THE PENNSYLVANIA STATE UNIVERSITY
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PARTITION IDENTITES USING MODULAR YOUNG DIAGRAMS AND CONGRUENCES AMONG FRACTIONAL PARTITION FUNCTIONS

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Abstract

In this work we look at the properties of the partition function in two different lights. In the second chapter, we look at proving partition identities using modular Young diagrams, which give each block a weight. We prove a result which generalizes a famous identity of Euler, and look to generalize further for multiple residue classes modulus 3. We also introduce the Lecture Hall Theorem and how modular diagrams may be applicable to this result. In the third chapter, we use results about generating functions and modular forms to look at the fractional partition functions, which are a generalization of the usual partition function. We prove results about ℓ-balanced congruences for the fractional partition function, finding conditions that guarantee the existance of these congruences. Furthermore, we look for necessary conditions, showing cases where congruences cannot exist.

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Chapter 1

Introduction to Integer Partitions

A *partition* of a positive integer n is a non-increasing sequence of positive integers that sum to n. We denote the number of partitions of n by $p(n)$. As convention, we define $p(0) = 1$.

Example. For $n = 4$, there are five partitions as follows.

$$\begin{aligned} 4 &= 1+1+1+1 \\ &= 2+1+1 \\ &= 2+2 \\ &= 3+1 \\ &= 4. \end{aligned}$$

Thus $p(4) = 5$.

Integer partitions are largely studied in mathematics due to having an easily manipulated combinatorial structure and a simple generating function. The generating function for $p(n)$ was first discovered by Euler [2]:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

This result, combined with the combinatorial interpretation of $p(n)$, is the basis of the rich study of partitions. In this work, we will be looking at properties derived both from the combinatorial aspects of partitions and from the properties of the generating function. More specifically, in Chapter 2, we will study three variants of Euler's partition theorem on partitions into distinct parts and odd parts. Also, the partition function $p(n)$ possesses interesting arithmetic properties, one of which is the famous Ramanujan's partition congruences. In Chapter 3, we will study congruences for a generalized partition function, namely the fractional partition function.

1.1 Euler's identity and its generalizations

One classical partition identity is Euler's identity, which can be proven both combinatorially and with generating function manipulations.

Theorem (Euler's identity [2])**.** For an integer $n \geq 1$, the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Euler's theorem is an example of a result involving the partition function that can be proved using a bijection. The proof of this is simple. Beginning with a partition that has odd parts, we merge parts that have the same size until we reach a partition that has distinct parts. Note that the result of merging any two odd parts results in a part of even size. That means that when we have our partition into distinct parts, we can divide any even parts into two equal parts until we are returned to all odd parts. In this case, we get back the exact partition we originally began with.

Example. Let $n = 19$, and $\lambda = (5, 3, 3, 3, 3, 1, 1)$. Odd into distinct parts:

$$\begin{aligned} 5+3+3+3+3+1+1 &= 5+(3+3)+(3+3)+(1+1) \\ &= 5+(6+6)+2 \\ &= 5+12+2. \end{aligned}$$

Distinct into odd parts:

$$\begin{aligned} 12+5+2 &= (6+6)+5+(1+1) \\ &= (3+3)+(3+3)+5+1+1 \\ &= 3+3+3+3+5+1+1. \end{aligned}$$

There are several generalizations of Euler's identity. Recently, Keith and Xiong proved the following generalization [14].

Theorem. Let m be a positive integer. Then the number of partitions of n with each part not congruent to 0 mod m is equal to the number of partitions n with parts repeating at most $m-1$ times.

When $m = 2$, this theorem yields Euler's identity.

Combinatorial proofs of partition identities are of special interest because of their concrete explanations for abstract concepts. They explain concepts on the deepest level in a way that is simple to understand. The proof of Keith and Xiong is combinatorial but very complicated. This motivated us to look for a simpler proof. In Chapter 2, we will introduce modular Young diagrams and reprove the theorem of Keith and Xiong for the $m = 3$ case using modular Young diagrams, and will work towards proving other results using these diagrams.

1.2 Ramanujan's partition congruences and fractional partition functions

The generating function, or q-series, found by Euler for the partition function has led to much of the research behind this function. In fact, Euler originally proved the identity above using the representation as a generating function. In the third chapter of this work, we will prove results about the fractional partition functions, which derive from this generating function. By manipulating q-series, more results can be proved about the functions they represent.

One result that, although eventually proven using combinatorial methods, was originally proved using q-series are the Ramanujan congruences. In the early 20th century, Ramanujan observed the following congruences among arithmetic progressions for the partition function

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Ramanujan conjectured that these congruences were special, and were the only congruences where the modulus of the arithmetic progression is the same as the modulus of the congruence. Furthermore, he conjectured that similar congruences existed for higher powers of 5, 7, and 11.

Watson [13] and Atkin [3] were able to confirm Ramanujan's intuition, and proved the following theorem using q-series.

Theorem 1.2.1 (Watson (1938), Atkin (1967))**.** For prime $\ell \geq 5$ and positive integer r, define $0 \leq c_{\ell,r} < \ell^r$ such that $24c_{\ell,r} \equiv 1 \pmod{\ell^r}$. Then for every nonnegative integer n, we have

$$\begin{aligned} p(5^r n + c_{5,r}) &\equiv 0 \pmod{5^r}, \\ p(7^r n + c_{7,r}) &\equiv 0 \pmod{7^{\lfloor r/2 \rfloor + 1}}, \\ p(11^r n + c_{11,r}) &\equiv 0 \pmod{11^r}. \end{aligned}$$

Later, Ahlgren and Boylan [1] confirmed that these were the only congruences of this form for the partition function.

In Chapter 3, we define ℓ^r*-balanced* congruences to be those of the form

$$p(\ell^r n + c) \equiv 0 \pmod{\ell^r}$$

for prime ℓ and positive integer r. The congruences observed by Ramanujan for all powers of 5 and 11 are ℓ^r-balanced.

In a 2018 paper, Chan and Wang introduced the fractional partition functions [6]. The *fractional partition functions* $p_\alpha(n)$ are defined for rational $\alpha = a/b$, for coprime integers a, b and $b \geq 1$ as

$$\sum_{n=0}^{\infty} p_\alpha(n) q^n := \prod_{n=1}^{\infty} (1 - q^n)^\alpha.$$

When $\alpha = -1$, we return the usual partition function $p(n)$.

Chan and Wang showed that the values of $p_\alpha(n)$ are ℓ-integral for any $\ell \nmid b$, and hence we can study congruences whenever ℓ is coprime to the denominator b. Chan and Wang proved the following formula about ℓ-balanced congruences among the fractional partition functions.

Theorem 1.2.2 (Chan-Wang (2018))**.** Let $\alpha = \frac{a}{b}$, and ℓ be a prime not dividing b, and suppose any of the following conditions hold:

1. $\alpha \equiv 1 \pmod{\ell}$ and $24c + 1$ is a quadratic non-residue modulo ℓ;
2. $\alpha \equiv 3 \pmod{\ell}$ and $8c + 1$ is a quadratic non-residue modulo ℓ or $8c + 1 \equiv 0 \pmod{\ell}$;
3. $\alpha \equiv 4, 8, 14 \pmod{\ell}$, $\ell \equiv 5 \pmod{6}$, and $24c \equiv -\alpha \pmod{\ell}$;
4. $\alpha \equiv 6, 10 \pmod{\ell}$, $\ell \equiv 3 \pmod{4}$, $\ell \geq 5$, and $24c \equiv -\alpha \pmod{\ell}$;
5. $\alpha \equiv 26 \pmod{\ell}$, $\ell \equiv 11 \pmod{12}$, and $24c \equiv -\alpha \pmod{\ell}$.

Then for $n \geq 0$,

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell}.$$

In Chapter 3, we will work to find congrueneces for $p_\alpha(n)$ using more general results about modular forms, setting out to create a framework for where these fractional partition functions exist.

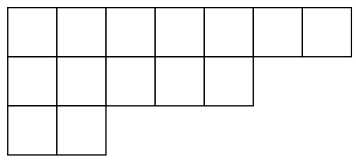
Chapter 2

Modular Young Diagrams and their Applications to Partition Identities

2.1 Definitions and Motivation

As mentioned in the introduction, Keith and Xiong generalized Euler's partition identity from the modulus 2 to any arbitrary modulus m. Before them, Pak and Postnikov considered partitions into parts from one congruence class mod m [11]. In this chapter, we will first prove the result of Pak and Postnikov combinatorially, and then prove the $m = 3$ case of the result of Keith and Xiong.

Young diagrams are a way to represent integer partitions via a graph. For the partition $\lambda = (7, 5, 2)$, we get the following diagram.



In a Young diagram, each row represents an integer, with the total number of squares in a row equal to the size of that integer. The total number of squares in the graph is the size of the partition. If you have a Young diagram and manipulate the squares into another shape that is also a Young diagram, you get a partition of the same integer. This property makes Young diagrams useful for representing and proving properties about the partition function. For instance, by exchanging the rows and columns in Young diagram, we get another Young diagram. We call this new diagram the conjugate of the original one. The conjugate of $\lambda = (7, 5, 2)$ is $(3, 3, 2, 2, 2, 1, 1)$.

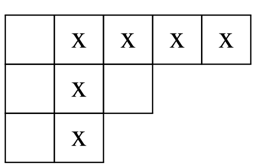
Next, we give the definitions of hooks and rim hooks of a partition, which will be essential in describing bijections in our proofs.

Definition 2.1.1. The *hook* of a partition λ at block (i, j), denoted $H_\lambda(i, j)$, is the subset of cells to the right of, and including, cell (i, j), plus the cells beneath (i, j) in the Young diagram of λ. In other words,

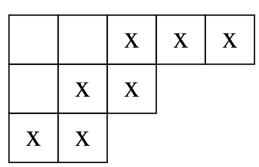
$$H_\lambda(i, j) := \{(a, b) : a = i \text{ and } b \geq j, \text{ or } a \geq i \text{ and } b = j\}.$$

Definition 2.1.2. A *rim hook* of a partition λ is a connected subset of cells that does not contain a two-by-two square, such that removing the rim-hook yields again a partition.

For instance, in the figure on the right below, the set of cells marked with x is the hook of $\lambda = (5, 3, 2)$ at block $(1, 2)$, and the figure on the right shows the longest rim hook of the same partition λ, which is the set of cells marked with x, and any connected subsets of the longest rim hook are rim hooks of λ.



$H_\lambda(1, 2)$



Longest rim hook of λ

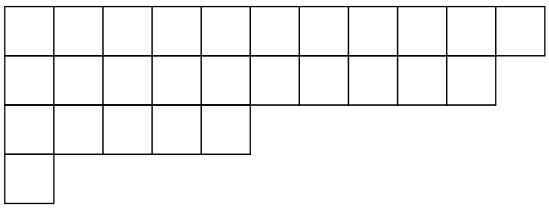
In order to denote partitions that have parts repeated multiple times, we will use the form $\lambda = (n_1^{r_1}, n_2^{r_2}, \ldots n_\ell^{r_\ell})$ where $n_i^{r_i}$ means that there are r_i parts of size n_i. As an example, we would represent the partition $\lambda = (5, 5, 4, 4, 3, 3, 3, 2, 1)$ as $(5^2, 4^2, 3^3, 2, 1)$.

In the next section, we introduce modular Young diagrams. In Section 2.3, we state the results, and in Section 2.4, we will give our proofs.

2.2 m- and m^*-modular Young diagrams

For a positive integer m, we define an m-modular Young diagram to be a Young diagram with blocks having horizontal weight m or c with $1 \leq c < m$. In a given row, all blocks except for the rightmost block will have weight m, while the last block will have weight c. This ensures that each row of an m-modular Young diagram corresponds to an integer that is congruent to c mod m.

Example. Let $m = 3$ and $\lambda = (11, 10, 5, 1)$. Using 3-modular Young diagrams, we get the two following equivalent representations of λ.



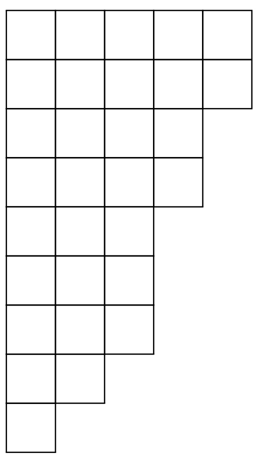
$\longleftrightarrow$

3	3	3	2
3	3	3	1
3	2		
1			

Young diagram 3-modular diagram

In m-modular Young diagrams, the weight of each block is distributed by row. We define the m^*-modular Young diagrams to also have blocks of weight m or c with $1 \leq c < m$. Instead of the weight being distributed by row, the weight is distributed vertically along a column.

Example. Let $m = 3$. Let $\lambda = (5, 5, 4, 4, 3, 3, 3, 2, 1)$. Then we get the following two equivalent representations of λ using a m^*-modular Young diagram.



$\longleftrightarrow$

3	3	3	3	2
3	3	3	1	
3	2	1		

Young diagram 3^*-modular diagram

From the definitions, we can easily see that the m-modular Young diagram of a partition λ is the same as the m^*-modular Young diagram of the conjugate of λ.

We will make use of m-modular and m^*-modular Young diagrams to create bijections to prove theorems similar to and including the lecture hall partition theorem above.

2.3 Statement of Results

First, we look to prove a result that is much like Euler's identity for generalized congruence classes. Instead of looking at partitions using odd parts, we look at partitions with each part belonging to the same congruence class for some modulus. We say that a partition is of type $(a, b, a, b, ...)$ if it alternates between having parts repeat a and b times.

We prove the following theorem using m-modular Young diagrams.

Theorem 2.3.1. [11] The number of partitions of n with each part congruent to c mod m is equal to the number of partitions of n of type $(c, m-c, c, m-c, ...)$.

When $m = 2, c = 1$, this theorem is equivalent to Euler's identity. We attempt to generalize Theorem 2.3.1 to find an identity with multiple residue classes mod m using m-modular Young diagrams. While we were not able to prove a more general theorem, we prove the following result modulus 3.

Theorem 2.3.2. [14] The number of partitions of n with each part congruent to 1 or 2 mod 3 is equal to the number of partitions n with parts repeating 1 or 2 times.

In Section 2.5, we will introduce the Lecture Hall Theorem, which is a refinement of Euler's identity. We again attempt to prove this result in a simple manner using m-modular Young diagrams, although we are unable to find a suitable bijection in this case. Using m-modular Young diagrams to prove this result may be of interest for future work in this area.

2.4 Proofs

2.4.1 Proof of Theorem 2.3.1

Proof of Theorem 2.3.1. Let λ be a partition of n with each part congruent to c mod m. Then $\lambda = (a_1 \cdot m + c, a_2 \cdot m + c, \dots, a_\ell \cdot m + c)$ where $a_1 \geq a_2 \geq \cdots \geq a_\ell \geq 0$. We define a bijection to a partition of type $(c, m-c, c, m-c, \dots)$ as follows:

Construct an m-modular Young diagram of λ such that row i contains a_i blocks of weight m and 1 block of weight c. The number of blocks in each row, $(a_1 + 1 \geq a_2 + 1 \geq \cdots \geq a_\ell + 1)$ is itself a partition. We will construct an m^*-modular diagram from this diagram that will represent a partition of type $(c, m-c, c, m-c, \dots)$.

We create an m^*-modular diagram from our original diagram by taking the blocks along the diagonal hooks of our m-modular diagram and creating new rows from each diagonal hook. That is to say, $H_\lambda(1,1)$ will become row 1 of the new diagram, $H_\lambda(2,2)$ becomes row 2, and so on, until there are no remaining blocks in λ. We define μ to be the partition whose m^*-modular diagram is this new modular diagram.

We now verify that μ is a partition of type $(c, m-c, c, m-c, \dots)$. Since $(a_1+1, a_2+1, \dots, a_\ell+1)$ is a partition, our new diagram taken without weights is also a partition. Each row in our new diagram will contain at least one block of weight c because each row must contain the rightmost block of some row from our diagram for λ. Additionally, the length of each row in the new diagram will necessarily be less than the number of blocks of weight m in the previous row. This is due to the following two facts: A) Each hook is at least two blocks shorter than the hook before it; and B) Any blocks of weight c beyond the one coming from the horizontal part must be in the vertical part of the hook, and cannot have blocks of weight c to the right of them. Thus, μ is indeed of the form $(c, m-c, c, m-c, \dots)$.

We can uniquely reverse the bijection as follows:

Express μ as an m^*-modular Young diagram. We will shift the rows in our m^*-modular diagram such that we can fold it and return the m-modular diagram for λ. We know each row must contain at least one block of weight c. We position one block of weight c on the right of the row and then any additional c's on the left with all m's sandwiched in the middle. We place the following row beneath the previous row such that the leftmost block in the lower row is directly underneath the leftmost block of weight m of the above row.

We will fold our shifted diagram for μ along a diagonal to return our original m-modular diagram. The diagonal will begin in the bottom row of m^*-modular diagram. We choose the block that is to the right of the last extra block of weight c. That is, if the bottom row contains all blocks of weight c, we choose the block on the right. If the bottom row contains any number of blocks of weight m, we will choose the rightmost block of weight m. This means that any blocks of weight c in the row beyond the required one will be to the left of our folding axis. We then create a diagonal axis from this block by choosing blocks that are one row up and one column to the right of the previous block. We fold our shifted diagram for μ along this diagonal so that this diagonal becomes the main diagonal in our new diagram. This method ensures that there is exactly one block of weight c in each row in the returned m-modular diagram, and returns λ. □

Example. Let $m = 3, c = 1$, and $n = 36$. Consider the partition $\lambda = (13, 7^2, 4^2, 1)$. Taking the hooks of our m-modular diagram, we demonstrate the forward direction of the bijection:

m-modular diagram

3	3	3	3	1
3	3	1		
3	3	1		
3	1			
3	1			
1				

⟷

m^*-modular diagram

3	3	3	3	3	3	3	3	1	1
3	3	1	1	1					
1									

This operation return the partition $\mu = (10, 8^2, 5, 2^2, 1)$ which is of type $(1, 2, 1, 2, 1)$.

Now we can demonstrate the reverse of the bijection by shifting the rows and folding along the diagonal axis:

m^*-modular diagram

1	3	3	3	3	3	3	3	3	1
	1	1	3	3	1				
			1						

⟷

m-modular diagram

3	3	3	3	1
3	3	1		
3	3	1		
3	1			
3	1			
1				

This returns the partition $\lambda = (13, 7^2, 4^2, 1)$.

2.4.2 Proof of Theorem 2.3.2

Proof of Theorem 2.3.2. Let λ be a partition of n into parts congruent to 1 or 2 mod 3. We will create a 3-modular diagram to represent λ, but we will shift the rows so that we can perform the forward direction of our bijection.

To create our 3-modular diagram, we will create a row for each part of λ such that all blocks are of weight 3 except the rightmost block which is either of weight 1 or 2. We will position each row on top of the other so that it is right-justified unless the rightmost block of the row we are adding is of weight 2 and the rightmost block of the previous row was of weight 1. In this case, we will shift the row by one block to the left. We will then continue to add rows that are now right-justified to the rightmost block of this row. We will continue in this matter until all parts of λ have been added to our diagram.

The goal of the above is to create a diagram that from the right to the left has a column of 2's followed by 1's, and then a column with 3's that go as far down as the last 1, then 2's followed by 1's, until eventually the columns are only 3's. At this point, the columns can become shorter as necessary.

Once we create a 3-modular diagram in this format, we can begin to form the rows of our 3^*-modular diagram for μ by taking modified rim hooks. Recall that typically, rim hooks are connected subsets of cells that do not contain two-by-two squares, and that when we remove a rim hook, we will still have a partition. When we build our modular Young diagram in the way described above, we want to take the largest rim hooks from the left of our diagram, such that the remaining diagram is still a partition and still follows the rules above. In order to form our 3^*-modular diagram, we take the longest modified rim hooks of λ to be each row of μ.

Example. Consider the partition $\lambda = (11, 8, 5, 4)$. We will show the longest rim hook of the 3-modular diagram for λ when considering our modified rim hooks, and when considering the usual rim hook. The rim hooks are displayed in red.

3	3	3	2
	3	3	1
	3	2	
	3	1	

Longest modified rim hook of λ

3	3	3	2
3	3	1	
3	2		
3	1		

Longest usual rim hook of λ

By taking rim hooks, we ensure that each smaller row in our diagram for μ has at most the total number of squares as the number of squares of weight 3 above it. Additionally, we ensure that each row has at least one 1 or 2 at the end. It is possible that a row has exactly the same number of total squares as the number of 3's in the row above it. In this case, the lower row must end in a block with weight 1, while the upper row must have at least one block of weight 2. While this means that you will get two parts of the same size from two different rows, each part will only be repeated once from its respective row, so it will be repeated a total of 2 times. Thus we get that μ is a partition of n with all parts repeated 1 or 2 times.

Once we have μ, we can reconstruct the 3^*-modular diagram by grouping the parts of μ into 3's from largest to smallest and forming a row from each group of 3. To get the reverse of the bijection, we can exactly reform our 3-modular diagram from the 3^*-modular diagram for μ.

To state the reverse of the bijection, we define the *tail* of a row to be all of the blocks of weight 2 or of weight 1. We take the bottom row of the 3^*-modular diagram and add it into our 3-modular diagram such that the tail of the row is vertical in the rightmost column and all 3's are placed to the left of this column in the top row. We will then add all remaining rows from the m^*-modular diagram from smallest to largest with placement based on the tail of the row.

Take the next row of the 3^*-modular diagram. There are 3 different cases for our 3-modular diagram for λ.

A) The rightmost column of the 3-modular diagram has no blocks of weight 1. In this case, we will place the tail of this next row in this column, and then drape the rest of the row to the left.

B) The rightmost column of the 3-modular diagram has any blocks of weight 1 but our tail only consists of blocks of weight 1. In this case, we will do the same as in case (A).

C) The rightmost column of the 3-modular diagram has blocks of weight 1 and our row has blocks of weight 2. In this case, we will start in the row one to the left and place the tail such that it begins one row beneath where the column to the right of it ends. We will then drape the rest of the 3's in this row to the left.

We will then repeat this process with all remaining rows, considering whichever column furthest to the left that has any squares of weight 1 or 2 to be the new rightmost column. This will return λ. □

Example. Let $n = 39$ and $\lambda = (11, 10, 5^2, 4^2)$. We create the following 3-modular diagram for λ and take rim hooks to create a 3^*-modular diagram for μ:

3-modular diagram

3	3	3	2
3	3	3	1
	3	2	
	3	2	
	3	1	
	3	1	

⟷

3^*-modular diagram

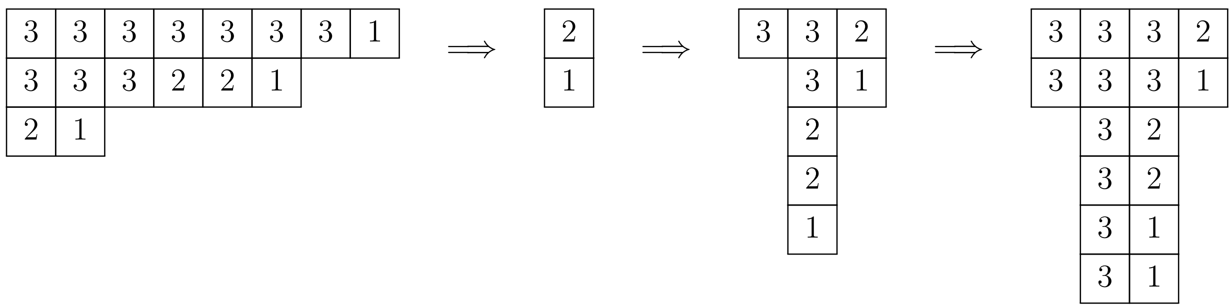
3	3	3	3	3	3	3	1
3	3	3	2	2	1		
2	1						

We get the partition $\mu = (8, 7^2, 5, 4, 3, 2^2, 1)$, which has all parts repeated one or two times. In order to recreate λ, we go through the following reversal of the bijection row-by-row.

We can recreate the 3^*-modular diagram for μ by grouping our parts into groups of 3 and creating rows in the 3^*-modular diagram. Starting with the smallest row of our 3^*-modular diagram, we begin to recreate the 3-modular diagram for λ. This row is placed on the right side of our diagram vertically. Because the next row of μ has a block of weight 2 and our rightmost column of the 3-modular diagram has a block of weight 1, we do not want to have any of this row drape onto the right column. We also want to align the first 2 from this row to be one block below the last 1 in our right column. In this case, that means that we have to place one block of weight 3 in its own column to the left.

Finally, we look to add the bottom row of the 3^*-modular diagram. This row contains no blocks of weight 2, so we place all blocks of weight 1 in column 2, and then snake all blocks of weight 3 up and then to the left as necessary.

The following diagram shows the recreation of the m-modular diagram for $\lambda = (11, 10, 5^2, 4^2)$.



2.5 Further work using m-modular diagrams on lecture hall partitions

One famous partition identity is known as the lecture hall partition theorem. Lecture hall partitions are derived from the idea of creating rows in lecture halls where each row can see the lecturer. Fixing a number of rows N, the set $\mathcal{L}_N$ is the set of partitions such that each row would be able to see over the row in front of it, based on a ratio condition. Formally, we define the set of lecture hall partitions of length $N \geq 1$ as

$$\mathcal{L}_N = \{\lambda_1 + \cdots + \lambda_N : \frac{\lambda_N}{N} \geq \frac{\lambda_{N-1}}{N-1} \geq \cdots \geq \frac{\lambda_1}{1} \geq 0\},$$

where some number of the N parts are allowed to be empty. We call the property

$$\frac{\lambda_N}{N} \geq \frac{\lambda_{N-1}}{N-1} \geq \cdots \geq \frac{\lambda_1}{1} \geq 0$$

the lecture hall criterion. When we disregard the empty parts, we get an integer partition in strictly decreasing order. When $\lambda_N \leq N$, the partitions of this type are all integer partitions of strictly decreasing order with up to N parts. However, once $\lambda_N > N$, the ratio rule means certain partitions with distinct parts do not satisfy the lecture hall criterion. For instance, if $N = 3$, the partition $\lambda = (5, 3, 1)$ satisfies the criterion, but the partition $\lambda = (5, 4, 1)$ does not as $\frac{5}{3} < \frac{4}{2}$.

The following identity for lecture hall partitions was proved by Bousquet-Mélou and Eriksson [4], and is a refinement of Euler's identity, which is introduced in Introduction.

Theorem 2.5.1 (Lecture hall partition theorem)**.** For a fixed length N of the lecture hall, the number of lecture hall partitions equals the number of partitions into odd parts smaller than 2N. In other words,

$$p(n \mid \text{lecture hall of length } N) = p(n \mid \text{odd parts } < 2N)$$

As N tends towards infinity, we return Euler's identity. Unlike Euler's identity, there is no simple bijection that returns the lecture hall partitions. However, several proofs of the lecture hall partition theorem using more sophisticated bijections exist, including the proof by Bousquet-Mélou and Eriksson [4] and a proof by Yee [15]. We looked for a more simplified proof of the lecture hall partition theorem using m-modular Young diagrams, but were unable to recover the theorem using these methods. What follows is an attempt at the forward direction of a bijection to prove the lecture hall theorem. While the map produces a lecture hall partition of length N from a partition of n into odd parts of size less than $2N$, future work is necessary to find a reverse map.

2.5.1 Map from partitions of odd parts to lecture hall partitions

Let λ be a partition with odd parts of size less than $2N$. We will create a lecture hall partition μ of length N from λ. Create an 2-modular diagram for λ. We will create an 2^*-modular diagram for μ via the following map.

We take the rows of λ's 2-modular diagram and successively add them to μ's diagram from the largest to smallest. We determine how to add all following rows of λ to μ based on the existing rows of μ. Suppose that i is the smallest such that row i of μ's diagram has length that is congruent to 0 mod $(N-2i+2)$. If the row of λ being added has length less than i, then we add the row vertically to μ's diagram so that exactly one block is added to each row from the largest and the block of weight 1 is placed at the bottom. Otherwise, we hook the row of λ onto μ from the left, so that it adds one block to each of the first $i-1$ rows, and that any additional blocks are added to row i.

We claim that this new partition will be a lecture hall partition of size N. Firstly, in the construction of the 2^*-modular diagram of μ, the new length will have at most $\lceil N/2 \rceil$ parts. This is due to the construction limiting the number of rows you can hook on top of each other before needing to add an entire row to the first row. For example, if you add rows of size $\lceil N/2 \rceil$ or larger, you will run out of space in the first row before you can hook beyond a height of $\lceil N/2 \rceil$. On the other hand, if you add rows of size less than $\lceil N/2 \rceil$, you will not be able to add more than $\lceil N/2 \rceil$ in height. Moreover, if N is odd and there are boxes at row $\lceil N/2 \rceil$, then all the boxes in the row have to be boxes of weight of 1. This ensures that our new partition μ can have at most N parts.

Furthermore, the new partition will satisfy the lecture hall criterion. In order for the partition to be lecture hall, we want

$$\frac{\lambda_1}{N} \geq \frac{\lambda_2}{N-1} \geq \cdots \geq \frac{\lambda_n}{1} \geq 0.$$

Let $\tilde{\mu}$ be the partition we got by adding all but the last part of λ. Suppose that $\tilde{\mu}$ is lecture hall. Write each part $\tilde{\mu}_j$ of $\tilde{\mu}$ as

$$\tilde{\mu}_j = a_j(N-j+1) + r_j \tag{2.5.1}$$

for some $0 \leq r_j \leq N-j+1$ and $a_j \geq 0$. Then, since $\tilde{\mu}$ is lecture hall, we see that

$$a_1 \geq a_2 \geq \cdots \geq a_N, \quad \text{and if } a_j = a_{j+1}, \text{ then } r_j > r_{j+1}. \tag{2.5.2}$$

Now, let i be the smallest such that row i in the modular diagram of $\tilde{\mu}$ is congruent to 0 mod $(N-2i+2)$. This means that

$$0 < r_{N-j+1} < N-j+1 \ \text{ for } 1 \leq j \leq N-2i+1, \text{ and } r_{N-2i+2} = 0.$$

Let the weight of the last part of λ be $2k-1$, i.e., in 2-modular, it has $k-1$ boxes of 2 and one box of 1. If this was added vertically, then we see that

$$\mu_j = \begin{cases} \tilde{\mu}_j + 1 & \text{for } 1 \le j \le 2k-1, \\ \tilde{\mu}_j & \text{otherwise.} \end{cases} \tag{2.5.3}$$

Otherwise,

$$\mu_j = \begin{cases} \tilde{\mu}_j + 1 & \text{for } 1 \le j \le 2i-2, \\ \tilde{\mu}_{2i-1} + (k-i+1) & \text{for } j = 2i-1, \\ \tilde{\mu}_{2i} + (k-i) & \text{for } j = 2i, \\ \tilde{\mu}_j & \text{for } j > 2i. \end{cases} \tag{2.5.4}$$

Since $\tilde{\mu}$ satisfies the lecture hall criterion, we can easily check that μ also satisfies the criterion from (2.5.1)-(2.5.4).

Example. Let $N = 5, m = 2, c = 1$ and $\lambda = (7^2, 5^3, 1)$.

2-modular diagram for λ

2	2	2	1
2	2	2	1
2	2	1	
2	2	1	
2	2	1	
1			

$\longrightarrow$

2^*-modular diagram for μ

1	2	2	2	2	1	2	2	2	2	1
	2	2	1			2	2	1		
	1									

We get $\mu = (11, 8, 6, 4, 1)$ which is lecture hall, as $\frac{11}{5} \ge \frac{8}{4} \ge \frac{6}{3} \ge \frac{4}{2} \ge \frac{1}{1}$.

Chapter 3

Congruences for the Fractional Partition Functions

3.1 Statement of Results

Let us recall the fractional partition function $p_\alpha(n)$ given in the Introduction: for rational $\alpha = a/b$, for coprime integers a, b and $b \geq 1$,

$$\sum_{n=0}^{\infty} p(n)_\alpha q^n = (q;q)_\infty^\alpha.$$

Here and throughout this chapter, we adopt the following standard q-series notation:

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

Beginning with Ramanujan, balanced congruences have been studied widely for the partition function and its integral powers. In [9], Kiming and Olsson identified necessary conditions for p_α to admit an ℓ-balanced congruence for integer α. Their main result (Theorem 1 of [9]) can be stated as follows. Suppose that α is an integer and $\ell \geq 5$ is a prime such that $\alpha \not\equiv 1,3 \pmod{\ell}$. If $p_\alpha(\ell n + c) \equiv 0 \pmod{\ell}$ for all n, then $24c + \alpha \equiv 0 \pmod{\ell}$. This result explains the significance of the residue class $1/24 \pmod{\ell}$ in Ramanujan's congruences for the partition function. Our first theorem is an analog of Kiming and Olsson's result for fractional α. It shows that when $\alpha \not\equiv 1,3 \pmod{\ell}$, there is generically at most one residue class modulo ℓ corresponding to an ℓ-balanced congruence for p_α.

Theorem 3.1.1. Let $\alpha = a/b$, and let $\ell \geq 5$ be a prime not dividing b such that $\alpha \not\equiv 1,3 \pmod{\ell}$. If c is an integer such that

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell}$$

for all n, then $24c + \alpha \equiv 0 \pmod{\ell}$.

The cases where $\alpha \equiv 1,3 \pmod{\ell}$ yield far more ℓ-balanced congruences. These two cases are distinguished by two famous lacunary Fourier expansion identities for powers of the Dedekind eta-function $\eta(z) := q^{1/24}(q;q)_\infty$, where $q = e^{2\pi i z}$.

Recall that a power series $\sum_{n=0}^{\infty} a(n)q^n$ is called *lacunary* if

$$\lim_{N\to\infty} \frac{\#\{n \leq N : a(n) = 0\}}{N} = 1.$$

In [12], Serre studied and classified which even positive powers of the eta-function are lacunary. Explicitly, he found that for even integer $r > 0$, η^r is lacunary if and only if $r \in \{2,4,6,8,10,14,26\}$. In the case when r is odd, we have the following classical formulae due to Euler and Jacobi, respectively:

$$\eta(24z) = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2}, \tag{3.1.1}$$

$$\eta(8z)^3 = \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) n q^{n^2}, \tag{3.1.2}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. These formulae show that $\eta(24z)$ and $\eta(8z)^3$ are lacunary. Chan and Wang [6] used this lacunarity to obtain ℓ-balanced congruences for $p_\alpha(n)$ when $\alpha \equiv 1, 3 \pmod{\ell}$. Our second result is an extension of Chan and Wang's work to include congruences modulo ℓ^r for $r > 1$.

Theorem 3.1.2. Let $\alpha = a/b$, let ℓ be a prime not dividing b, and let c be an integer.

(i) If $\left(\frac{24c+1}{\ell}\right) = -1$, then for all n we have

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell^{\mathrm{ord}_\ell(\alpha-1)}}.$$

(ii) If $\left(\frac{8c+1}{\ell}\right) = -1$, then for all n we have

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell^{\mathrm{ord}_\ell(\alpha-3)}}.$$

Remark. By the Chinese Remainder Theorem, a simple consequence of Theorem 3.1.2 is that for any positive integer L, there exists $\alpha \in \mathbb{Q}$ such that for all n,

$$p_\alpha(Ln + c) \equiv 0 \pmod{L}.$$

In addition to the exceptional closed formulas for η and η^3, there are certain even powers of the eta-function arising from the theory of complex multiplication that are well known to share the property of lacunarity. By exploiting these even powers mentioned above, Chan and Wang [6] found ℓ-balanced congruences for $p_\alpha(n)$ when $\alpha \equiv 4, 6, 8, 10, 14, 26 \pmod{\ell}$. Their methods produce a large number of ℓ-balanced congruences, expanding the results of previous literature which only presented a handful.

We seek to obtain congruences through other general properties of modular forms. To do so, we make use of a different phenomenon which is universal for modular forms, thereby establishing a general framework for describing all known ℓ-balanced congruences for $p_\alpha(n)$. The advantage to this method is that we can obtain ℓ-balanced congruences for $p_\alpha(n)$ when α is not restricted to a finite set of residue classes modulo ℓ. Furthermore, our theory enables us to extend our results to obtain ℓ^r-balanced congruences for $r > 1$.

In order to exploit this phenomenon, we need the following definition. We say that a prime ℓ *is good for* $\alpha = a/b$ *with parameter* k if $\ell \nmid b$ and k is a positive integer such that

$$\ell \mid (24k - \alpha), \tag{3.1.3}$$
$$(\ell - 1) \mid (12k - m) \text{ for some } m \in \{4, 6, 8, 10, 14\}, \text{ and} \tag{3.1.4}$$
$$\ell \nmid N_{12k}(\mathcal{D}_{12k}), \text{ the norm of the weight } 12k \text{ Hecke determinant (see Section 3.2.1).} \tag{3.1.5}$$

We say that ℓ *is good for* α if there exists such a parameter k.

Theorem 3.1.3. If ℓ is good for α with parameter k and $v \leq \mathrm{ord}_\ell(24k - \alpha)$ is a positive integer, then for all n, we have

$$p_\alpha(\ell^v n - k) \equiv 0 \pmod{\ell^v}.$$

Remark. If ℓ is a prime that is good for α with parameter k, then $k < \ell$ (see Proposition 3.3.1) and $\ell \leq \max\{|26b - a|, |14b + a|\}$ for all but finitely many α (see Proposition 3.3.2). Thus, we conclude that for generic α, there are only finitely many primes ℓ good for α. We note that the α for which we cannot derive a bound on ℓ as in Proposition 3.3.2 are $\alpha \in \{0, 2, 4, 6, 8, 10, 14, 26\}$, aligning with the lacunary even powers of the eta-function classified by Serre.

The first condition (3.1.3) ensures that the conditions of Theorem 3.1.1 are met. The second and third conditions are to ensure the ℓ-non-ordinairty of normalized eigenforms, which will be discussed to a greater degree in Section 3.2.1. Theorem 3.1.3 represents a complete classification of the situations for which congruences of the fractional partition functions arise “automatically” because every normalized eigenform in S_{12k} is ℓ-non-ordinary. Combining results from Theorem 3.1.2 and Theorem 3.1.3, we can deduce all congruences achieved by the main theorem of Chan and Wang (Theorem 1.2 of [6]) up to the third condition of ℓ being good for α with parameter k.

Example. Consider Theorem 1.2 (5) of [6], which equivalently states that if $\ell \equiv 11 \pmod{12}$ and $\alpha \equiv 26 \pmod{\ell}$, then for any integer c satisfying $24c + 26 \equiv 0 \pmod{\ell}$, we have

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell} \tag{3.1.6}$$

for all n. We can prove this result using Theorem 3.1.3 up to Condition (3.1.5). To do so, we choose $1 \leq k < \ell$ such that $k \equiv -c \pmod{\ell}$. We verify the first two conditions of ℓ being good for α with parameter k. Clearly $24k - \alpha \equiv -24c - 26 \equiv 0 \pmod{\ell}$, which is (3.1.3). Because $\ell \equiv 11 \pmod{12}$, it can be checked that $12k - 13 = \ell$. Therefore $(\ell - 1) \mid (12k - 14)$, which is (3.1.4). If (3.1.5) also holds, we can conclude by Theorem 3.1.3 that $p_\alpha(\ell n - k) \equiv 0 \pmod{\ell}$ for all n, which is equivalent to (3.1.6).

Example. We use Theorem 3.1.3 to recover the ℓ-balanced congruences for the usual partition function. By Proposition 3.3.1 and Proposition 3.3.2, if a prime ℓ is good for $\alpha = -1$ with parameter k, then $k < \ell$ and $\ell \leq 27$. One can check that the only values of $\ell \leq 27$ which are good for $\alpha = -1$ are $\ell \in \{5, 7, 11\}$, aligning with the results of Ahlgren and Boylan [1]. For example, $\ell = 11$ is good for $\alpha = -1$ with parameter $k = 5$ because $11 \nmid N_{60}(\mathcal{D}_{60})$. To compute the norm of the Hecke determinant $N_{60}(\mathcal{D}_{60})$, we use the function `CuspForms(1,60).hecke_algebra().discriminant()` on CoCalc.

Example. Let $\ell = 367$ and $\alpha = 3/37$. Then, ℓ is good for α with parameter $k = 31$ because $367 \nmid N_{372}(\mathcal{D}_{372})$. We conclude from Theorem 3.1.3 that for all n,

$$p_{\frac{3}{37}}(367n - 31) \equiv 0 \pmod{367}.$$

Example. We show that the set of v identified by Theorem 3.1.3 is sharp for certain choices of α and ℓ. Let $\ell = 17$ and $\alpha = 57/61$. Then, ℓ is good for α with parameter $k = 3$ because $17 \nmid N_{36}(\mathcal{D}_{36})$. Since $\mathrm{ord}_{17}(24 \cdot 3 - \frac{57}{61}) = 2$, we conclude from Theorem 3.1.3 that for all n,

$$p_{\frac{57}{61}}(17n - 3) \equiv 0 \pmod{17},$$
$$p_{\frac{57}{61}}(17^2 n - 3) \equiv 0 \pmod{17^2}.$$

However, we do not have an analogous 17^3-balanced congruence because

$$p_{\frac{57}{61}}(17^3 - 3) = p_{\frac{57}{61}}(4910) \equiv \frac{1445}{2052} \not\equiv 0 \pmod{17^3}.$$

We now return to our study of necessary conditions for fractional partition function congruences. For a given α, we want to restrict the set of primes ℓ for which p_α can admit an ℓ-balanced congruence. Ahlgren and Boylan [1] showed that for the usual partition function, ℓ-balanced congruences exist only for $\ell \in \{5, 7, 11\}$. Boylan [5] later extended this result to show that for a negative odd integer α, there are finitely many primes ℓ for which p_α admits an ℓ-balanced congruence. Using the same framework as Boylan, we extend this result to even $\alpha < 0$ and odd $\alpha > 3$.

Theorem 3.1.4. Let α be an integer that is either even and < 0 or odd and > 3. If there exists an integer δ such that

$$p_\alpha(\ell n - \delta) \equiv 0 \pmod{\ell}$$

for all n, then $\ell \leq |\alpha| + 4$. In particular, p_α admits finitely many ℓ-balanced congruences.

Remark. It is unsurprising that this theorem does not extend to even positive α. When $\alpha = 24$, for example, congruences for p_α are equivalent to congruences for the Delta-function. According to Lemma 3.2.2, congruences for Δ would be implied by ℓ-non-ordinarity of Δ at different primes ℓ. Whether Δ is non-ordinary at infinitely many primes is a famous open problem [7] that has been studied extensively.

The methods used by Ahlgren and Boylan [1], and later by Boylan [5], do not extend in the same way to all rational α. Instead, we obtain a residue class restriction on the primes ℓ for which p_α can admit an ℓ-balanced congruence. To state our result, it is convenient to introduce the following notation. If m is a positive integer and β is a rational number whose denominator in lowest terms is coprime to m, we denote by $\Psi_m(\beta)$ the unique integer in the set $\{0, 1, \ldots, m-1\}$ that is congruent to β modulo m.

Theorem 3.1.5. Let $\alpha = a/b$ be a rational number that is not an even integer ≥ 0. If $\ell \geq |a| + 5b$ is a prime for which p_α admits an ℓ-balanced congruence, then

$$\Psi_{2b}\left(\frac{a}{\ell}\right) \geq b.$$

Remark. This theorem shows that for sufficiently large ℓ, half of the primes classified by residue class modulo $2b$ cannot be the modulus of a balanced congruence for p_α due to Dirichlet's Theorem.

3.2 Preliminaries

The following facts about modular forms are well-known and can be found in any standard text, such as [10]. For an integer k, denote by M_k (resp. S_k) the $\mathbb{C}$-vector space of holomorphic modular forms (resp. cusp forms) of weight k on $\mathrm{SL}_2(\mathbb{Z})$.

Recall that for any positive integer m, the mth Hecke operator $T_{m,k}$ (often abbreviated to T_m when the weight is clear) is an endomorphism on M_k. Its action on a Fourier expansion $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is given by the formula

$$f(z) \mid T_{m,k} = \sum_{n=0}^{\infty} \left(\sum_{d \mid (m,n)} d^{k-1} a(mn/d^2) \right) q^n.$$

For $m = \ell$ prime, this reduces to

$$f(z) \mid T_{\ell,k} := \sum_{n=0}^{\infty} \left(a(\ell n) + \ell^{k-1} a(n/\ell)\right) q^n.$$

We call a modular form $f(z) \in S_k$ a *Hecke eigenform* if it is an eigenvector of $T_{m,k}$ for all $m \geq 1$, i.e. there exist $\lambda(m) \in \mathbb{C}$ such that

$$f(z) \mid T_{m,k} = \lambda(m) f(z).$$

We say that a Hecke eigenform $f = \sum_{n=0}^{\infty} a(n)q^n$ is *normalized* if $a(1) = 1$.

We make extensive use of modular forms throughout this paper to prove congruences for the fractional partition functions. As a starting point, we require two key results from Chan and Wang's work [6]. The first result (Theorem 1.1 of [6]) identifies which congruences are meaningful to study.

Theorem (Thm 1.1 [6])**.** Let $\alpha = a/b$. Then the denominator of $p_\alpha(n)$ when written in lowest terms is given by

$$\text{denom}(p_\alpha(n)) = b^n \prod_{p \mid b} p^{\text{ord}_p(n!)}.$$

From this theorem, we conclude that for a given rational number α, we can study congruences for p_α modulo L whenever $\gcd(L, b) = 1$. The second result that we require is a technical lemma (Lemma 2.1 of [6]) that is a consequence of applying the Frobenius endomorphism. This lemma allows us to move exponents through q-Pochhammer symbols, which is a crucial step in several proofs.

Lemma 3.2.1 (Lemma 2.1 [6])**.** Let $\alpha = a/b$. Let ℓ be a prime not dividing b. Then for any $r \geq 1$,

$$(q;q)_\infty^{\ell^r \alpha} \equiv (q^\ell; q^\ell)_\infty^{\ell^{r-1}\alpha} \pmod{\ell^r}.$$

3.2.1 ℓ-non-ordinary primes

We now introduce the related notions of non-ordinarity at ℓ and ℓ-non-ordinarity. Throughout this subsection, we denote by L a number field with ring of integers $\mathcal{O}_L$. Recall that a modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k \cap \mathcal{O}_L[[q]]$ is said to be *non-ordinary* at ℓ if there exists a prime ideal $\mathfrak{l}$ above ℓ such that

$$a(\ell) \equiv 0 \pmod{\mathfrak{l}}.$$

For our purposes, it will be convenient to work with a strengthened form of non-ordinarity. We say that f is ℓ-non-ordinary if $a(\ell) \equiv 0 \pmod{\ell \mathcal{O}_L}$. In the case when f is a normalized Hecke eigenform, ℓ-non-ordinarity extends to congruences for many Fourier coefficients of f modulo powers of ℓ.

Lemma 3.2.2. Let $k \geq 4$ be an even integer and let ℓ be prime. Suppose $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k \cap \mathcal{O}_L[[q]]$ is a normalized Hecke eigenform. If f is ℓ-non-ordinary, then for all $r, n \geq 1$,

$$a(\ell^r n) \equiv 0 \pmod{\ell^r \mathcal{O}_L}.$$

Proof. We fix n and prove the result by induction on r. Because f is a normalized Hecke eigenform, we have

$$a(\ell n) = a(\ell)a(n) - \ell^{k-1}a(n/\ell) \equiv 0 \pmod{\ell\mathcal{O}_L}.$$

This establishes the case $r = 1$. For the inductive step, we apply the last equation with n replaced by $\ell^r n$ to find

$$a(\ell^{r+1}n) = a(\ell)a(\ell^r n) - \ell^{k-1}a(\ell^{r-1}n).$$

By the induction hypothesis for r and $r-1$, we conclude that

$$a(\ell^{r+1}n) \equiv 0 \pmod{\ell^{r+1}\mathcal{O}_L}.$$

□

It remains an open question [7] whether a generic normalized Hecke eigenform is non-ordinary at an infinite number of primes. Jin, Ma, and Ono [8] proved that if S is a finite set of primes, there are infinitely many normalized Hecke eigenforms on $\mathrm{SL}_2(\mathbb{Z})$ that are ℓ-non-ordinary for each $\ell \in S$. Their key result (Theorem 2.5 of [8]) identifies a sufficient condition to guarantee the ℓ-non-ordinarity of all cusp forms with Fourier coefficients in $\mathcal{O}_L$. Combining Theorem 2.5 and Proposition 2.1 of [8], we obtain the following result, which is crucial in the formulation of ℓ being good for α.

Lemma 3.2.3. Let $k \geq 12$ be an even integer and let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k \cap \mathcal{O}_L[[q]]$ be a normalized Hecke eigenform. Let ℓ be a prime such that $(\ell - 1) \mid (k - m)$ for some $m \in \{4, 6, 8, 10, 14\}$. Then, f is ℓ-non-ordinary.

Proof. In Theorem 2.5 of [8], we take integral $a \geq 0$ sufficiently large so that

$$k - 2 \leq (m-2)\ell^a.$$

Clearly, $\mathrm{ord}_\infty(f) \geq 1 > -\ell^a$, so applying Theorem 2.5 of [8] shows that

$$a(\ell^a) \equiv -\frac{2m}{B_m}a(0) \equiv 0 \pmod{\ell\mathcal{O}_L},$$

where B_m are the Bernoulli numbers defined by

$$\sum_{m=0}^{\infty} B_m \cdot \frac{t^m}{m!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \dots.$$

By Proposition 2.1 of [8], we conclude that $a(\ell) \equiv 0 \pmod{\ell\mathcal{O}_L}$, i.e. f is ℓ-non-ordinary. □

Given an even weight $k \geq 12$, it is known that the space of cusp forms S_k has a canonical basis of normalized Hecke eigenforms that is unique up to reordering. The coefficients of their Fourier expansions lie in the ring of integers of a number field L_k. Thus, any cusp form $f \in S_k \cap \mathcal{O}_{L_k}[[q]]$ can be written as an L_k-linear combination of normalized eigenforms. If each normalized eigenform is ℓ-non-ordinary, then f will also be ℓ-non-ordinary as long as the denominators of the coefficients in the linear combination are coprime to ℓ.

To control these denominators, we introduce the *weight k Hecke determinant* for the space of cusp forms S_k, denoted by $\mathcal{D}_k$. Write $d_k := \dim S_k$, so that S_k has a canonical basis of d_k normalized eigenforms. We put the first d_k coefficients of their Fourier expansions in each column of a $d_k \times d_k$ matrix. We denote by $\mathcal{D}_k \in \mathcal{O}_{L_k}/\{\pm 1\}$ the determinant of this matrix. Because the basis of normalized eigenforms can be reordered, $\mathcal{D}_k$ is only defined up to a sign. Due to Cramer's rule, $\mathcal{D}_k$ controls the denominators of the coefficients of a linear combination of the normalized eigenforms (see the proof of Lemma 3.2.4). For this reason, we are interested in primes ℓ for which the ideals $\mathcal{D}_k\mathcal{O}_{L_k}$ and $\ell\mathcal{O}_{L_k}$ are coprime. This condition is equivalent to the more easily verifiable condition $\ell \nmid N_k(\mathcal{D}_k)$, where N_k is the norm over the extension $L_k/\mathbb{Q}$. When $\ell \nmid N_k(\mathcal{D}_k)$, the ℓ-non-ordinarity of normalized eigenforms extends through linear combinations.

Lemma 3.2.4. Let $k \geq 12$ be an even integer and let ℓ be a prime such that $\ell \nmid N_k(\mathcal{D}_k)$ and $(\ell - 1) \mid (k - m)$ for some $m \in \{4, 6, 8, 10, 14\}$. Then for all $g = \sum_{n=1}^{\infty} a_g(n)q^n \in S_k \cap \mathcal{O}_{L_k}[[q]]$, and all positive integers r and n, we have

$$a_g(\ell^r n) \equiv 0 \pmod{\ell^r \mathcal{O}_{L_k}}.$$

Proof. Let $f_1, \ldots, f_d$ be some ordering of the canonical basis of normalized Hecke eigenforms for S_k. Write $L := L_k$ for the number field generated by their Fourier coefficients. By Lemma 3.2.3, each f_i is ℓ-non-ordinary. Write

$$f_i = \sum_{n=1}^{\infty} a_i(n)q^n \in \mathcal{O}_L[[q]].$$

There exist $\beta_i \in L_k$ such that

$$g = \sum_{i=1}^{d} \beta_i f_i.$$

Therefore, we have the matrix equation

$$\begin{bmatrix} a_1(1) & a_2(1) & \cdots & a_d(1) \\ a_1(2) & a_2(2) & \cdots & a_d(2) \\ \vdots & \vdots & \ddots & \vdots \\ a_1(d) & a_2(d) & \cdots & a_d(d) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_d \end{bmatrix} = \begin{bmatrix} a_g(1) \\ a_g(2) \\ \vdots \\ a_g(d) \end{bmatrix}.$$

By Cramer's rule, we can write $\beta_i = \gamma_i/\mathcal{D}_k$ (where we fix some sign for $\mathcal{D}_k$) for some $\gamma_i \in \mathcal{O}_L$. Thus,

$$\begin{aligned} a_g(\ell^r n) &= \beta_1 a_1(\ell^r n) + \cdots + \beta_d a_d(\ell^r n) \\ &= \frac{1}{\mathcal{D}_k} \left(\gamma_1 a_1(\ell^r n) + \cdots + \gamma_d a_d(\ell^r n) \right). \end{aligned} \tag{3.2.1}$$

Because each f_i is ℓ-non-ordinary, Lemma 3.2.2 implies that for all i,

$$a_i(\ell^r n) \equiv 0 \pmod{\ell^r \mathcal{O}_L}.$$

Because $\ell \nmid N_k(\mathcal{D}_k)$, we know that $\mathcal{D}_k\mathcal{O}_L$ is coprime to $\ell\mathcal{O}_L$. Therefore, by (3.2.1), we conclude that $a_g(\ell^r n) \equiv 0 \pmod{\ell^r \mathcal{O}_L}$. ∎

3.2.2 Modular forms modulo ℓ

To study modular forms modulo ℓ, we make use of the Serre's theory of filtration. We collect the set of modular forms modulo ℓ of weight k into the space

$$M_{k,\ell} := \{f(z) \pmod{\ell} : f(z) \in M_k \cap \mathbb{Z}[[q]]\}.$$

For an integer weight modular form f on $\mathrm{SL}_2(\mathbb{Z})$ with ℓ-integral rational coefficients, we define the *filtration of f modulo ℓ* by

$$\omega_\ell(f) := \inf\{k : f(z) \pmod{\ell} \in M_{k,\ell}\}.$$

We conclude this section with two important lemmas used in the proof of Theorem 3.1.4. The first lemma (Proposition 2.44 of [10]) is a fundamental result that describes the effect of the Ramanujan Theta-operator on filtration. The Θ-operator is defined on power series in q by

$$\Theta\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \sum_{n=0}^{\infty} na(n)q^n.$$

In terms of differentials, we have $\Theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$.

Lemma 3.2.5. If $\ell \geq 5$ is a prime and $f \in M_k \cap \mathbb{Z}[[q]]$, then $\Theta(f) \pmod{\ell}$ is the reduction of a modular form modulo ℓ. Moreover,

$$\omega_\ell(\Theta f) = \omega_\ell(f) + (\ell+1) - s(\ell-1)$$

for some integer $s \geq 0$, with $s = 0$ if and only if $\ell \nmid \omega_\ell(f)$.

The second lemma (Corollary of Proposition 2.56 of [10]) controls the filtrations of powers of Δ acted on by the Θ-operator.

Lemma 3.2.6. If $\ell \geq 5$ is a prime and δ is a positive integer, then for any $m \geq 0$,

$$\omega_\ell(\Theta^m \Delta^\delta) \geq \omega_\ell(\Delta^\delta) = 12\delta.$$

3.3 Proofs

3.3.1 Proofs of Theorem 3.1.1 and Theorem 3.1.2

The key input for Theorem 3.1.1 is a detailed study of the Θ-operator on Serre filtrations, which Kiming and Olsson used to classify when a power of Δ is fixed by a power of Θ [9].

Proof of Theorem 3.1.1. Since $\ell \nmid b$, we may write $\alpha = 24k + \ell u$ for some $k \geq 1$ and $u \in \mathbb{Z}_{(\ell)}$, where $\mathbb{Z}_{(\ell)}$ denotes the ring of ℓ-integral rational numbers. By Lemma 3.2.1, we have

$$\sum_{n=0}^{\infty} p_\alpha(n)q^{n+k} = q^k(q;q)_\infty^{24k+\ell u} = \Delta^k(q;q)_\infty^{\ell u} \equiv \Delta^k(q^\ell;q^\ell)_\infty^u \pmod{\ell}.$$

Throughout the rest of this paper, we write $\Delta^k =: \sum_{n=0}^{\infty} \tau_k(n)q^n$. We can rewrite the above congruence in the form

$$(q^\ell; q^\ell)_\infty^{-u} \sum_{n=0}^{\infty} p_\alpha(n)q^{n+k} \equiv \sum_{n=0}^{\infty} \tau_k(n)q^n \pmod{\ell}. \tag{3.3.1}$$

We extract terms of the form $q^{\ell n+c+k}$ on both sides of (3.3.1) and use the assumption that $p_\alpha(\ell n + c) \equiv 0 \pmod{\ell}$ for all n to find

$$\tau_k(\ell n + c + k) \equiv 0 \pmod{\ell}$$

for all n. It follows from Fermat's little theorem that

$$\Theta^{\ell-1}\left(q^{-(c+k)}\Delta^k\right) \equiv q^{-(c+k)}\Delta^k \pmod{\ell}.$$

Since $\alpha \not\equiv 1, 3 \pmod{\ell}$, we may apply Theorem 3 of [9] to conclude[1] that $c + k \equiv 0 \pmod{\ell}$. Consequently,

$$24c + \alpha \equiv 24(c + k) \equiv 0 \pmod{\ell}.$$

□

The proof of Theorem 3.1.2 makes use of Euler's and Jacobi's identities (3.1.1) and (3.1.2) for $\eta(24z)$ and $\eta(8z)^3$, respectively. We express the generating function for p_α in terms of these formulae and compare coefficients to obtain congruences. This provides a small extension to Theorem 1.2 (1) and (2) from [6].

Proof Theorem 3.1.2. Write $r := \mathrm{ord}_\ell(\alpha - 1)$ so that $\alpha - 1 = \ell^r u$ for some $u \in \mathbb{Z}_{(\ell)}$. It follows that

$$\sum_{n=0}^{\infty} p_\alpha(n)q^{24n+1} = q(q^{24}; q^{24})_\infty^{1+\ell^r u} = \eta(24z)(q^{24}; q^{24})_\infty^{\ell^r u}.$$

Substituting for $\eta(24z)$ with Euler's identity (3.1.1) and applying Lemma 3.2.1, we conclude that

$$\sum_{n=0}^{\infty} p_\alpha(n)q^{24n+1} \equiv (q^{24\ell}; q^{24\ell})_\infty^{\ell^{r-1}u} \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2} \pmod{\ell^r}.$$

The term $(q^{24\ell}; q^{24\ell})_\infty^{\ell^{r-1}u}$ is a power series in q^ℓ, while the term $\sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2}$ contains only monomials in whose exponents are perfect squares. Thus, if $24n + 1$ is not a square modulo ℓ, the preceding equation implies that $p_\alpha(n) \equiv 0 \pmod{\ell^r}$. This proves part (i) of Theorem 3.1.2. Part (ii) follows along the same lines by defining $r := \mathrm{ord}_\ell(\alpha - 3)$ and using Jacobi's identity (3.1.2). □

3.3.2 Proof of Theorem 3.1.3

Theorem 3.1.3 uses the ℓ-non-ordinarity of modular forms to obtain ℓ^r-balanced congruences for p_α. To do so, we express the generating function for p_α in terms of a power of Ramanujan's Delta-function. Under suitable hypotheses, this power of Δ is ℓ-non-ordinary, from which we obtain ℓ^r-balanced congruences for p_α.

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[1] As phrased in [9], this theorem can only be applied in our setting when $c + k \in \{0, 1, \ldots, \ell - 1\}$. However, it is clear from the proof of the theorem that it applies for $c + k$ an arbitrary integer.

Proof of Theorem 3.1.3. Write $r := \mathrm{ord}_\ell(24k-\alpha)$ so that $\alpha - 24k = \ell^r u$ for some $u \in \mathbb{Z}_{(\ell)}$. We prove by induction that for all $1 \le i \le r$, we have

$$\sum_{n=0}^{\infty} p_\alpha(\ell^i n - k)q^n \equiv (q;q)_\infty^{\ell^{r-i}u} \sum_{n=0}^{\infty} \tau_k(\ell^i n)q^n \pmod{\ell^r}. \tag{3.3.2}$$

For the base case $i = 1$, we use Lemma 3.2.1 to write

$$\sum_{n=0}^{\infty} p_\alpha(n)q^{n+k} = q^k(q;q)_\infty^{24k+\ell^r u} = \Delta^k(q;q)_\infty^{\ell^r u} \equiv \Delta^k(q^\ell;q^\ell)_\infty^{\ell^{r-1}u} \pmod{\ell^r}.$$

Extracting terms of the form $q^{\ell n}$ on both sides of the last equation and replacing q^ℓ by q, we obtain

$$\sum_{n=0}^{\infty} p_\alpha(\ell n - k)q^n \equiv (q;q)_\infty^{\ell^{r-1}u} \sum_{n=0}^{\infty} \tau_k(\ell n)q^n \pmod{\ell^r}.$$

This establishes the base case.

For the inductive step, we note that the conditions of Lemma 3.2.4 are satisfied because ℓ is good for α with parameter k. Therefore, by Lemma 3.2.4 we have

$$\tau_k(\ell^i n) \equiv 0 \pmod{\ell^i}.$$

In addition, Lemma 3.2.1 gives

$$(q;q)_\infty^{\ell^{r-i}u} \equiv (q^\ell;q^\ell)_\infty^{\ell^{r-(i+1)}u} \pmod{\ell^{r-i}}.$$

Thus, we can rewrite (3.3.2) in the form

$$\sum_{n=0}^{\infty} p_\alpha(\ell^i n - k)q^n \equiv (q^\ell;q^\ell)_\infty^{\ell^{r-(i+1)}u} \sum_{n=0}^{\infty} \tau_k(\ell^i n)q^n \pmod{\ell^r}.$$

Extracting terms of the form $q^{\ell n}$ on both sides of the previous equation and replacing q^ℓ by q, we find

$$\sum_{n=0}^{\infty} p_\alpha(\ell^{i+1} n - k)q^n \equiv (q;q)_\infty^{\ell^{r-(i+1)}u} \sum_{n=0}^{\infty} \tau_k(\ell^{i+1} n)q^n \pmod{\ell^r}.$$

This completes the induction.

To finish the proof, we take $i := v$ in (3.3.2). By Lemma 3.2.4, we know that $\tau_k(\ell^v n) \equiv 0 \pmod{\ell^v}$ for all n. Thus, we conclude that

$$\sum_{n=0}^{\infty} p_\alpha(\ell^v n - k)q^n \equiv (q;q)_\infty^{\ell^{r-v}u} \sum_{n=0}^{\infty} \tau_k(\ell^v n)q^n \equiv 0 \pmod{\ell^v}. \qquad \square$$

Theorem 3.1.3 yields an ℓ^r-balanced congruence for p_α for all primes ℓ good for α. In fact, for all but finitely many α, there are only finitely many primes ℓ that are good for α. Furthermore, the associated parameter k is strictly bounded by ℓ from above.

Proposition 3.3.1. If a prime ℓ is good for α with parameter k, then $k < \ell$.

Proof. Suppose for contradiction that $k \geq \ell$. By Lemma 3.2.3, we know that all normalized Hecke eigenforms in S_{12k} are ℓ-non-ordinary. Therefore, each entry in the ℓth row of the matrix defining the weight $12k$ Hecke determinant $\mathcal{D}_{12k}$ is divisible by ℓ. Consequently, $\ell \mid N_{12k}(\mathcal{D}_{12k})$, in contradiction to (3.1.5). This contradiction proves that $k < \ell$. □

We can use the preceding proposition to obtain an upper bound on a prime ℓ that is good for $\alpha = a/b$ in terms of a and b. This proves that for all but finitely many α, Theorem 3.1.3 identifies only finitely many congruences for p_α.

Proposition 3.3.2. Suppose that ℓ is good for $\alpha = a/b$. If α is not an even integer in the range $[-14, 26]$, then $\ell \leq \max\{|26b - a|, |14b + a|\}$.

Proof. Suppose that ℓ is good for α with parameter k. If $\ell \in \{2, 3\}$, the result is obvious. Thus, suppose that $\ell \geq 5$. By (3.1.4), we can write $12k - m = c(\ell - 1)$ for some $m \in \{4, 6, 8, 10, 14\}$ and $c \in \mathbb{Z}$. Because $k \geq 1$, it is easy to see that $c \geq 1$. By Proposition 3.3.1, we have

$$12(\ell - 1) \geq 12k = c(\ell - 1) + m.$$

Thus, $c < 12$. It follows that $c \leq 11$. We now use (3.1.3) to obtain

$$\begin{aligned} 0 \equiv b(24k - \alpha) &= 24kb - a \\ &= 2b(c(\ell - 1) + m) - a \\ &\equiv 2b(m - c) - a \pmod{\ell}. \end{aligned}$$

Note that $-14 \leq 2(m - c) \leq 26$. Because α is not an even integer in the range$[-14, 26]$, we know that $2b(m-c)-a \neq 0$. It follows that $|2b(m-c)-a| \geq \ell$. Testing the extremal values of $2(m-c)$, we obtain the desired result. □

Remark. By checking the possible values of $2(m-c)$ and using Theorem 3.1.5 to handle the case of $\alpha < 0$, the preceding proposition can easily be extended to all $\alpha \notin \{0, 2, 4, 6, 8, 10, 14, 26\}$. This set of exceptional α corresponds to the set of even lacunary powers of the eta-function classified by Serre.

3.3.3 Proofs of Theorem 3.1.4 and Theorem 3.1.5

For our next proof, we employ similar strategies to Ahlgren and Boylan [1], and Boylan [5], taking advantage of Serre filtrations of modular forms. We restate an ℓ-balanced congruence for p_α equivalently in terms of a modular form f_ℓ being fixed by $\Theta^{\ell-1}$. We then study the sequence of weights $\omega_\ell(\Theta^i f_\ell)$ to gain further information.

Proof of Theorem 3.1.4. Suppose for contradiction that p_α admits an ℓ-balanced congruence of the given form for some prime $\ell > |\alpha| + 4$. It follows that $\ell \geq 5$ and $\alpha \not\equiv 0, 2, 4 \pmod{\ell}$. Without loss of generality, we take δ to be a positive integer. By Theorem 3.1.1, we know that $24\delta \equiv \alpha \pmod{\ell}$. Let $24\delta = \alpha + \ell u$ for some $u \in \mathbb{Z}_{(\ell)}$. It follows from Lemma 3.2.1 that

$$\Delta^\delta = q^\delta (q; q)_\infty^{\alpha + \ell u} = (q; q)_\infty^{\ell u} \sum_{n=0}^{\infty} p_\alpha(n - \delta) q^n \equiv (q^\ell; q^\ell)_\infty^u \sum_{n=0}^{\infty} p_\alpha(n - \delta) q^n \pmod{\ell}.$$

Put $f_\ell := \Delta^\delta$. By Fermat's Little Theorem, we have

$$\Theta^{\ell-1} f_\ell - f_\ell = \sum_{n=0}^{\infty} n^{\ell-1} p_\alpha(n-\delta) q^n - \sum_{n=0}^{\infty} p_\alpha(n-\delta) q^n \equiv \sum_{n \equiv 0 \pmod{\ell}} p_\alpha(n-\delta) q^n \pmod{\ell}.$$

Thus, the ℓ-balanced congruence in the theorem statement is equivalent to

$$\Theta^{\ell-1} f_\ell \equiv f_\ell \pmod{\ell},$$

Let $c := \Psi_\ell(-12\delta)$. Because $-2c \equiv \alpha \not\equiv 2, 4 \pmod{\ell}$, we have $c < \ell - 2$. Repeatedly applying Lemma 3.2.5, we observe that

$$\omega_\ell(\Theta^c f_\ell) = \omega_\ell(f_\ell) + c(\ell+1) \equiv 12\delta + c \equiv 0 \pmod{\ell}.$$

By Lemma 3.2.5, there exists an integer $s \geq 1$ such that

$$\begin{aligned} \omega_\ell(\Theta^{c+1} f_\ell) &= \omega_\ell(\Theta^c f_\ell) + (\ell+1) - s(\ell-1) \\ &= 12\delta + (c+1)(\ell+1) - s(\ell-1). \end{aligned} \tag{3.3.3}$$

Applying Lemma 3.2.6, we deduce that

$$s \leq \frac{(c+1)(\ell+1)}{\ell-1} < c + 3.$$

Hence, $s \leq c + 2$. We now choose $j \geq 1$ minimal such that

$$\omega_\ell(\Theta^{c+j} f_\ell) \equiv 0 \pmod{\ell}.$$

Such a j exists and satisfies $j \leq (\ell-2) - c$ by the following argument. If $\omega_\ell(\Theta^{\ell-2} f_\ell) \not\equiv 0 \pmod{\ell}$, then by Lemma 3.2.5, we get

$$\omega_\ell(\Theta^{\ell-2} f_\ell) = \omega_\ell(\Theta^{\ell-1} f_\ell) - (\ell+1) = \omega_\ell(f_\ell) - (\ell+1) < \omega_\ell(f_\ell),$$

in contradiction to Lemma 3.2.6. Thus, it is imperative that $\omega_\ell(\Theta^{\ell-2} f_\ell) \equiv 0 \pmod{\ell}$. This proves that j exists and $j \leq (\ell-2) - c$. We now observe that

$$0 \equiv \omega_\ell(\Theta^{c+j} f_\ell) = 12\delta + (c+j)(\ell+1) - s(\ell-1) \equiv j + s \pmod{\ell}.$$

By the bounds on j and s, we must have $(j, s) = (\ell - c - 2, c + 2)$. Rewriting (3.3.3), we obtain

$$\omega_\ell(\Theta^{c+1} f_\ell) = 12\delta + 2c - \ell + 3.$$

By Lemma 3.2.6, we deduce that

$$2c - \ell + 3 \geq 0. \tag{3.3.4}$$

If $\alpha < 0$ is even, then $2c = 2\Psi_\ell\left(-\frac{\alpha}{2}\right) = -\alpha$. Hence

$$2c - \ell + 3 < -\alpha - (|\alpha| + 4) + 3 < 0.$$

If $\alpha > 3$ is odd, then $2c = 2\Psi_\ell\left(-\frac{\alpha}{2}\right) = \ell - \alpha$. Hence

$$2c - \ell + 3 < (\ell - \alpha) - \ell + 3 < 0.$$

In either case, we derive a contradiction to (3.3.4). This contradiction proves that $\ell \leq |\alpha| + 4$. $\square$

Theorem 3.1.4 cannot be easily extended for non-integral α, because $c = \Psi_\ell\left(-\frac{\alpha}{2}\right)$ oscillates from 0 to $\ell - 1$ as ℓ varies. However, we can use the proof of Theorem 3.1.4 for rational α to constrain the set of primes ℓ for which p_α can admit an ℓ-balanced congruence.

Proof of Theorem 3.1.5. Assume that $\ell > |a| + 5b$ and write $\ell = 2bk + r$, where $r := \Psi_{2b}(\ell)$. Clearly $\gcd(r, 2b) = 1$, so we may define $s \in \mathbb{Z}$ such that

$$\Psi_{2b}\left(\frac{a}{r}\right) r = 2bs + a.$$

Put

$$c := \Psi_{2b}\left(\frac{a}{r}\right) k + s.$$

If α is a negative even integer, then $\Psi_{2b}\left(\frac{a}{r}\right) = 0$, and it is easy to check that $c = -\frac{\alpha}{2} \geq 0$. Otherwise $\Psi_{2b}\left(\frac{a}{r}\right) \geq 1$, and hence

$$c \geq k + s \geq k - \frac{a}{2b} = \frac{\ell - (r + a)}{2b} \geq 0.$$

Furthermore, we check that

$$2bc = \Psi_{2b}\left(\frac{a}{r}\right)(2bk) + 2bs \equiv \Psi_{2b}\left(\frac{a}{r}\right)(-r) + 2bs = -a \pmod{\ell}.$$

Thus, $c \equiv -\frac{\alpha}{2} \pmod{\ell}$. Suppose for contradiction that $\Psi_{2b}\left(\frac{a}{r}\right) < b$. It follows that

$$2s - r = \frac{\Psi_{2b}\left(\frac{a}{r}\right) r - a}{b} - r < -\frac{a}{b}.$$

Therefore, we estimate

$$\begin{aligned} 2c - \ell + 3 &= k\left(2\Psi_{2b}\left(\frac{a}{r}\right) - 2b\right) + (2s - r) + 3 \\ &< -2k - \frac{a}{b} + 3 \\ &= \frac{(3b + r - a) - \ell}{b} < 0. \end{aligned} \tag{3.3.5}$$

In particular, $c < \ell$ so $c = \Psi_\ell\left(-\frac{\alpha}{2}\right)$. Because $\ell \geq |a| + 5b > |\alpha| + 4$ and $\alpha \notin \{0, 2, 4\}$, the proof of Theorem 3.1.4 ensures that (3.3.4) still holds. However, (3.3.5) contradicts (3.3.4). This contradiction proves that if p_α admits an ℓ-balanced congruence for $\ell \geq |a| + 5b$, then $\Psi_{2b}(\frac{a}{r}) \geq b$. □

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Number Theory REU at Emory University **Emory University**
Researcher June 2019 – July 2019
- Worked with Dr. Ken Ono from the University of Virginia to conduct research on integer partitions.
- Studied congruences among the fractional partition functions, constructing a framework to find congruences using the theory of modular forms and non-ordinary primes.
- Synthesized results from a large range of papers to apply to the fractional partition functions, using the theory of Theta operators and Serre filtration to discover necessary conditions for congruences.
- Drafted and edited a research paper with two other students that has been submitted for publication.

Summer Undergraduate Applied Mathematics Institute **Carnegie Mellon University**
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- Participated in summer research program designed to emulate the graduate school experience; conducted research while taking an advanced undergraduate course in combinatorial optimization.
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- Volunteered as a web developer for the THON Information Network (THINK), a website that acts as an information hub for THON's 16,500 student volunteers and for families affected by pediatric cancer.
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- Assisted in running a 60-members club that acts as the ambassadors for the Eberly College of Science.
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Springfield, Benefitting THON **Pennsylvania State University**
Donor and Alumni Relations Chair April 2018 – April 2019
- Led an organization of 100 students in philanthropic efforts for families fighting pediatric cancer.
- Spent 15-30 hours a week coordinating fundraising campaigns, community outings and other events for the organization in order to facilitate a successful fundraising season.
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CONFERENCES AND PRESENTATIONS

- Young Mathematicians Conference Talk (2019) "Ramanujan Congruences for Fractional Partition Functions."
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