laplacian and rewrite using gradients. We know that

$$
L = -\Delta + \frac{d(d-2)}{4}.
$$

Consider

$$
\begin{aligned}
& \int_{s^d} |Y|^{\frac{d-2}{d-1}} L\,(1-\cos\phi)\,Y^{\frac{d}{d-1}}\,\mathrm{d}volg_s \\
&= \int_{s^d} |Y|^{\frac{d-2}{d-1}} (-\Delta + \frac{d(d-2)}{4})\,(1-\cos\phi)\,Y^{\frac{d}{d-1}}\,\mathrm{d}volg_s \\
&= \int_{s^d} (|Y|^{\frac{d-2}{d-1}} (-\Delta)\,(1-\cos\phi)\,Y^{\frac{d}{d-1}} \\
&+ \frac{d(d-2)}{4} |Y|^{\frac{d-2}{d-1}}\,(1-\cos\phi)\,Y^{\frac{d}{d-1}})\,\mathrm{d}volg_s.
\end{aligned}
$$

We need only look at the first part of the equation involving the Laplacian.

$$\begin{aligned}
&\int_{s^d} (|Y|^{\frac{d-2}{d-1}} (-\Delta) (1-\cos\phi) |Y|^{\frac{d}{d-1}} \mathrm{d}vol g_s \\
&= \int_{s^d} \left\langle \nabla |Y|^{\frac{d-2}{d-1}}, \nabla (1-\cos\phi) |Y|^{\frac{d}{d-1}} \right\rangle \mathrm{d}vol g_s \\
&= \int_{s^d} \left\langle \frac{d-2}{d-1} |Y|^{\frac{d-2-(d-1)}{d-1}} \nabla Y, \nabla (1-\cos\phi) |Y|^{\frac{d}{d-1}} \right\rangle \mathrm{d}vol g_s \\
&= \int_{s^d} \left\langle \frac{d-2}{d-1} |Y|^{\frac{d-2-(d-1)}{d-1}} \nabla Y, (1-\cos\phi) \nabla |Y|^{\frac{d}{d-1}} + |Y|^{\frac{d}{d-1}} \nabla (1-\cos\phi) \right\rangle \mathrm{d}vol g_s \\
&= \int_{s^d} \left\langle \frac{d-2}{d-1} |Y|^{\frac{-1}{d-1}} \nabla Y, (1-\cos\phi) \frac{d}{d-1} |Y|^{\frac{1}{d-1}} \nabla Y + |Y|^{\frac{d}{d-1}} \nabla (1-\cos\phi) \right\rangle \mathrm{d}vol g_s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t} f[u] = &\frac{8}{d(d-2)} |S^d|^{\frac{-2}{d}} \|Y\|_{L^{\frac{2d}{d-1}}}^{\frac{4}{d-1}} \Bigg(\int_{s^d} \Bigg\langle \frac{d-2}{d-1} |Y|^{\frac{-1}{d-1}} \nabla Y\ , \\
&(1-\cos\phi) \frac{d}{d-1} |Y|^{\frac{1}{d-1}} \nabla Y + |Y|^{\frac{d}{d-1}} \nabla (1-\cos\phi) \Bigg\rangle \\
&\quad + \frac{d(d-2)}{4} |Y|^{\frac{d-2}{d-1}} (1-\cos\phi) |Y|^{\frac{d}{d-1}} \mathrm{d}vol g_s \Bigg) \\
&\quad - \int_{s^d} 2((1-\cos\phi) Y^{\frac{d}{d-1}}) |Y|^{\frac{d+2}{d-1}} \, \mathrm{d}vol g_s \\
= &\frac{8}{d(d-2)} |S^d|^{\frac{-2}{d}} \|Y\|_{L^{\frac{2d}{d-1}}}^{\frac{4}{d-1}} \Bigg(\int_{s^d} \Bigg\langle \frac{d-2}{d-1} |Y|^{\frac{-1}{d-1}} \nabla Y\ , \\
&(1-\cos\phi) \frac{d}{d-1} |Y|^{\frac{1}{d-1}} \nabla Y \Bigg\rangle \\
&+ \left\langle \frac{d-2}{d-1} |Y|^{\frac{-1}{d-1}} \nabla Y, |Y|^{\frac{d}{d-1}} \nabla (1-\cos\phi) \right\rangle \\
&\quad + \frac{d(d-2)}{4} |Y|^{\frac{d-2}{d-1}} (1-\cos\phi) |Y|^{\frac{d}{d-1}} \mathrm{d}vol g_s \Bigg) \\
&\quad - \int_{s^d} 2((1-\cos\phi) |Y|^{\frac{d}{d-1}}) |Y|^{\frac{d+2}{d-1}} \, \mathrm{d}vol g_s
\end{aligned}$$

Let us consider part of the above equation:

$$\int_{s^d} \left\langle \frac{d-2}{d-1} |Y|^{\frac{-1}{d-1}} \nabla Y, |Y|^{\frac{d}{d-1}} \nabla (1-\cos\phi) \right\rangle \mathrm{d}vol g_s$$

$$= \int_{s^d} \left\langle \frac{d-2}{d-1} |Y|^{\frac{-1}{d-1}} Y^{\frac{d}{d-1}} \nabla Y, \nabla\left(1-\cos\phi\right) \right\rangle \mathrm{d}volg_s$$
$$= \int_{s^d} \left\langle \frac{d-2}{d-1} Y \nabla Y, \nabla\left(1-\cos\phi\right) \right\rangle \mathrm{d}volg_s$$
$$= \int_{s^d} \left\langle \frac{d-2}{2(d-1)} \nabla Y^2, \nabla\left(1-\cos\phi\right) \right\rangle \mathrm{d}volg_s$$
$$= \int_{s^d} -\frac{(d-2)}{2(d-1)} Y^2 \Delta\left(1-\cos\phi\right) \mathrm{d}volg_s.$$

According to Peterson [6],$-\Delta(1-\cos\phi) = -d\cos\phi = d(1-\cos\phi) - d$. Hence, the above equation reduces to,

$$\int_{s^d} -\frac{-d(d-2)}{2(d-1)} Y^2((1-\cos\phi)-1)\mathrm{d}volg_s$$
$$= \int_{s^d} \frac{d(d-2)}{2(d-1)} Y^2\left(1-\cos\phi\right) - \frac{d(d-2)}{2(d-1)} Y^2 \mathrm{d}volg_s$$

Therefore, putting all the previous results together we can see the following:

$$\frac{\partial}{\partial t} f[u] = \frac{8}{d(d-2)} |s^d|^{\frac{-2}{d}} \|Y\|^{\frac{4}{d-1}}_{L^{\frac{2d}{d-1}}} \Bigg(\int_{s^d} \Bigg\langle \frac{d-2}{d-1} |Y|^{\frac{-1}{d-1}} \nabla Y,$$
$$\left(1-\cos\phi\right) \frac{d}{d-1} |Y|^{\frac{1}{d-1}} \nabla Y \Bigg\rangle$$
$$+ \frac{d(d-2)}{2(d-1)} Y^2\left(1-\cos\phi\right) - \frac{d(d-2)}{2(d-1)} Y^2$$
$$+ \frac{d(d-2)}{4} |Y|^{\frac{d-2}{d-1}} \left(1-\cos\phi\right) Y^{\frac{d}{d-1}} \Bigg) \mathrm{d}volg_s$$
$$- \int_{s^d} 2((1-\cos\phi) Y^{\frac{d}{d-1}}) Y^{\frac{d+2}{d-1}} \mathrm{d}volg_s$$

$$= \frac{8}{d(d-2)} |s^d|^{\frac{-2}{d}} \|Y\|^{\frac{4}{d-1}}_{L^{\frac{2d}{d-1}}} \int_{s^d} \frac{d(d-2)}{(d-1)^2} |\nabla Y|^2 \left(1-\cos\phi\right)$$
$$+ \frac{d(d+1)(d-2)}{4(d-1)} Y^2\left(1-\cos\phi\right) - \frac{d(d-2)}{2(d-1)} Y^2 \mathrm{d}volg_s$$
$$- \int_{s^d} 2((1-\cos\phi) Y^{\frac{2(d+1)}{d-1}}) \mathrm{d}volg_s. \qquad \square$$

3.2 Proof of Theorem 2.1.2

Proof. The general idea of the proof is to take the GNS inequality (1.2.3) and pull it back by stereographic projection.

$$C_{GNS}(\int_{\mathbb{R}^d} |g|^{\frac{2d}{d-1}} \mathrm{d}volg_{\mathbb{R}})^{\frac{2}{d}} \int_{\mathbb{R}^d} g(-\Delta)g\, \mathrm{d}volg_{\mathbb{R}} \geq \int_{\mathbb{R}^d} |g|^{\frac{2(d+1)}{d-1}} \mathrm{d}volg_{\mathbb{R}}$$

Next we change variables using stereographic projection. The constant remains unchanged.

$$\begin{aligned} &C_{GNS}(\int_{s^d} |\Phi^* g|^{\frac{2d}{d-1}} (1-\cos\phi)^{-d} \,\mathrm{d}volg_s)^{\frac{2}{d}} \\ &\times \int_{s^d} \Phi^* g\,(1-\cos\phi)^{\frac{d+2}{2}} L[(1-\cos\phi)^{\frac{-(d-2)}{2}}\, \Phi^* g]\,(1-\cos\phi)^{-d}\, \mathrm{d}volg_s \geq \\ &\int_{s^d} |\Phi^* g|^{\frac{2(d+1)}{d-1}} (1-\cos\phi)^{-d}\, \mathrm{d}volg_s \end{aligned} \tag{3.2}$$

By the work of Case (see Proposition 3.3 in [3]), the result in Theorem 1.2.3 is equivalent to (3.2), since stereographic projection is a conformal transformation. Therefore, we can conclude that Theorem 1.2.4 from [2] implies that Theorem 2.2.1 is true. □

Chapter 4

Characterization of Steady State Solutions

Lemma 4.0.1. *Consider the map* Φ *as defined in 1.3.1 and where* h *is given by 1.2. Then the conformal factor* $(1 - x^{n+1})$ *to the power* $p = 1 - 2d + \lambda$ *is such that,*

$$(1 - x^{n+1})^p h(\Phi(x)) = \frac{(1 - x^{n+1})}{[(2(1 - x^{n+1})]^{\frac{2d-\lambda}{2}}}.$$

Proof. We have

$$\begin{aligned} h(\Phi(x)) &= (\frac{1}{1 + |\Phi(x)|^2})^{\frac{2d-\lambda}{2}} \\ &= (\frac{1}{1 + |\bar{x}|^2 |\frac{1}{1-x^{n+1}}|^2})^{\frac{2d-\lambda}{2}} \\ &= (\frac{(1 - x^{n+1})^2}{(1 - x^{n+1})^2 + |\bar{x}|^2})^{\frac{2d-\lambda}{2}}. \end{aligned}$$

Next, we need to find the appropriate power of the conformal factor $(1 - x_{n+1})$ in which the choice of power cancels out exactly the $(1 - x_{n+1})^p$ term in the numerator.

$$(1-x^{n+1})^p h(\Phi(x)) = (1-x^{n+1})^p (\frac{(1-x^{n+1})^2}{(1-x^{n+1})^2+|\bar{x}|^2})^{\frac{2d-\lambda}{2}}$$
$$= \frac{(1-x^{n+1})^{p+2d-\lambda}}{[(1-x^{n+1})^2+|\bar{x}|^2]^{\frac{2d-\lambda}{2}}}$$
$$= \frac{(1-x^{n+1})^{p+2d-\lambda}}{[(2(1-x^{n+1})]^{\frac{2d-\lambda}{2}}}$$

We desire the numerator of the above equation to be $(1-x^{n+1})$,

$$p+2d-\lambda = 1 \implies p = 1-2d+\lambda$$ □

The conformal group on the sphere is generated by dilations and rotations. Specifically, [5] defined this as the general conformal transformation with denoting O as the rotations and D as the dilations.

$$\Psi_{O,\delta} = \Phi^{-1} D_{\delta(t)} O_t \Phi$$

Lemma 4.0.2. *The steady- state solutions found by Carlen, Carillo, and Loss [2] is the following:*

$$u(x, e^{\beta t}) = Cs^{-d}(t)\left[h\left(\frac{x}{s(t)} - x_o(t)\right)\right]^{\frac{d+2}{d-1}}. \tag{4.1}$$

Therefore, the general steady state solution on the sphere is $v(t) = (\Phi\Psi_{O,\delta})^* u^{CCL}$ *where* u^{CCL} *is* (4.1). *Hence the general solution is the same steady state solutions as in [2] except with the pullback.*

Remark 4.0.3. *Therefore,* $v(t) = Cs^{-d}(t)\big[h(O_t D_{s(t)}\Phi)\big]$ *where h is the same h in Lemma 4.0.1.*

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Academic Vita

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Education

The Pennsylvania State University August 2014- May 2019

B.S. Mathematics and B.S. Statistics

Work Experience

Department of Defense, U.S. Army Armaments Research, Development and Engineering Center

Summer Intern, SMART Scholarship Program Summer 2018

- Responsible for using statistical analysis and creating mathematical algorithms for data sets.
- Organized advanced processes
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Learning Center, Penn State Berks

Mathematics, Statistics, Economics, and Chemistry Tutor January 2016- August 2016

- Aided students in learning computational thinking in regards to problem solving.
- Broke down complex problems to individual learning levels.
- Analyzed student comprehension and prepared goals.

Academic Dean's Office, Penn State Berks

Research Aide for Academic Dean Spring 2016-December 2016

- Assisted in editing professional and research papers for publication.
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Research

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- The fast diffusion equation is $u_t=-\Delta u^{\beta}$ for some $0<ß<1$. There is a conformally invariant analogue of the fast diffusion equation which has an interesting relationship to sharp Gagliardo--Nirenberg inequalities. This project involves carrying out a detailed study of this flow on the round spheres and using the insights gained to study the asymptotic behavior of the flow on general manifolds In particular, there seems to be an interesting relationship to static metrics that is worth exploring.

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Statistical Research Assistant for Mathematics faculty Nate Brown PhD. Spring 2017- Spring 2018

- Responsible for using statistical analysis and creating mathematical algorithms for data sets.
- Assisted faculty in creating research questions and criteria for study of possible gender inequalities in mathematics.
- Awarded the *Penn State Women in Math Research Scholarship* for continued research work.

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Statistical Research Assistant for Chemistry faculty member Spring 2016 – Spring 2017

- Created mathematical algorithms through coding in R.
- Used Data Science to come up with statistical correlations and possible causations from the data.
- Entrusted with private student information and data.
- Found significant factors to predict student's performance in chemistry classes

Presentations, Conferences, and Posters

Research Poster, PSU Undergrad Research Exhibition	April 2018
Research presenter, BCCE conference	August 2016
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Student speaker and workshop leader for Careers with Math Options Conference	May 2016

Honors and Awards

Society of Distinguished Alumni protégée program 2018
Penn State University Mass Program, Fall 2018
Best Project Award in Mass Program, Fall 2018
Member of Penn State Schreyer Honors College
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Systems; Microsoft Office Suite, Windows Operating System, Mac OSX,
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