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A COMPARATIVE ANALYSIS OF CONVERTIBLE BOND PRICING MODELS  

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Abstract

Convertible bonds have existed for over 150 years, and are academically interesting to research given that they have both stock- and bond-like components. In going through basic pricing models for each component of a convertible bond, including stocks, bonds, and options, a rudimentary pricing model is presented for convertible bonds. The 1997 Goldman Sachs convertible bond pricing model is also presented, after which the two models are compared and discussed. The rudimentary pricing model presented has some problematic assumptions but thoroughly explains each component of a convertible bond, while the Goldman Sachs model is simpler and easier to understand, but is less applicable to other areas of finance.
Acknowledgements

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I would also like to lovingly acknowledge my mother, Mary Conway, and my sister, Kathleen Conway, for their unconditional support.

⋄ ⋄ ⋄

This thesis is humbly dedicated to my late father, Arthur Conway, from whom my passion for mathematics was born.
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1 Introduction

The goal of this thesis is to provide a start-to-finish approach to a rudimentary convertible bond pricing model, an explanation of the Goldman Sachs convertible bond pricing model, and a comparison of the two models.

Convertible bonds have existed for more than 150 years[1]. They are typically issued by companies with low credit ratings as a way to generate capital more cheaply than they would issuing regular bonds, since investors also get equity upside with convertibles[2]. This is explained in further detail in Subsection 1.1, below. The motivation behind this thesis lies in the complexity of convertible bonds; this single asset class contains both fairly straightforward pricing theory behind straight bonds with more theoretical and hotly debated stock valuation aspects. Additionally, there is far less research available on convertible bond pricing models than there is for their stock or straight bond cousins.

There are three main components to this research: the explanation of a convertible pricing model proposed by this paper and a hypothetical example, followed by an explanation of the 1997 Goldman Sachs model with the same hypothetical example, and a thorough comparison of the assumptions and mechanics underpinning each.

We will first explain the basics of stocks, bonds, options, and convertible bonds in Subsections 1.2 through 1.5. Once sufficient understanding of underlying securities is attained, bond, stock, and stock option pricing models will be discussed in the second section. Bond pricing models are fairly straightforward and universal and will be presented first, after which the more complex stock price model used will be created using the binomial tree method. After that, the Black-Scholes equation for valuing stock options will be presented and explained, so that it can be used to build a convertible bond pricing model.

In section three, a fairly basic convertible bond valuation model will be presented and will incorporate the models previously discussed. Subsequently, an explanation and derivation of the 1997 Goldman Sachs Convertible Bond Pricing Model will be presented in section four. An example will be provided to demonstrate how the model works.

Finally, the comparison of the two models will be presented. The assumptions underpinning each model will be thoroughly analyzed and explained. Subsequently, the actual mechanics of each model will be compared and discussed.

1.1 Financial Overview

In this section we will introduce and discuss the financial instruments the models serve. These instruments are called “securities,” which are defined as tradable, fungible financial assets. Securities come in a variety of forms, some of which are ownership in a company (known as equity or stock) or pieces of debt, including principal repayment and periodic interest payments (bonds)[3]. Convertible bonds are an example of a more complex security with both bond- and stock-like attributes.
By analyzing securities, one can determine good investment opportunities that provide capital gains and/or income via interest payments, and other benefits. For example, if you buy a security today that doubles in price tomorrow, you can then sell it and return double your original investment. Conversely, if you buy a security today that halves in price tomorrow, you have lost half of your investment. When making financial investments, one might attempt to buy securities expected to appreciate in value, or those one perceives as under-valued by the marketplace. Thus, using and understanding valuation methods for securities can be a crucial piece of investing.

1.2 Bonds

A bond is a debt instrument representing a loan from a lender to a borrower ("issuer" of the debt). There are three main types of bonds: corporate bonds, issued by corporations, sovereign bonds, issued by national governments, and municipal bonds, issued by local governments (such as the Chicago Board of Education, or the State of New Jersey)[4].

The initial amount of most bonds issued by US corporations is $1000, and corporations can issue thousands and thousands of bonds on the primary market in order to raise billions of dollars from many individual lenders. After bonds are issued, they are free to be bought and sold on the secondary market, and change hands quite frequently[5]. Issuers pay periodic interest payments, and after a predetermined period of time, the bondholders are paid back their initial investment by the issuer. A summary of basic bond characteristics and terminology may be found below:

1. Par Value: The initial price of a bond; $1000 for most corporations, as described above. Also known as face value or principal

2. Interest payments: made periodically during the life of a bond. The size of interest payments is measured as a percentage of the principal value of the bond (interest rate). Issuers are motivated to have lower interest rates to make their debt less expensive to borrow; lenders have the opposite motivation.

3. Maturity: The date on which a bond’s principal will be paid back

4. Price: The price a bond currently trades at on the secondary market, independent of the initial price or par value.

5. Time to Maturity (TTM): The amount of time until a bond matures. Bonds with higher TTM is usually accompanied by a higher interest rate because they have higher interest rate risk, or more time for interest rates to potentially rise and thus devalue the bond.

6. Credit rating: a measure of the trustworthiness of a borrower to pay back their debt in a timely manner. Credit ratings are produced by credit rating agencies, such as Fitch, Standard and Poor’s, or Moody’s[6]. Bonds
with lower credit ratings tend to pay higher interest rates as a way to compensate for additional risk.

7. Yield: moves inversely to bond price, a measure of interest payments relative to how much a bond is trading for on the markets

8. Call Provisions: An option that allows the issuer to “call back”, or repurchase the bonds at a predetermined price. One reason an issuer might want to do this is because if interest rates start lowering, the issuer can call their outstanding bonds and issue new ones at a newer, lower rate. Call provisions provide protection to the issuer and put upward pressure on interest payments required by investors[7].

9. Put Provisions: An option that allows the bondholders to sell the bonds back to the issuer at a predetermined price. Bondholders would do this in a rising-rate scenario, or if the bonds rapidly start losing value. Put provisions provide protection to investors and thus put downward pressure on interest rates investors will accept[8].

1.3 Stocks

Stocks are a type of security that give fractional ownership in a corporation. Stocks are also known as equities; owners of stocks are called “stockholders” or “shareholders”.

All companies have stock, since stock simply represents ownership, but not all stocks are available to the public for purchase. Companies are either classified as private or public; public corporations have shares available for purchase to the general public, while private companies have shares not available to the public. Public shares are bought and sold on a stock exchange, which is an electronic marketplace designed to instantaneously receive and fill orders for buyers and sellers of stocks. Commonly known stock exchanges include the New York Stock Exchange (NYSE) and the NASDAQ of the United States and the London Stock Exchange of the UK, but stock exchanges exist in almost every major metropolis worldwide[9].

One way for a corporation to raise money is to go from private to public, and offer their shares for sale to the public in a process known as an Initial Public Offering (IPO). In an IPO, banks team up with a corporation to gauge market demand for their equity, eventually deciding at a price each share will be offered at and how many total shares will be sold. As this process happens, a symbol for the corporation, known as a “ticker,” is assigned within the stock exchange, and the shares become publicly available, and proceeds from this process go directly to the corporation issuing the shares (minus a cut from the investment banks facilitating the process)[10].

The price of stocks varies over time depending on market supply and demand. The supply and demand for specific stocks in the market is largely influenced by news specific to a single stock, such as corporate earnings reports or management statements, and general economic data, such as changes in interest rates or tax
policy. Some stocks’ prices can change drastically over a short period of time, while others remain largely stable for long periods of time; the statistical concept of volatility relates to the tendency of certain stocks to vary wildly in price and others to hardly move at all[11]. This will be discussed in further detail in section 3.1.

Owning stocks come with a variety of rights and privileges, including the right to vote in shareholder meetings, which give shareholders a degree of control in how a company is managed. Shareholders receive dividends, which are a distribution of profits returned on a per-share basis[12]. Finally, and perhaps obviously, stock ownership gives owners the right to sell their shares on an exchange for higher (lower) than the original purchase price, which results in capital appreciation (depreciation) for the seller.

1.4 Options

Options are a derivative (a security that derives its value from the value of another security) on stocks that give the right but not the obligation for the holder to buy or sell stocks at a predetermined price, known as the “strike price”[13]. There are two basic classes of options: puts and calls. Puts grant the right but not the obligation to sell at the strike price; calls grant the right but not the obligation to buy at the strike price. This subsection will deal with call options exclusively because of their relation to convertible bonds. The action of using the call option to purchase the stock at the strike price is known as ”exercising”. Call options have an expiry date, or a date after which they expire; thus, any decisions by call option owners to trade or exercise their options must be made on or before this date[14].

Two of the most common types of options are known as ”European” and ”American” options; these distinctions have nothing to do with geographical location. European options are the most basic, and may be exercised at the time of expiration but not before, but may be traded at any time. American options may be traded or exercised at any time on or prior to expiration. When the stock price exceeds the strike price of an option, it is known as ”in-the-money”. Conversely, when the strike price exceeds the stock price, the option is known as ”out-of-the-money”.

The profit (or loss) from a position is the payoff less the price paid for the option if the option is in-the-money. If the option is out-of-the-money, the loss is only the price paid for the option, since it would just expire without being exercised. Therefore, there is limited downside and virtually unlimited upside when purchasing call options. The most one can lose is the price paid for the option, and profits are limited only by how high the stock price can grow[15].

It is never optimal to exercise an American call option on non-dividend paying stock prior to expiration[16]. To demonstrate the rationale behind this statement, consider an investor who plans to exercise an in-the-money option prior to expiration and hold on to it for the remainder of the life of the option. Exercising prior to expiration would mean the investor would forego potential interest earned on the cash used to buy the stock. Additionally, if the option
were to become out-of-the-money, it would be advantageous to have waited to buy the stock on the market at a cheaper price. Furthermore, consider an investor who feels the underlying the stock is over-priced and wants to sell. He, too, would be better off trading the option rather than exercising and selling the stock. This is because there always exists buyers in the options market who feel the stock is appropriately priced and would want to buy and hold the option; if none such buyers existed, the stock price would fall until demand increased.

We will now consider certain factors that effect the value of call options, both American and European. The value of call options increase with an increasing stock price and a decreasing strike price, because both of these things directly improve payoff. Additionally, the value of options also increase with more time to expiration. This is because more time to expiration represents more opportunities to trade or exercise the option. Finally, volatility in the underlying stock price also increases the value of a call option. As volatility increases in the stock price, the chance that the stock appreciates or depreciates rapidly becomes greater. Because there is limited downside with call options, the option owner benefits more from the upside chance of the stock price increasing.

1.5 Convertible Bonds

Convertible bonds are a hybrid security that display both bond- and stock-like features. A convertible can be compared to a combination of two different securities: a straight bond issued from a corporation, and a call option on that bond that allows one to trade it for equity. This is because a convertible bond is essentially a bond, with a principal amount and periodic interest payments, which may be exchanged for a predetermined number of shares of the issuer’s stock prior to the maturity of the bond [17]. When a convertible bond is converted to stock, all remaining interest payments and principal repayment are foregone.

A summary of terms relating to convertibles may be found below:

1. Maturity: The date on which a bond’s principal will be paid back
2. Principal amount: The defined face amount for the bond.
3. Conversion Price: the price at which the bond may be converted to stock
4. Conversion Ratio: the number of shares underlying each bond.
5. Conversion Value (Parity): Current value of the shares underlying each convertible bond; may be calculated by multiplying the conversion ratio by the current stock price.
6. Call Provisions: An option that allows the issuer to “call back”, or repurchase the convertible bonds at a predetermined price.
7. Put Provisions: An option that allows the convertible bondholders to sell the convertible bonds back to the issuer at a predetermined price.
A corporation might decide to issue convertibles because they give the issuer a lower cost of debt. Convertibles allow the issuer to have lower interest rates on the bonds they issue because the built-in call option also adds value to the bond. From an investor perspective, one would accept lower interest payments in exchange for the potential to convert to equity during the life of bond. Investors might also be interested in convertibles because they give the potential to be equity investors in high-growth companies with greater stability of income than straight equity offers[18].
2 Pricing Models

Note: For ease of reading, we will denote partial derivatives as such: \( f_t = \frac{\partial f}{\partial t} \) is the partial derivative of \( f \) with respect to \( t \); \( f_{tt} = \frac{\partial^2 f}{\partial t^2} \) is the second partial derivative of \( f \) with respect to \( t \).

2.1 Bond Pricing

Since bond interest and principal payments are fixed in time, and thus theoretically known at every point during the life of the bond, pricing bonds is a fairly straightforward and objective matter. A bond’s value is simply the present value of the remaining coupon payments plus principal repayment, discounted at the risk-free rate plus a spread accounting for default risk, or the possibility that the issuer may not be able make good on promised payments.

In practice, this credit-adjusted discount rate could be determined as a spread off of the US Treasury of similar maturity, as US Treasuries are considered risk-free assets. A wider spread between the credit-adjusted rate and the base rate would reflect increased credit risk, which would lower the value of the bond; the converse is true for a tighter spread.

Thus, for a bond with principal \( P \) and annual coupon payments \( C \), given risk-free rate \( r \) and risk-adjusted credit spread \( s \), compounding over \( n \) periods, the value \( B \) of a bond is given by the equation below, borrowed from Jean Folger[19]:

\[
B = C \left(1 + s\right) + \frac{C}{(1 + r + s)^2} + ... + \frac{C}{(1 + r + s)^n} + \frac{P}{(1 + r + s)^n} \quad (1)
\]

Or, equivalently:

\[
B = C \times \left(1 - \frac{1}{(r+s)^n}\right) + \frac{P}{(r+s)^n} \quad (2)
\]

Thus, the value of a bond is hugely determined by one’s view on a company’s credit risk. By setting \( B \) equal to a bond’s market price and solving for the \( s \) term, one can determine the view of the market as a whole on a bond’s credit risk, in the form of a spread off the risk-free rate. In this case, the \((r+s)/r\) term is known as the bond’s yield. This illustrates the inverse relationship between a bond’s yield and price. When analysts say that a bond’s yields have been rising, the periodic payments of the bond’s coupons and principal are being discounted by a higher rate in the equation above, thus lowering the value or price of the bond.

Let us consider an example. Suppose a bond has $100 principal with an interest rate and yield of 10%, compounded annually, and matures in 5 years. According to (2), the current value of the bond is:

\[
B = 10 \times \frac{(1 - \frac{1}{1.1^5})}{.1} + \frac{100}{(1.1)^5} \quad (3)
\]
Which turns out to be $100.

If the bond has four years to maturity, all else equal, then the current value of the bond is:

\[ B = 10 \times (1 - \left(\frac{1}{(1.1)^4}\right) + \frac{100}{(1.1)^4} \]

which turns out to also be $100

### 2.2 Stock and Option Pricing

In this section, stock price theory and a model for stock pricing will be presented, upon which the Black-Scholes equation for option pricing will build. The basic idea of the stock price theory presented here assumes that stock prices roughly follow a mean growth rate, with random "white noise" deviations from the mean rate. This is a concept known as the Wiener process, which is a continuous-time stochastic process named in honor of Norbert Wiener\(^{[20]}\).

A Wiener process \( W_t \) is characterized by four conditions\(^{[21]}\):

1. \( W_0 = 0 \)
2. \( W_t \) is almost surely continuous
3. \( W_t \) has independent increments
4. \( W_t - W_s \approx N(0, t - s) \) for \( 0 \leq s \leq t \)

This is roughly in agreement with the Efficient Markets Hypothesis (EMH), upon which most of modern financial theory lies. EMH is a theory that presumes stock prices are based upon all relevant, publicly available information, and thus, all stocks trade at their fair value and it is impossible to consistently beat the market\(^{[22]}\). While there is a large body of evidence that supports EMH, detractors may point to a few extremely abnormal events, such as Black Monday of the 1987 stock market crash, when the stock market fell by more than 20% in a single trading session, or the volatility in the markets in calendar year 2018, as evidence to the contrary\(^{[23]}\).

### 2.3 Stochastic Stock Price model

Borrowed from John Hull’s *Options, Futures, and Other Derivatives*\(^{[24]}\). Suppose a stock price at time \( t, S_t \) follows the stochastic differential equation

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (4) \]

where \( W_t \) is a standard Brownian motion and for constants \( \mu \) and \( \sigma \). Consider also a function \( f \) on the stock at time \( t \), given by \( f(t, S_t) \). Ito’s lemma states\(^{1}\):

\[ df = [f_t + \mu S_t f_S + \frac{\sigma^2}{2} f_{SS}] dt + \sigma S_t f_S dW_t, \quad (5) \]

\(^1\)An informal proof of Ito’s Lemma is provided to the reader in the Appendix
If we define $G = \ln(S_t)$ and apply Ito’s Lemma, we get:

$$dG = [G_t + \mu S_t G_S + \frac{\sigma^2 S_t^2}{2} G_{SS}]dt + \sigma S_t G_w dW_t.$$ 

By noticing that:

1. $G_t = \frac{\partial G}{\partial t} = \frac{\partial \ln(S_t)}{\partial t} = 0$
2. $G_S = \frac{\partial G}{\partial S_t} = \frac{1}{S_t}$
3. $G_{SS} = \frac{\partial^2 G}{\partial S_t^2} = -\frac{1}{S_t^2}$

We get:

$$dG = [\mu S_t \frac{1}{S_t} - \frac{\sigma^2 S_t^2}{2} \frac{1}{S_t^2}]dt + \sigma S_t \frac{1}{S_t} dW_t.$$ 

Or,

$$d(\ln(S_t)) = \ln(S_t) - \ln(S_{t-1}) = \ln\left(\frac{S_t}{S_{t-1}}\right) = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t$$

Therefore, we can arrive at the formula for a stock price:

$$\ln(S_t) = \ln(S_{t-1}) + (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t \quad (6)$$

or

$$S_t = S_{t-1} \exp[(\mu - \frac{\sigma^2}{2})dt + \sigma dW_t] \quad (7)$$

There remains one problem, however. This formula contains both deterministic $dt$ and stochastic $dW_t$ term. In order to be of use in determining future stock prices, we must come up with an approximation for $dW_t$ [25]:

$$dW_t = W_{t+\delta t} - W_t$$

Additionally, for some standard normal $\epsilon$:

$$1 = \text{Var}(\epsilon) = E(\epsilon^2) - E(\epsilon)^2 = E(\epsilon^2)$$

And in accordance with the fourth condition for a Wiener process from subsection 2.2,

$$E(dB_t^2) = E(\Delta t \epsilon^2) = \Delta t E(\epsilon^2) = \Delta t$$

$$\text{Var}(dW_t^2) = \Delta t^2 \text{Var}(\epsilon^2)$$

and is of order $\Delta t^2$. Thus in the limit as $\Delta t$ approaches zero:

$$\text{Var}(dW_t^2) << \Delta t = E(dW_t^2)$$
so $dW_t^2$ converges to $\Delta t^2$. Thus, we can use $\pm \sqrt{\Delta t}$ as an approximation for $dW_t$. Equation (7) then becomes:

$$S_t = S_{t-1} \exp[(\mu - \frac{\sigma^2}{2})\Delta t \pm \sigma \sqrt{\Delta t}]$$  \hfill (8)

Note that the $\pm$ term originates from an assumption that the stock will appreciate at the mean rate and deviate off that mean rate with equal probability above or below the mean rate.

### 2.4 Options Pricing: The Black-Scholes Equation

The Black-Scholes Equation builds upon the previous sections to model the prices for European call option. Additionally, because it was demonstrated in 1.4 that it is never optimal to exercise an American call option prior to expiration, an argument can be made this equation also holds for American call options on non-dividend paying stock[13]. Borrowed from *The Pricing of Options and Corporate Liabilities* by Fischer Black and Myron Scholes[26].

For one European call option on an underlying stock with price $S_t$, we make the following assumptions:

1. The stock follows Brownian motion; that is, $dS_t = \mu S_t dt + \sigma S_t dW_t$ from equation (4).
2. Risk free rate $\mu$ and volatility $\sigma$ are known with certainty
3. No transaction costs exist, no dividends will be paid out from the underlying stock, and no arbitrage opportunities exist.
4. Continuous trading is possible with short-selling permitted, and one can buy or sell any number of any assets, not necessarily an integral number.

For call option with value $V$ that is a function of stock price $S_t$ and time to expiration $t$, we can use Ito’s lemma, as derived in the Appendix 5.1, to arrive at the following equation:

$$dV = (V_t + \mu S_t V_S + \frac{\sigma^2 S_t^2}{2} V_{SS})dt + \sigma S_t V_S dW_t$$  \hfill (9)

Next, we construct a portfolio with value $\pi$ that is long exactly one option and short $\Delta$ number of shares of the underlying stock. Remember that according to the assumptions, $\Delta$ can be any real number and is not limited to the set of all positive integers. The value of the portfolio is:

$$\pi(t) = V - \Delta S_t$$  \hfill (10)

And the change in the value of the portfolio in one time-step $dt$ is:

$$d\pi(t) = dV - \Delta dS_t$$  \hfill (11)

\footnote{The mathematical symbol "<<" means "is of smaller order than," or "the growth is bounded by"}
Substituting $dS$ from equation (4) and $dV$ from equation (12) into equation (14), we get:

$$d\pi(t) = (V_t + \mu S_t V_S + \frac{\sigma^2 S_t^2}{2} V_{SS}) dt + \sigma S_t V_S dt - \mu S_t dt - \sigma S_t dW_t$$

or,

$$d\pi(t) = (V_t + \mu S_t V_S + \frac{\sigma^2 S_t^2}{2} V_{SS} - \Delta S_t) dt + (\sigma S_t V_S - \Delta S_t) dW_t$$

(12)

Note that there are two terms in the above equation, a deterministic term ending in $dt$ and a stochastic term ending in $dW_t$. We can simplify this equation through our choice in $\Delta$. By choosing $\Delta = V_S$, we can eliminate the stochastic term, leaving us with a completely deterministic equation:

$$d\pi = (V_t + \frac{\sigma^2 S_t^2}{2} V_{SS}) dt$$

(13)

Now, according to the no arbitrage assumption, this portfolio should appreciate in value at the risk free rate. To motivate this statement, let us consider the contrary. If the portfolio were to grow at higher than the risk-free rate, then investors could borrow money from the bank, invest in the portfolio and return the difference between the two rates, essentially getting risk-free money, which violates the no arbitrage assumption. If the portfolio were to return less than the risk-free rate, investors could short shares of the portfolio and invest proceeds in the bank and get risk-free money in that way, again violating the no-arbitrage assumption.

Thus, assuming the portfolio grows at the risk free rate, we can replace $\frac{d\pi}{dt}$ with $r\pi$; thus, equation (16) becomes:

$$r\pi = V_t + \frac{\sigma^2 S_t^2}{2} V_{SS}$$

(14)

Now, substituting $\pi$ for $V - \Delta S_t$ by equation (9) and $V_S$ for $\Delta$ as before, we are left with:

$$r (V - V_S S_t) = V_t + \frac{\sigma^2 S_t^2}{2} V_{SS}$$

(15)

which can be rearranged into

$$V_t + \frac{\sigma^2 S_t^2}{2} V_{SS} + r S_t V_S - r V = 0$$

(16)

which is the Black Scholes Partial Differential Equation. When (15) is solved with boundary conditions depicting a European call option with strike price $K$, $f(S, T) = \max(S - K, 0)$, we get the Black Scholes price of the option:

$$c = c(K, r, S_t, t, T, \sigma) = S_t \phi(d_1) - Ke^{-r(T-t)} \phi(d_2)$$

(17)

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = d_1 - \sigma \sqrt{T-t}$$

where:
2.5 An example

Let us now consider an example, using the formulas derived for stock and European call option prices. Consider the stock satisfying the parameters in Table 1.

<table>
<thead>
<tr>
<th>Current stock price</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility</td>
<td>10% per year</td>
</tr>
<tr>
<td>Riskless rate</td>
<td>5% per year</td>
</tr>
</tbody>
</table>

Table 1: A Hypothetical Stock

Given our equation,

\[ S_t = S_{t-1} \exp[(\mu - \frac{\sigma^2}{2})\Delta t + \sigma \sqrt{\Delta t}] \]

we can solve the equation for the price of the stock one time-step into the future:

\[ S_1 = $100 \exp[(5\% - \frac{(10\%)^2}{2})](1) \pm 10\% \sqrt{T} \]

which equals $115.60 or $94.65 for the upward/downward movements.

Now let us build on this example by considering the value of an option on the stock given in Table 1, satisfying the parameters given in Table 2:

<table>
<thead>
<tr>
<th>T</th>
<th>5 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>various</td>
</tr>
<tr>
<td>Strike Price</td>
<td>$100</td>
</tr>
</tbody>
</table>

Table 2: A Hypothetical Option on the Hypothetical Stock

The value \( V \) of an option is given by equation (17):

\[ c = c(K, r, S_t, t, T, \sigma) = S_t \phi(d_1) - Ke^{-r(T-t)} \phi(d_2) \]
\[
\begin{align*}
d1 &= \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \\
d2 &= d1 - \sigma \sqrt{T-t}
\end{align*}
\]

At the onset, we have \((T-t) = 5 - 0 = 5\) and \(S_0 = $100\). Plugging these and the inputs given in Table 1, we arrive at a European Call Price of $23.42^3. One time step (in this case, one year) into the future, we have \((T-t) = 5 - 1 = 4\) and \(S_1 = $115.60\) or \($94.65\). With these inputs, we arrive at European Call Option Prices of $34.06 or $15.18, respectively.

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^3Left for the reader to verify, if desired
3 Proposed Convertible Bond Pricing Model

As stated in Section 1, convertibles can be thought of as a combination of a straight bond and a call option on the underlying stock. Therefore, we propose the model for valuing convertibles as the sum of the bond and call option components. Consider the convertible satisfying the following parameters:

| Principal | $100 |
| Coupon | 10% per year, compounded annually |
| Maturity | 5 years |
| Conversion Ratio | 1 |

| Starting stock price | $100 |
| Volatility | 10% per year |
| Riskless rate | 5% per year |

Table 3: A Hypothetical Convertible Bond

Let us consider the value of the convertible with 5 years to maturity: the bond price we calculated in Section 2.1 was $100, independent of time to maturity. This makes up one component of our convertible bond price. The initial option price we calculated in Section 2.5, which has the same parameters as in this section, was $23.42.

Our proposed method of pricing convertible bonds is to simply add these two components. This would lead us to arrive at a convertible bond price of $123.42 initially.

Let us now consider one year into the future, when the underlying stock price is either $115.60 or $94.65 with equal probability. The bond price is still $100, but the call option price is now $34.06 or $15.18, depending on whether the underlying stock appreciated or depreciated in value. This leaves us with convertible bond prices of $134.06 or $115.18, respectively. But in all time periods subsequent to the initial time period, we also factor in the $10 coupon being paid from the bond, and thus add $10 to each node, which leaves us with $144.06 or $125.18, respectively.

The only caveat in this proposed method of pricing convertible bonds is they must be handled differently at the time of expiration. Consider the options left to the convertible bond holder at expiration: either they convert, in which their payoff is the equity value, or they hold the bond to maturity, in which their payoff is the $100 principal plus $10 accrued interest.

One way to represent the stock, call option, and convertible bond prices is to use a tree containing all possible upward and downward movements in the stock, option, and convertible prices over a specified time period. You will find three such trees on the following pages, modelling the stock prices (Figure 1) call option prices (Figure 2), and convertible bond prices (Figure 3) over a five
year period using the equations derived here and in the preceding chapters, which will be of use in later sections; these trees are explained in further detail in Section 4.

Note that for creating the convertible bond tree, the $100 bond price and $10 coupon is added to the call option for that node in the tree to come up with the convertible bond price for all nodes subsequent to the initial node and prior to expiration, and the larger between the underlying equity value and bond principal plus accrued interest is used for the convertible bond price at expiration, as explained above.

Figure 1: 5-Period Stock Tree
Figure 2: 5-Period Call Option Tree
Figure 3: 5-Period Convertible Bond Tree
4 1994 Goldman Sachs Convertible Bond Pricing Model

In this section, we present and explain the 1994 Goldman Sachs Convertible Bonds Pricing Model[27] for the purposes of comparing it to the one previously presented in this thesis.

There are a number of preliminary assumptions made in this model, highlighted as follows:

1. The distribution of future stock prices is lognormal with known volatility
2. All future interest rates, including the risk-free rate and the issuer’s credit spread, are known with certainty
3. All information about default risk is accurately contained within the credit spread for the issuer’s bonds.

![Figure 4: One-period stock tree](image)

The first step is to create a stock price tree using the Cox-Ross-Rubinstein method[28], as demonstrated above in Figure 4. To create this stock price tree, assume a stock starts up at price $S_t$. After time period $t$, the stock price can move to either $uS$ or $dS$ with equal probability. The size of the upward or downward jumps is determined by the stock’s volatility, which is known with certainty and held constant throughout the life of the tree. From each node in the tree, we can create upward and downward jumps until the desired length of the tree has been achieved.

![Figure 5: One-period convertible tree](image)
Once a stock price tree has been created, we can create a corresponding convertible bond tree, according to Figure 5 above. The value of the convertible can be determined by evaluating the choices available to the investor at each node, keeping in mind that there are only two possible future values for the convertible exactly one time-step into the future.

We define the holding value $H$ of the convertible bond as the value the investor expects to get by holding on to the convertible bond one more period without converting to stock. It is the expected present value of the two future nodes $V_u, V_d$ plus any interest payments made in the time step between $V$ and $V_u, V_d$.

Alternately, if the investor chooses to convert at a node, they value they expect would be parity, $R$, or the conversion ratio multiplied by the market price of the stock.

The value $V$ of the convertible bond is, then, $Max[H, R]$, provided there are no put or call provisions active.

If the convertible bond is putable at price $P$, then the investor’s choices are to hold, convert, or put the bond for the put value $P$. Then the value of the convertible becomes $Max[H, R, P]$.

If the convertible bond is callable at price $C$ and putable at price $P$, then the issuer gains the ability to call the bond when that would be cheaper than paying remaining interest payments and principal. The value of the convertible, then, becomes $Max[R, P, Min[H, C]]$.

### 4.1 Discount Rate

As stated above, the holding value $H$ is the expected present value of the convertible one step into the future plus any interest to be accrued in the next period. We will now consider how to choose an appropriate discount rate to determine these present values.

In section 2.3, we made the argument that stocks (and thus options) grow at the approximately risk-free rate. However, convertible bonds have both stock- and bond-like properties, so this is not the entirely appropriate rate to use. Consider the case where the stock price is above the conversion price. The convertible, in this case, is likely to be converted, and should be treated as a stock option. Therefore, the risk-free rate, $r$, is the most appropriate interest rate choice.

Alternately, consider the case where the stock price is below the conversion price. In this instance, the convertible will likely not be converted, in which case it would be more appropriate to discount it as a debt instrument. The most appropriate rate choice, then, would be the risk free rate plus a credit spread, or $r + s$. 
4.2 An Example

| Principal | $100 |
| Coupon | 10% per year, compounded annually |
| Maturity | 5 years |
| Conversion Ratio | 1 |
| Calls | $115 in year 2, declining by $5 every year to maturity |
| Puts | Putable at $120 in year 3 |
| Current stock price | $100 |
| Volatility | 10% per year |
| Riskless rate | 5% per year |
| Stock loan rate | 5% per year |
| Credit spread | 500 basis points |

Table 4: A Hypothetical Convertible Bond

Let us now consider an example to demonstrate how to use this model. First, we create a stock tree for the underlying stock, with one-year time steps in between nodes, seen in Figure 6. This stock tree is created using the up and down formulas for stock pricing derived in section two, beginning in the original stock price given in Table 4. After the stock tree is constructed, a corresponding convertible bond tree may be constructed, as seen in Figure 7. At each node in the convertible tree, the convertible price is given, along with the decision made at the node, followed by the probability of conversion, which is used to determine the appropriate discount rate.

The decision made by the investor is abbreviated as defined in Table 5.

Let us sample some of the nodes and show the calculations behind the ending convertible value. In the beginning node of the stock price tree, the value is $100.00. In one time period, the stock either appreciates in value to $115.60 or depreciates to $94.65. This corresponds exactly to an average growth at the risk-free rate of 5% and a volatility from this mean of 10%, consistent with the initial conditions.

Let us now consider the convertible tree. At maturity, the convertible can either be redeemed for $110.00 (principal plus interest from the final period) or converted for stock. Thus, at each node, the value is the maximum between the stock price or the bond value, with corresponding probability of conversion at either 1.0 or 0.0, accordingly. When the probability of conversion is 1.0, the appropriate discount rate to be used is the risk free rate; when the probability of conversion is 0.0, the appropriate discount rate is the credit-adjusted rate, or the risk free rate plus the credit spread (in this example, 10%).

Now look at the lowest convertible node one period prior to expiry. The holding value of the convertible is the expected present value of two proceeding nodes discounted at the appropriate rate, plus any interest accrued in the period. Each of the proceeding nodes is $110.00 with a probability of conversion of
0.0; therefore, the expected present value is \( .5(\frac{110.00}{1.1}) + .5(\frac{110.00}{1.1}) + 10(\text{interest}) = 100 \). Because the call value in year 4 is $115, including interest, the issuer will not call the bond. There are no applicable put options in year 4. Converting to stock would yield the stock price from the corresponding node in the stock tree, or $80.25. Therefore, the maximum payoff for the investor is to hold the convertible one more period at the calculated holding value, $110.00, and the conversion probability at this node is the average of the conversion probabilities at the two preceding nodes.

For another example, consider the $103.56 node on the stock price tree in year 3. On the convertible tree, the holding value is given by the sum of the coupon paid during the period ($10) plus the expected present value of two future nodes discounted at the appropriate rate. The upper value of $119.15 on the convertible tree has a conversion probability of 1.0, so the appropriate discount rate is 5%. The lower value of $97.55 has a conversion probability of .5, so the appropriate discount is \( .5(5\%) + .5(10\%) = 7.5\% \). Therefore, the holding value is \( .5(\frac{119.15}{1.05}) + .5(\frac{97.55}{1.075}) + 10(\text{interest}) = 119.60 \). There is no call option active in this period. The investor has a put option in this period worth $120 plus accrued interest of $10; thus the total put option is worth $130. Finally, conversion would give the investor the stock price at that node, or $103.56. The put option more than the holding value or conversion value to the investor; therefore, the investor exercises the put option, and the corresponding conversion probability is then 0.0.

This process can be repeated for all nodes in the tree, starting at maturity and working backwards towards the initial node.

| X: | investor converts to stock |
| P: | investor exercises put     |
| C: | issuer exercises call      |
| H: | investor holds convertible for one more period |
| R: | issuer redeems convertible at maturity |

Table 5: Letter Codes for Figure 7
Figure 6: 5-Period Stock Tree
Figure 7: 5-Period Convertible Bond Tree
5 Comparison of the Models

5.1 Critique and Comparison of Assumptions

The assumptions for the preliminary model include the underlying assumptions of the bond, stock, and call option pricing models, and are as follows:

1. All future interest rates, including the risk-free rate and the issuer’s default risk are known with certainty and accurately reflected in the credit spread throughout the life of the bond.

2. All information about default risk is accurately contained within the credit spread for the issuer’s bonds.

3. The underlying stock follows Geometric Brownian motion with known and constant drift (the risk free rate) and volatility.

4. The European call option pricing model derived in the Black-Scholes Equation can be used as a substitute for American call option pricing.

5. No transaction costs exist, no dividends will be paid out from the underlying stock, and no arbitrage opportunities exist.

6. Continuous trading is possible with short-selling permitted, and one can buy or sell any number of assets, not necessarily an integral number.

As with most models that serve to model real-life future events, a number of assumptions must be made to simplify and make estimations possible. The first three assumptions would fall into this category; for example, of course future interest rates are not known with certainty! Additionally, a stock’s volatility often changes, as does a debt issuer’s credit risk, as more news and events hit the markets.

The next assumption, that European call option valuation can be used as a proxy for American call options, is born from the inclination to use the Black Scholes equation for convertible bonds which in practice can be converted at any time, similar to American call options. The logic behind this assumption is that because it is never advantageous to exercise an American call option prior to expiration, they can be priced similarly to European call options, which do not allow for the ability to exercise prior to expiration. While this logically makes sense, it fails to account for the value that the ability to exercise at any time adds exists in American call options.

The final two assumptions, similar to the first three, are necessary simplifications that are not reflective of real life. The no-arbitrage assumption in particular is the foundation of much of modern financial theory and can possibly be debated[29].

The explicit assumptions for the Goldman Sachs model are more or less the same as the above, and as such, there is no need to further discuss them. Notably, however, this model lacks the no-arbitrage and European/American call option assumptions.
Overall, both models contain assumptions that are not entirely realistic. However, the Goldman Sachs model has slightly more reasonable assumptions. This is due to the absence of any option pricing theory in the model, the source of more troubling assumptions for the preliminary model.

5.2 Critique of the Model Mechanics

In this subsection we will critique and compare the actual ideas behind and mechanics of the two models. In the preliminary model, the convertible is treated as the simple sum of its parts: a bond, an American call option treated as a European call option, and accrued interest (when applicable).

The benefit of this approach is that it is relatively simple and easy to understand; it also incorporates well-known and widely-studied ideas, most notably the Black Scholes Equation. The drawbacks of this approach also lie in its simplicity, however. The model presented here does not allow for call and put options on the convertible bond, which, while uncommon, do exist. Additionally, model does not allow for early conversion into equity, which is a common occurrence in the real world.

In the Goldman Sachs model, the end equity and bond payoffs are considered to create the choice investors would face in this scenario, with the optimal choice creating the value of the convert. The model works backwards from there, creating risk-adjusted and equal-probability weighted valuations prior to the end nodes from the two potential future outcomes, and weighs that outcome against the payoff of immediate conversion. Call and put options are also possible and considered.

There are a number of benefits to this model. First, it allows for the additional complexity of call and put options. Second, it has a game theory approach in that it is primarily based off of a series of decisions and optimal choices, which is easy for the reader to understand with little to no math training. There are also a few critiques of this model. When call and put options exist, this model does not account for the added (positive or negative) value they add to the convertible prior to their exercise. The protection that the ability to put a convertible bond provides certainly adds value to investor but is not reflected in the model, and the same goes for the value that a call option adds for the convertible bond issuer.

5.3 Conclusion

In conclusion, each model brings to the table its own merits and drawbacks. The preliminary model is useful in that implicit in it are a number of pricing models for related assets, including stocks, bonds, and options. While the inclusion of the Black Scholes equation could be somewhat problematic, the entire model creates a working knowledge of more than one asset class. The Goldman Sachs model does not contain such a technical math-heavy approach, however, so it may appeal to a broader audience than the preliminary model. Additionally, in comparison to the preliminary model, the Goldman Sachs model allows for call
and put options, but not completely, so this is not much of a value add relative to the preliminary model.
Appendix

Ito’s Lemma

An informal proof is given below for Ito’s Lemma, used in section 2.2, and borrowed from the “Stochastic Differential Equations” presentation by Florian Herzog[30].

Suppose a stock price follows the stochastic differential equation

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \]  

(18)

where \( W_t \) is a standard Brownian motion and for constants \( \mu \) and \( \sigma \). Consider also the value of an option on the stock at time \( t \), given by \( f(t, S_t) \). Ito’s lemma states:

\[ df = [f_t + \mu S f_S + \frac{\sigma^2}{2} f_{SS}] dt + \sigma S f_S dW_t, \]  

(19)

for \( f_t \) and \( f_S \) being the partial derivative of \( f \) with respect to \( t \) and \( S_t \), respectively, and \( f_{SS} \) being the second partial derivative of \( f \) with respect to \( S_t \).

Proof: If \( f(t, S_t) \) is twice differentiable, consider its Taylor series expansion around \((a, b)\):

\[ f(t, S_t) = f(a, b) + f_t(a, b)(t - a) + f_S(a, b)(S_t - b) + \frac{1}{2} f_{SS}(a, b)(S_t - b)^2 ... \]

or, rearranging:

\[ f(t, S_t) - f(a, b) = f_t(a, b)(t - a) + f_S(a, b)(S_t - b) + f_{SS}(a, b)(S_t - b)^2 ... \]

In the limit as \((a, b)\) approaches \((t, S_t)\), all second order and higher partial derivatives tend to zero very quickly and drop out of the equation, except for \( dS_t^2 \). This is because \( dS_t \) is of the same order as \( \sqrt{\Delta t} \), so \( dS_t^2 \) is of the same order as \( dt \) and is not dropped.

Thus, in the limit, we get:

\[ df = f_t dt + f_S dS_t + \frac{1}{2} f_{SS} dS_t^2 \]

And using equation 18 to substitute for \( dS_t \), we get:

\[ df = f_t dt + f_S(\mu S dt + \sigma S dW_t) + \frac{1}{2} f_{SS}(\mu^2 dt^2 + 2\mu\sigma dt dW_t + \sigma^2 dW_t^2) \]

Noting that \( dt^2 \) and \( dt dW_t \) both have higher order and thus tend to zero much faster than \( dt \), we set those equal to zero, and additionally noting that the order of \( dS_t^2 \) is the same order as \( dt \) and thus can be substituted for \( dt \), we get:

\[ df = [f_t + \mu S f_S + \frac{\sigma^2}{2} f_{SS}] dt + \sigma S f_S dW_t, \]  

(20)

which is Ito’s lemma, as required.
References


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Academic Vita | Colleen E. Conway

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JPMorgan Chase & Co.  
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- Completed 9-week internship with North American Credit Trading group, built relationships with Distressed, High Yield and Investment Grade traders and desk analysts to learn fundamental and technical analyses of a variety of credit products  
- Pitched weekly credit trade ideas including long, short, and pair trades in the Distressed, High Yield, Investment Grade, Leveraged Loan, CDS, and credit index markets; created slide decks on ESG investing and the CDS auction process  
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- Used Bloomberg to assemble daily market primers containing summaries of news, economic releases, and market data; gave weekly presentations on Fed statements and other relevant events  
- Shadowed syndicate group and municipal, high-yield, and investment-grade corporate bond desks  
Summit, NJ  
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Wall Street Boot Camp II  
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- Chosen by competitive application process on technical skills to succeed on Wall Street, culminating in building a comparable company analysis, financial statement projections, and DCF for a selected company  
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- Taught at a 6-week GED program to men and women who never attended high school and were dealing with extensive drug and alcohol issues; coursework ranged from elementary school to high school mathematics, writing, and science  
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LEADERSHIP AND INVOLVEMENT

Student-Athlete Advisory Board  
Chair, Athlete Welfare Committee  
- Elected to SAAB Executive Board to spearhead all welfare campaigns spanning mental health, sexual assault, diversity and inclusion, and professional development for Penn State’s 800 varsity student-athletes  
- Represented SAAB to student population as a whole and partnered with UPUA to benefit entire Penn State student body  
University Park, PA  
2019-2020 School Year

USA Women’s Field Hockey National Indoor Team  
Development Team Member  
Penn State Varsity Field Hockey Team  
Student-Athlete  
- Competed as member of 2016 Big Ten championship team; consistently ranked in the top 5 nationally  
Pottstown, PA  
2019-Present

Athletic Director’s Leadership Institute  
Veteran Leader  
- Nominated via a competitive process by a head coach to participate in a program designed to further develop leadership skills among fellow student athletes via monthly presentations by notable alumni and professional athletes  
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