THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF MATHEMATICS

MORSE–BOTT THEORY AND DIAMETERS OF IMMERSED SUBMANIFOLDS OF EUCLIDEAN SPACE

Victor Ginsburg
Spring 2022

A thesis
submitted in partial fulfillment
of the requirements
for a baccalaureate degree
in Mathematics
with honors in Mathematics

Reviewed and approved* by the following:

Sergei Tabachnikov
Professor of Mathematics
Thesis Supervisor

Nathanial Brown
Professor of Mathematics
Honors Adviser

*Electronic approvals are on file.

Abstract

We lay the groundwork for Morse–Bott theory, which we then apply to obtain two lower bounds for the number of diameters of an immersed submanifold of Euclidean space. The sharper of the two estimates is due to Pushkar' [Pus97], and essentially marks the conclusion to this classical problem.

Table of Contents

List of Figures

Acknowledgements

My deepest gratitude goes to Sergei Tabachnikov for his constant encouragement and mentorship throughout my undergraduate education.

Chapter 1

Morse–Bott theory

1.1 Introduction

Let M be a smooth manifold and let $f : M \to \mathbb{R}$ be a real-valued smooth function. The points $p \in M$ for which the differential df_p vanishes are called *critical points*, and their images under f are called *critical values*. We denote by $\mathrm{Crit}(f)$ the set of critical points of f. Broadly speaking, Morse theory describes quantitative relationships between the topology of a smooth manifold M and the properties of the critical points of a "nondegenerate" smooth function $f : M \to \mathbb{R}$. Morse–Bott theory extends these relationships to "nondegenerate" functions whose differentials vanish on submanifolds of M of arbitrary dimension.

Morse–Bott theory has many classical applications. One example from homotopy theory is the celebrated Bott periodicity theorems for the unitary and orthogonal groups. These theorems are proven with a combination of variational and Morse-theoretic arguments that establish connections between the topology of the path space of a Riemannian manifold M and the geometry of M. We refer the reader to [Mil63] and [Bot82] for details. In Chapter 2, we describe some applications of Morse–Bott theory to finding lower bounds on the number of diameters of a generic submanifold of Euclidean space,

Beyond applications, there are several ways to establish the basic tools of Morse–Bott theory, each of which has led to fruitful generalizations. In dynamical systems, Conley index theory is a powerful tool for analyzing dynamics via the topology of invariant sets. The Conley index of a hyperbolic fixed point recovers the Morse index (Definition 1.4) of that point, so the Conley index can be seen as a generalization of the Morse index. Some of the foundation of Conley index theory rests on the use of dynamics to construct homotopy equivalences of various sets, a technique which the reader may compare to our proofs of Theorems 1.9 and 1.10. See [Mis95] and [Mis99] for details on Conley index theory. Building on Conley index theory, in [Flo89], Floer made significant progress on the Arnol'd conjecture in symplectic geometry by introducing Floer homology, an infinite-dimensional analogue of Morse homology (Section 1.6). Finally, Witten used analytic tools from quantum field theory to obtain Morse theory in his celebrated paper [Wit82], thereby connecting theoretical physics to Morse theory.

It is worth noting that there are other methods, not directly related to Morse–Bott theory, of attacking variational problems using homotopy invariants. A prime example is the Lusternik–Schnirelmann category—see [CLOT03] for an introduction.

We will now discuss the basic ideas and results of Morse–Bott theory, beginning with Morse theory.

1.2 Morse functions

We recall that a smooth vector field X on a smooth manifold M can be regarded as a derivation of the ring of smooth functions $C^\infty(M)$ by defining $Xf = df(X)$ for $f \in C^\infty(M)$.

Lemma 1.1. Let $f : M \to \mathbb{R}$ be a smooth function and let $p \in M$ be a critical point of f. Suppose X, X', Y, Y' are smooth vector fields satisfying $X(p) = X'(p)$ and $Y(p) = Y'(p)$. Then

$$XYf(p) = X'Y'f(p) = YXf(p).$$

Proof. Since p is a critical point of f, we have

$$[X,Y]f(p)=0,$$

so $XYf(p)=YXf(p)$. Since $(X-X')(p)=0$, we have $(X-X')Yf(p)=0$ and thus

$$XYf(p)=X'Yf(p).$$

Repeating these calculations for other combinations of X,Y,X',Y', we conclude

$$X'Yf(p)=YX'f(p)=Y'X'f(p)=X'Y'f(p),$$

proving the claim. □

Let $p\in M$ be a critical point of f. We define the *Hessian* of f at p by

$$H_{f,p}:T_pM\times T_pM\to\mathbb{R}$$

by $H_{f,p}(v,w)=XYf(p)$, where X,Y are any smooth vector fields such that $X(p)=v$ and $Y(p)=w$. By Lemma 1.1, $H_{f,p}$ is well-defined and is a symmetric bilinear form.

Definition 1.2. If the Hessian at p is a nondegenerate bilinear form, then p is called a *nondegenerate* critical point, and otherwise p is called *degenerate*.

Definition 1.3. A smooth function with no degenerate critical points is called a *Morse function*.

Let $x=(x_1,\dots,x_n)$ be coordinates centered at a critical point p. Any smooth vector fields X,Y can be locally written $X=\sum_i X^i\partial_{x_i}$ and $Y=\sum_i Y^i\partial_{y_i}$. Then in the x-coordinates, we have

$$H_{f,p}(X,Y)=\sum_{i,j}h_{ij}X^iY^j,$$

where $h_{ij}=\partial_{x_i}\partial_{x_j}f(p)$. The Hessian $H_{f,p}$ is a nondegenerate bilinear form if and only if $\det(h_{ij})\neq 0$, and this condition does not depend on the choice of coordinates x. We remark that the function $H_{f,p}(x):=\sum_{i,j}h_{ij}x_ix_j$ appears in the Taylor expansion of f at p:

$$f(x)=f(p)+\frac{1}{2}H_{f,p}(x)+O\left(\sum_{i,j,k}x_ix_jx_k\right). \tag{1.1}$$

We will return to this discussion shortly.

Suppose p is a nondegenerate critical point. By the Sylvester law from linear algebra, there exist coordinates $x=(x_1,\dots,x_n)$ centered at p such that for any $v=\sum_i v^i\partial_{x_i}(p)\in T_pM$, we have

$$H_{f,p}(v,v)=-(|v^1|^2+\dots+|v^\lambda|^2)+|v^{\lambda+1}|^2+\dots+|v^n|^2. \tag{1.2}$$

The Sylvester law assures us that the number λ does not depend on the coordinates x. Therefore, we may define:

Definition 1.4. The number $\lambda=\lambda(f,p)$ in (1.2) is called the *Morse index* of f at p.

We now state the *Morse lemma*, a foundational fact of Morse theory, which the reader should compare to (1.1).

Lemma 1.5 (Morse lemma)**.** Let p be a nondegenerate critical point of f. Then there exist coordinates $y = (y_1, \dots, y_n)$ centered at p such that in the y-coordinates, we have

$$f(y) = f(p) - (y_1^2 + \cdots + y_{\lambda(f,p)}^2) + y_{\lambda(f,p)+1}^2 + \cdots + y_n^p.$$

Proof. The most straightforward proof is an application of linear algebra to a coordinate representation (h_{ij}) of the Hessian $H_{f,p}$. See Lemma 2.2 in [Mil63] for details. □

Corollary 1.6. Let $f : M \to \mathbb{R}$ be a Morse function on a closed manifold M. Then $\mathrm{Crit}(f)$ is finite.

Proof. By Lemma 1.5, there is a coordinate chart $(U, (y_1, \dots, y_n))$ centered at p such that the Jacobian at p of the map $U \to \mathbb{R}^n$ defined by $q \mapsto (\partial_{y_1} f(q), \dots, \partial_{y_n} f(q))$ is a diagonal matrix with diagonal entries ± 2. Shrinking U if necessary, we can apply the inverse function theorem to conclude that df does not vanish on $U \setminus \{p\}$. Therefore $\mathrm{Crit}(f)$ is discrete, and since M is compact we conclude that $\mathrm{Crit}(f)$ is finite. □

Example 1.7. Consider the torus $\mathbb{T}^2 \subset \mathbb{R}^3$ positioned on its side, so that the positive z-axis passes through its center of mass and so that it is tangent to the xy-plane at the origin (see Figure 1.1). Let $f : \mathbb{T}^2 \to \mathbb{R}$ be the projection $(x, y, z) \mapsto z$, or in other words, the height above the xy-plane. Then f has four nondegenerate critical points: its minimum and maximum, and the highest and lowest point on the shortest longitude of $\mathbb{T}^2$. The minimum has Morse index 0, the two intermediate points have Morse index 1 (they are saddle points), and the maximum has Morse index 2.

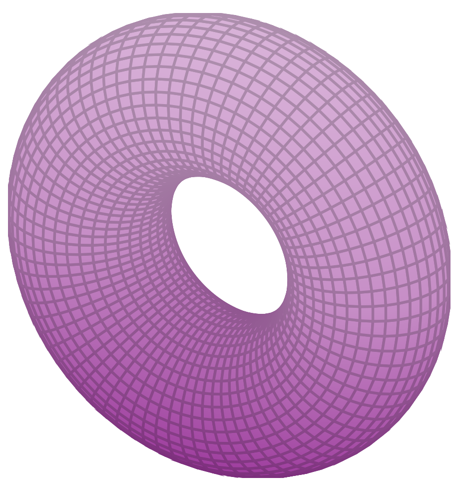

Figure 1.1: Vertical torus

1.2.1 Morse functions are generic

As we prepare to prove several facts about Morse functions, it is natural to ask how common Morse functions are, or even whether every manifold admits them. The applicability of Morse theory is in large part due to the ubiquity of Morse functions:

Theorem 1.8. The set of Morse functions on a smooth manifold M is an open dense subset of $C^\infty(M)$ in the C^2-topology.

Proof. See Section 1.2 of [Nic11] for the proof and for more precise genericity statements. □

1.3 Structure of sublevel sets

For the remainder of this chapter, we will assume that M is a closed manifold. Let $f : M \to \mathbb{R}$ be a Morse function. In this section, we address ourselves to relating the homotopy type of M to the critical points of f.

Given $a \in \mathbb{R}$, we denote by M^a the sublevel set $f^{-1}((-\infty, a])$.

Theorem 1.9. Suppose there are real numbers $a < b$ such that $f^{-1}([a, b])$ contains no critical points of f. Then M^a and M^b are diffeomorphic. Moreover, M^a is a strong deformation retract of M^b.

Proof. Equip M with a Riemannian metric $\langle \cdot, \cdot \rangle$. Recall that for a smooth function $f : M \to \mathbb{R}$, we may define the *gradient* ∇f to be the unique smooth vector field on M such that for any smooth vector field X, we have $\langle X, \nabla f \rangle = df(X)$. We wish to flow from $f^{-1}(\{b\})$ to $f^{-1}(\{a\})$ along the trajectories of $-\nabla f$. More precisely, since $f^{-1}([a, b])$ contains no critical points, we can find a smooth function $h : M \to [0, \infty)$ that is equal to $1/\|\nabla f\|^2$ on $f^{-1}([a, b])$. Let Φ be the flow of the vector field $-h\nabla f$. Fix $p \in M$. We have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} f(\Phi^t(p)) &= \left\langle -h(\Phi^{t_0}(p))\nabla f(\Phi^{t_0}(p)), \nabla f(\Phi^{t_0}(p)) \right\rangle \\
&= -h(\Phi^{t_0}(p)) \left\|\nabla f(\Phi^{t_0}(p))\right\| \\
&\leq 0.
\end{aligned}$$

The map $t \mapsto f(\Phi^t(p))$ is therefore decreasing, and it has derivative equal to -1 at any t_0 for which $\Phi^{t_0}(p) \in f^{-1}([a, b])$. It follows that $\Phi^{b-a}(M^b) = M^a$, which establishes the first claim.

The one-parameter family of maps $r_t : M^b \to M^b$ given by

$$r_t(p) = \begin{cases} p, & p \in M^a \\ \Phi^{t(f(p)-a)}(p), & p \in f^{-1}([a, b]) \end{cases}$$

is a homotopy between the identity on M^b and the retract $\Phi^{b-a}|_{M^b} : M^b \to M^a$. Its restriction to M^a is the identity for all t. This proves the second claim. □

Recall that for a nonnegative integer m, an m-cell is by definition a closed m-disk D^m.

Theorem 1.10. Let c be a critical value for f with one corresponding critical point p. Let $\lambda = \lambda(f, p)$ denote the Morse index of p. Then for all sufficiently small $\varepsilon > 0$, the set $M^{c+\varepsilon}$ is homotopic to $M^{c-\varepsilon}$ with a λ-cell attached.

Example 1.11. Before proving Theorem 1.10, we heuristically verify it for the height function $f : \mathbb{T}^2 \to \mathbb{R}$ defined in Example 1.7. Let p, q, r, s be its four critical points with critical values $0 = f(p) < f(q) < f(r) < f(s)$ and respective Morse indices $0, 1, 1, 2$. Notice that these indices match the dimensions of the cells in the standard CW decomposition of $\mathbb{T}^2$. Let $\varepsilon > 0$ be smaller than the smallest distance between two distinct critical values of f.

Since f is the height above the xy-plane, we see that $M^{-\varepsilon}$ is empty, M^0 consists of one point, and M^ε is bowl-shaped and homotopic to a point. By Theorem 1.9, the latter holds for M^t with any $0 \leq t < f(q)$. At the critical value $f(q)$, the level set $M^{f(q)}$ now contains a meridian of $\mathbb{T}^2$ attached to the disk, so M^q is homotopic to the circle. But the circle is homotopic to $M^{f(q)-\varepsilon}$ with a 1-cell attached. The same is true for $M^{f(q)+\varepsilon}$. Continuing in this way, we see that $M^{f(r)+\varepsilon}$ is the wedge $S^1 \vee S^1$, which is just $M^{f(r)-\varepsilon}$ with a 1-cell glued to the 0-cell. Similarly, $M^{f(s)+\varepsilon} = \mathbb{T}^2$ is homotopic to $S^1 \vee S^1$ with the 2-disk D^2 glued along its boundary, but this is just $M^{f(r)+\varepsilon}$ with a 2-cell attached along $M^{f(s)-\varepsilon}$. □

Proof of Theorem 1.10. For simplicity we assume $c = 0$. By Lemma 1.5 and Corollary 1.6, there exist $\varepsilon > 0$ and a coordinate chart $(U, y = (y_1, \dots, y_n))$ centered at p such that the following hold.

- $f^{-1}([-\varepsilon, +\varepsilon])$ contains no critical points besides p,
- $y(U) \subset \mathbb{R}^n$ contains the closed disk $D := \{\sum_i y_i^2 \leq 2\varepsilon\}$, and
- $f|_D(y) = -y_1^2 - \dots - y_\lambda^2 + y_{\lambda+1}^2 + \dots + y_n^2$.

We write $y_- := \sum_{i=1}^\lambda y_i^2$ and $y_+ := \sum_{j=\lambda+1}^n y_j^2$, and we extend these to smooth functions $M \to \mathbb{R}$ such that $(y_- + y_+)|_{M \setminus D} > 2\varepsilon$. We will also use D to denote the coordinate disk $y^{-1}(D) \subset M$.

We want to apply Theorem 1.9. To do this, we need to define a Morse function F that agrees with f on $M \setminus D$ and takes values slightly smaller than the critical value 0 on the disk D. Fix a smooth cutoff function $\mu : [0, \infty) \to \mathbb{R}$ such that:

$$\mu(0) > \varepsilon, \quad 0 \geq \mu'(t) > -1 \text{ for all } t \geq 0, \quad \text{and } \mu'(0) = \mu(t) = 0 \text{ for all } t \geq 2\varepsilon.$$

Define $F : M \to \mathbb{R}$ by $F = f - \mu(y_- + 2y_+)$. Our extensions of y_- and y_+ guarantee that $F|_{M\setminus D} = f$, and by definition we have

$$F|_D = -y_- + y_+ - \mu(y_- + 2y_+).$$

The following claims are straightforward consequences of the properties of μ.

- F is Morse with $\mathrm{Crit}(F) = \mathrm{Crit}(f)$,
- $F(p) < -\varepsilon$ and $F(q) = f(q)$ for all $q \in \mathrm{Crit}(f) \setminus \{p\}$, and
- for all $a \in \mathbb{R}$, the f-sublevel set M_f^a is contained in the F-sublevel set M_F^a. Moreover, $M_f^b = M_F^b$ for all $b \geq \varepsilon$.

It follows that $F^{-1}([-\varepsilon, \varepsilon])$ contains no critical points, and Theorem 1.9 then implies that $M_F^{-\varepsilon}$ is diffeomorphic to $M_F^\varepsilon = M_f^\varepsilon$. It therefore suffices to show that $M_F^{-\varepsilon}$ is homotopic to $M_f^{-\varepsilon}$ with a λ-cell attached.

Define

$$H := \overline{M_F^{-\varepsilon} \setminus M_f^{-\varepsilon}} = \{q \in M : F(q) \leq -\varepsilon \leq f(q)\},$$

so that $M_F^{-\varepsilon} = M_f^{-\varepsilon} \cup H$.

For $q \in M_F^{-\varepsilon} \setminus M_f^{-\varepsilon}$, we have $F(q) < f(q)$ and therefore $\mu(y_-(q) + 2y_+(q)) > 0$. This is true only if $y_-(q) + 2y_+(q) < 2\varepsilon$, which implies $q \in D$ because y_- and y_+ are nonnegative. Since D is closed, we conclude that $H \subset D$. Moreover, H contains the λ-cell e^λ defined by

$$e^\lambda := \{q \in M : y_-(q) \leq \varepsilon, y_+(q) = 0\}.$$

Intuitively, H is a λ-handle (i.e. $D^\lambda \times D^{n-\lambda}$) from which e^λ is obtained by first contracting H onto e^λ transversely to the level sets of y_+, and then by shrinking the rest of $H \setminus e^\lambda$ to the boundary $\partial e^\lambda = y_-^{-1}(\{\varepsilon\}) \cap y_+^{-1}(\{0\})$. Indeed, we claim that $M_f^{-\varepsilon} \cup e^\lambda$ is a deformation retract of $M_F^{-\varepsilon} = M_f^{-\varepsilon} \cup H$.

To prove the claim, we need to construct a one-parameter family of continuous maps $r_t : M_F^{-\varepsilon} \to M_F^{-\varepsilon}$. On $M_F^{-\varepsilon} \setminus D$ and on $M_f^{-\varepsilon}$, define r_t to be the identity for all $0 \leq t \leq 1$. It remains to define r_t on H. It suffices to do so on the two regions R^1 and R^2, where R^1 is the set of points in H for which $y_- \leq \varepsilon$ and R^2 is the set of points in H for which $\varepsilon \leq y_- \leq y_+ + \varepsilon$. Indeed, if $y_+(q) + \varepsilon \leq y_-(q)$ for $q \in H$, then $q \in M_f^{-\varepsilon}$ and we have already defined $r_t(q) = q$.

On R^1, define r_t in the y-coordinates by

$$r_t(y_1, \ldots, y_n) = (y_1, \ldots, y_\lambda, ty_{\lambda+1}, \ldots, ty_n)$$

and pull the definition back to $M_F^{-\varepsilon}$. A computation shows that $\frac{\partial F}{\partial y_+} > 0$, so r_t is well-defined as a map $M_F^{-\varepsilon} \to M_F^{-\varepsilon}$ for all $0 \leq t \leq 1$. By construction, r_1 is the identity on R^1 and r_0 contracts R^1 to e^λ.

Given $(y_1, \ldots, y_n)$ in the region of H defined by $\varepsilon \leq y_- < y_+ + \varepsilon$, define a number $0 \leq s_t \leq 1$ by

$$s_t := t + (1-t)\sqrt{(y_- - \varepsilon)/y_+}$$

and extend s_t to a continuous function on R^2. Define r_t on R^2 in y-coordinates by

$$r_t(y_1, \ldots, y_n) = (y_1, \ldots, y_\lambda, s_t y_{\lambda+1}, \ldots, s_t y_n).$$

Then r_1 is the identity on R^2 and r_0 maps R^2 into $f^{-1}(\{-\varepsilon\})$:

$$f(r_0(y)) = -y_- + \frac{(y_- - \varepsilon)}{y_+} y_{\lambda+1}^2 + \cdots + \frac{(y_- - \varepsilon)}{y_+} y_n^2 = -\varepsilon.$$

It is straightforward to check that these definitions of r_t agree on the overlaps of $M_f^{-\varepsilon}$, R^1, and R^2, and give rise to a one-parameter family of continuous maps. By construction, r_1 is the identity and r_0 has image $M_f^{-\varepsilon} \cup e^\lambda$, with e^λ attached to $f^{-1}(\{-\varepsilon\})$ along ∂e^λ by the inclusion map. □

In Example 1.11 we saw that Theorem 1.10 actually reflected the data of the standard CW structure of $\mathbb{T}^2$. Something more general is true:

Corollary 1.12. Let $f : M \to \mathbb{R}$ be a Morse function on a closed manifold M. Then M is homotopic to a CW complex consisting of one $\lambda(f,p)$-cell for each $p \in \mathrm{Crit}(f)$.

Proof. We present a heuristic argument and refer the reader to Theorem 3.5 in [Mil63] for a rigorous proof. Let c be a critical value of f whose preimage $f^{-1}(\{c\})$ consists of critical points $p_1, \dots, p_k$ with Morse indices $\lambda_1, \dots, \lambda_k$. The argument in the proof of Theorem 1.10 can be extended to show that for small $\varepsilon > 0$, the sublevel set $M^{c+\varepsilon}$ is obtained up to homotopy by attaching cells $e^{\lambda_1}, \dots, e^{\lambda_k}$ to $M^{c-\varepsilon}$. By compactness, $M^t = \varnothing$ for all $t \ll 0$. Up to some topological technicalities, the corollary follows by letting t increase to ∞ and applying Theorem 1.10 whenever t reaches a critical value of f. □

1.4 Morse inequalities

At the heart of Morse theory are algebraic relationships between the topology of a manifold and the critical points of a Morse function. In this section we discuss one. For a Morse function $f : M \to \mathbb{R}$, define the *Morse polynomial* of f by

$$P_f(t) := \sum_{p \in \mathrm{Crit}(f)} t^{\lambda(f,p)}.$$

Note that the sum is finite by Corollary 1.6.

For a compact topological space X and a field $\mathbb{F}$, define the *Poincaré polynomial* by

$$P_X(t) = P_{X;\mathbb{F}}(t) := \sum_i t^i \dim H_i(X; \mathbb{F}).$$

Similarly, we can associate to a pair (X, A) the relative Poincaré polynomial $P_{X,A;\mathbb{F}}$.

Theorem 1.13. For every Morse function $f : M \to \mathbb{R}$, there exists a polynomial Q with nonnegative coefficients such that

$$P_f(t) - P_M(t) = (1 + t)Q(t).$$

Proof. Let $c_1 < c_2 < \cdots < c_k$ be the critical values of f. Write

$$t_0 = c_1 - 1, \quad t_k = c_k + 1, \quad \text{and } t_j = \frac{1}{2}(c_j + c_{j+1}) \text{ for } 0 < j < k.$$

The sublevel set M^{t_j} contains all critical points with critical values $c_1, \dots, c_j$ and contains no critical points with critical values $c_{j+1}, \dots, c_k$. We suppress the field $\mathbb{F}$ from our notation and consider the long exact sequence in relative homology of the pair $(M^{t_j}, M^{t_{j-1}})$:

$$\cdots \to H_\ell(M^{t_{j-1}}) \xrightarrow{i_{\ell *}} H_\ell(M^{t_j}) \xrightarrow{j_{\ell *}} H_\ell(M^{t_j}, M^{t_{j-1}}) \xrightarrow{\partial_{\ell *}} H_{\ell-1}(M^{t_{j-1}}) \to \cdots .$$

Combining the rank-nullity theorem and the exactness of the sequence, we obtain the following system of equations:

$$\begin{cases} \dim H_\ell(M^{t_{j-1}}) & = \dim \mathrm{Ker}(i_{\ell *}) + \dim \mathrm{Ker}(j_{\ell *}) \\ \dim H_\ell(M^{t_j}) & = \dim \mathrm{Ker}(j_{\ell *}) + \dim \mathrm{Ker}(\partial_{\ell *}) \\ \dim H_\ell(M^{t_j}, M^{t_{j-1}}) & = \dim \mathrm{Ker}(\partial_{\ell *}) + \dim \mathrm{Ker}(i_{(\ell-1)*}). \end{cases}$$

From this system we obtain

$$\dim H_\ell(M^{t_{j-1}}) - \dim H_\ell(M^{t_j}) + \dim H_\ell(M^{t_j}, M^{t_{j-1}}) = \dim \operatorname{Ker}(i_{\ell *}) + \dim \operatorname{Ker}(i_{(\ell-1)*}).$$

Multiplying both sides by t^ℓ and summing over ℓ gives

$$\begin{aligned} P_{M^{t_{j-1}}}(t) - P_{M^{t_j}}(t) + P_{M^{t_j}, M^{t_{j-1}}}(t) &= \sum_\ell t^\ell (\dim \operatorname{Ker}(i_{\ell *}) + \dim \operatorname{Ker}(i_{(\ell-1)*})) \\ &= (1+t) \sum_\ell t^\ell \dim \operatorname{Ker}(i_{\ell *}). \end{aligned}$$

We now sum over j. The sum of the terms $(P_{M^{t_{j-1}}}(t) - P_{M^{t_j}}(t))$ is telescoping, and we get

$$-P_{M^{t_k}}(t) + \sum_{j=1}^k P_{M^{t_j}, M^{t_{j-1}}}(t) = (1+t) \sum_{j=1}^k \sum_\ell t^\ell \dim \operatorname{Ker}\left(i_{\ell *} : H_\ell(M^{t_{j-1}}) \to H_\ell(M^{t_j})\right).$$

The critical points with critical values c_k are global maxima by compactness, so $M^{t_k} = M$ and in particular $P_{M^{t_k}}(t) = P_M(t)$. The polynomial

$$Q(t) := \sum_{j=1}^k \sum_\ell t^\ell \dim \operatorname{Ker}\left(i_{\ell *} : H_\ell(M^{t_{j-1}}) \to H_\ell(M^{t_j})\right)$$

has nonnegative coefficients. To prove the theorem, it now suffices to show that

$$P_f(t) = \sum_{j=1}^k P_{M^{t_j}, M^{t_{j-1}}}(t). \tag{1.3}$$

By excision and Theorem 1.10, the homology of the pair $(M^{t_j}, M^{t_{j-1}})$ satisfies

$$H_\ell(M^{t_j}, M^{t_{j-1}}) \cong \bigoplus_{\substack{p \in \operatorname{Crit}(f) \\ f(p) = c_j}} H_\ell(e^{\lambda(f,p)}, \partial e^{\lambda(f,p)}),$$

where $e^{\lambda(f,p)}$ is a $\lambda(f,p)$-cell. The pair $(e^{\lambda(f,p)}, \partial e^{\lambda(f,p)})$ has homology isomorphic to the reduced homology of the sphere $S^{\lambda(f,p)}$. Therefore the right-hand side above is isomorphic to a direct sum of copies of $\mathbb{F}$, one for each critical point with critical value c_j and Morse index ℓ. It follows that

$$P_{M^{t_j}, M^{t_{j-1}}}(t) = \sum_{\substack{p \in \operatorname{Crit}(f) \\ f(p) = c_j}} t^{\lambda(f,p)}.$$

Summing over j, we obtain (1.3). □

From Theorem 1.13, one can obtain many quantitative relationships between $\operatorname{Crit}(f)$ and the homology of M. Here are some examples.

Corollary 1.14. For every Morse function $f : M \to \mathbb{R}$, the inequality $P_f(t) \geq P_M(t)$ is actually a coordinate-wise inequality. That is, for each λ, the number of critical points of index λ is bounded below by $\dim H_\lambda(M; \mathbb{F})$. □

Corollary 1.15. For every Morse function $f : M \to \mathbb{R}$, we have

$$\sum_{p \in \mathrm{Crit}(f)} (-1)^{\lambda(f,p)} = \chi(M),$$

where χ is the Euler characteristic. □

Corollary 1.16. For every Morse function $f : M \to \mathbb{R}$, we have

$$\#\mathrm{Crit}(f) \geq \sum_i \dim H_i(M; \mathbb{F}).$$

□

1.5 Morse–Bott theory

In this section we sketch the basics of Morse–Bott theory following Section 2.6 in [Nic11]. Let M be a closed manifold and let $f : M \to \mathbb{R}$ be a smooth function.

Definition 1.17. An embedded submanifold $S \subset M$ is called a *nondegenerate critical submanifold* of f if the following hold.

- S is closed and connected.
- $S \subset \mathrm{Crit}(f)$.
- For all $s \in S$, we have $\mathrm{Ker}\, H_{f,s} = T_s S \subset T_s M$.

Definition 1.18. A smooth function $f : M \to \mathbb{R}$ is called *Morse–Bott* if $\mathrm{Crit}(f)$ is a union of nondegenerate critical submanifolds.

Suppose S is a nondegenerate critical submanifold of $f : M \to \mathbb{R}$. We wish to define an analogue of Morse index for S. For each $s \in S$, the Hessian $H_{f,s}$ descends to a nondegenerate quadratic form $q_{f,s}$ on $T_s M / T_s S$. Consider the bundle $E := (TM)|_S / TS$ over S. Although we have not fixed a Riemannian metric on M, topologically E is the normal bundle of S in M. We get a decomposition $E = E^+ \oplus E^-$ given by taking E^+ (resp. E^-) to be the maximal subbundle of E such that for all $s \in S$, the restriction $Q_{f,s}|_{E_s^+}$ is positive-definite (resp. negative-definite). Since S is connected, the bundles E^+ and E^- have constant rank.

Definition 1.19. The *Morse index* of S, denoted $\lambda(f, S)$, is defined to be the rank of E^-.

Definition 1.20. For a Morse–Bott function f and a field $\mathbb{F}$, define the *Morse–Bott polynomial* of f by

$$P_f(t) = P_{f;\mathbb{F}}(t) := \sum_{S \subset \mathrm{Crit}(f)} t^{\lambda(f,S)} P_{S;\mathbb{F}}(t),$$

where the sum is taken over critical submanifolds of f.

Let S be a nondegenerate critical submanifold of $f : M \to \mathbb{R}$. Fixing a section of $E = (TM)|_S/TS$, we can extend the inclusion $S \hookrightarrow M$ to an embedding of a neighborhood of $S \subset E$ into M. We can arrange that this embedding pulls $f : M \to \mathbb{R}$ back to a function $Q_f : E \to \mathbb{R}$ whose restriction to a fiber T_sM/T_sS is proportional to the quadratic form $Q_{f,s}$. Using the decomposition $E = E^+ \oplus E^-$, we get a Morse–Bott analogue of Lemma 1.5.

There is also a Morse–Bott version of Theorem 1.10. Roughly, for a critical value c satisfying conditions analogous to those in Theorem 1.10, there is a homotopy equivalence between $M^{c+\varepsilon}$ and the space obtained by gluing certain disk bundles to $M^{c-\varepsilon}$ for all sufficiently small $\varepsilon > 0$.

These two theorems lead to a Morse–Bott version of Theorem 1.13. As the Morse–Bott index is defined in terms of vector bundles, we must take care to avoid issues of orientation. Let $\pi : E \to B$ be a real vector bundle of rank n equipped with a smoothly varying inner product on the fibers. Let $D(E)$ denote the unit disk bundle with respect to the inner product, and $S(E)$ the unit sphere bundle. We define the *Thom space* $T(E) := D(E)/S(E)$.

Definition 1.21. Let $\mathbb{F}$ be a field. A real vector bundle $\pi : E \to B$ of rank n is called $\mathbb{F}$-*orientable* if there exists a class

$$u \in H^n(T(E); \mathbb{F})$$

such that its restriction to each fiber $T(E)_b$ generates the reduced cohomology $\widetilde{H}^n(T(E)_b; \mathbb{F})$.

Definition 1.22. A Morse–Bott function is called $\mathbb{F}$-*orientable* if for every critical submanifold S, the bundle E^- is $\mathbb{F}$-orientable.

The Thom isomorphism theorem says that if a class $u \in H^n(T(E); \mathbb{F})$ induces an $\mathbb{F}$-orientation, then in fact u globally generates reduced cohomology, in the sense that the map

$$H^i(E; \mathbb{F}) \to \widetilde{H}^{i+n}(T(E); \mathbb{F}), \quad x \mapsto x \smile u$$

is an isomorphism for all i. Here $\smile$ denotes cup product. The upshot of this discussion is that if $f : M \to \mathbb{R}$ is $\mathbb{F}$-orientable, then we can relate the homology of a critical submanifold to the homology of the disk bundles in the Morse–Bott version of Theorem 1.10 mentioned above. With these tools, we can formulate and prove the following.

Theorem 1.23. Let $f : M \to \mathbb{R}$ be an $\mathbb{F}$-orientable Morse–Bott function on a closed manifold M. Then there exists a polynomial Q with nonnegative coefficients such that

$$P_f(t) = P_M(t) + (1+t)Q(t).$$

□

Like Theorem 1.13, this readily yields many interesting formulas. Here is one that will be relevant in Chapter 2.

Corollary 1.24. For every $\mathbb{F}$-orientable Morse–Bott function $f : M \to \mathbb{R}$, we have

$$\sum_{S \subset \mathrm{Crit}(f)} \sum_i \dim H_i(S; \mathbb{F}) \geq \sum_i \dim H_i(M; \mathbb{F}),$$

with the first sum taken over critical submanifolds of f. □

Remark 1.25. Let $f : M \to \mathbb{R}$ be a Morse function with Morse polynomial P_f. It follows from the above definitions that f is Morse–Bott and $\mathbb{F}$-orientable, with Morse–Bott polynomial equal to P_f.

1.6 Morse homology

In this section we sketch an alternative approach to obtaining Morse theory. See [Hut02] or [Web06] for details and more general statements. Let M be a closed manifold and equip it with a Riemannian metric g. We will leverage the relationship between the critical points of a Morse function and the dynamics of its negative gradient flow to define a chain complex with homology isomorphic to singular homology of M.

1.6.1 Gradient flow dynamics

We recall some notions from dynamical systems. Let $f : M \to \mathbb{R}$ be Morse and let Φ be the flow of the negative gradient $-\nabla f$. For a critical point $p \in \mathrm{Crit}(f)$, the *stable manifold* of p is defined by

$$W^s(p) := \left\{ x \in M : \lim_{t\to\infty} \Phi^t(x) = p \right\}$$

and the *unstable manifold* of p by

$$W^u(p) := \left\{ x \in M : \lim_{t\to-\infty} \Phi^t(x) = p \right\}.$$

Let $\dim M = n$. By the stable manifold theorem, $W^s(p)$ and $W^u(p)$ are immersed open disks of dimension $\lambda(f,p)$ and $n - \lambda(f,p)$, respectively. In fact, since f is strictly decreasing along any trajectory of Φ that does not intersect $\mathrm{Crit}(f)$, it follows that $W^s(p)$ and $W^u(p)$ are in fact embedded. We will assume from now on that the pair (f,g) is *Morse–Smale*, meaning that f is Morse and that for all $p, q \in \mathrm{Crit}(f)$, the stable and unstable manifolds $W^s(p)$ and $W^u(q)$ are transverse.

Remark 1.26. Given that $\dim W^s(p)$ depends on p, the Morse–Smale condition may seem too restrictive. In fact, for a fixed Morse function f, a generic Riemannian metric g yields a Morse–Smale pair (f,g). Consider the example of the height function f on $\mathbb{T}^2 \subset \mathbb{R}^3$, where $\mathbb{T}^2$ is equipped with the Riemannian metric g induced by the Euclidean metric on $\mathbb{R}^3$ (Examples 1.7 and 1.11). The pair (f,g) is not Morse–Smale, as the stable manifold of the lower saddle point and the unstable manifold of the higher saddle intersect along two semicircles, but this can be remedied with a small perturbation of g.

Fix two critical points p, q and let $\gamma : \mathbb{R} \to M$ be an integral curve of X. We say that γ is a ***flow line*** from q to p if $\gamma(0) \in W^u(q) \cap W^s(p)$. In other words, γ is an integral curve with $\lim_{t\to\infty} \gamma(t) = p$ and $\lim_{t\to-\infty} \gamma(t) = q$. We are only interested in the images of flow lines, not their parametrizations. We therefore take the quotient of the space of flow lines from q to p by the $\mathbb{R}$-action induced by Φ, obtaining the moduli space $\mathcal{M}(q,p) := W^u(q) \cap W^s(p)/\mathbb{R}$. The intersection $W^u(q) \cap W^s(p)$ is transverse by the Morse–Smale assumption, so $\mathcal{M}(q,p)$ is a submanifold of M with

$$\dim \mathcal{M}(q,p) = \lambda(f,q) - \lambda(f,p) - 1$$

if $p \neq q$, and

$$\dim \mathcal{M}(p,p) = \lambda(f,p) - \lambda(f,p) = 0.$$

1.6.2 Chain complex

Suppose $p, q \in \mathrm{Crit}(f)$ are distinct. One can show that if $\dim \mathcal{M}(q,p) = 0$, i.e., if $\lambda(f,q) = \lambda(f,p)+1$, then $\mathcal{M}(q,p)$ is compact and hence finite. Let $\mathrm{Crit}_i(f)$ denote the set of critical points with Morse index i and let $C_i = C_i(f,g;\mathbb{Z}_2)$ be the $\mathbb{Z}_2$-vector space generated by $\mathrm{Crit}_i(f)$, where $\mathbb{Z}_2$ is the field with two elements. We define a differential $d : C_i \to C_{i-1}$ by setting

$$d(q) = \sum_{p \in \mathrm{Crit}_{i-1}(f)} \#\mathcal{M}(q,p)\, p, \qquad q \in \mathrm{Crit}_i(f)$$

and extending $\mathbb{Z}_2$-linearly. With some work we can show that $d^2 = 0$. We denote the homology of the chain complex $(C_*(f,g), d)$ by $H_*^{\mathrm{Morse}}(f,g;\mathbb{Z}_2)$ and call it ($\mathbb{Z}_2$-)*Morse homology*.

Remark 1.27. Morse homology can be defined with coefficients in $\mathbb{Z}$, say, but this requires orienting the moduli spaces $\mathcal{M}(q,p)$ in a coherent way and modifying of the definition of d accordingly.

Theorem 1.28. Let M be a closed manifold and (f,g) a Morse–Smale pair on M. Then there is a canonical isomorphism between $\mathbb{Z}_2$-Morse homology and singular homology with coefficients in $\mathbb{Z}_2$:

$$H_*^{\mathrm{Morse}}(f,g;\mathbb{Z}_2) \cong H_*(M;\mathbb{Z}_2).$$

The same holds for $\mathbb{Z}$-Morse homology and singular homology with coefficients in $\mathbb{Z}$. □

Example 1.29. We again return to the example of the height f function on the torus $\mathbb{T}^2$ (Examples 1.7 and 1.11). As in Remark 1.26, we can assume that $\mathbb{T}^2$ is equipped with a metric g such that g is close to the Euclidean metric and such that (f,g) is Morse–Smale. We compute $H_*^{\mathrm{Morse}}(f,g;\mathbb{Z}_2)$.

Let p,q,r,s denote the four critical points, with $f(p) < f(q) < f(r) < f(s)$ and respective Morse indices $0,1,1,2$. Then

$$C_0 = \mathbb{Z}_2 \cdot p, \quad C_1 = (\mathbb{Z}_2 \cdot q) \oplus (\mathbb{Z}_2 \cdot r), \quad \text{and} \quad C_2 = \mathbb{Z}_2 \cdot s.$$

The moduli space $\mathcal{M}(q,p)$ consists of two flow lines, namely the two open semicircles obtained by deleting p and q from the meridian in which they lie. The space $\mathcal{M}(r,p)$ is empty, as points in $W^u(r)$ flow towards q. The differential $d : C_1 \to C_0$ is therefore trivial:

$$d(q) = 2p = 0 \quad \text{and} \quad d(r) = 0.$$

An identical argument shows that $d : C_2 \to C_1$ is trivial. We conclude that

$$H_i^{\mathrm{Morse}}(f,g;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & i = 0,2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & i = 1 \end{cases},$$

which indeed agrees with the singular homology of $\mathbb{T}^2$. □

Some results in Section 1.4 can be proven with Theorem 1.28 by mimicking some of the ideas in the proof of Theorem 1.13.

Chapter 2

Diameters of immersed submanifolds of Euclidean space

2.1 Counting diameters

Let f be an immersion of an n-dimensional manifold M into $\mathbb{R}^{n+k}$. A *diameter* of $f(M)$ is a line segment joining distinct points $f(x), f(y) \in f(M)$ that is perpendicular to the tangent spaces $T_{f(x)}f(M)$ and $T_{f(y)}f(M)$. In this chapter we will apply Morse–Bott theory to the problem of estimating the number of diameters of $f(M)$ for a "generic" immersion f. We postpone defining genericity until later.

For a field $\mathbb{F}$, let $B_{\mathbb{F}}$ denote the sum of Betti numbers:

$$B_{\mathbb{F}} := P_{M;\mathbb{F}}(1) = \sum_i \dim H_i(M;\mathbb{F}).$$

Theorem 2.1. Let M be a closed n-dimensional manifold. For a generic embedding $f : M \to \mathbb{R}^{n+k}$, the number of diameters of $f(M)$ is at least

$$\frac{1}{2}(B_{\mathbb{F}}^2 - B_{\mathbb{F}}).$$

This estimate is straightforward to prove but not sharp, and the techniques cannot be fully adapted to the immersed case. In [TW72], Takens–White obtain estimates in the immersed and embedded case that are both sharper than Theorem 2.1, but they are still not optimal. The state of the art is due to Pushkar':

Theorem 2.2 ([Pus97])**.** Let M be a closed n-dimensional manifold. For a generic immersion $f : M \to \mathbb{R}^{n+k}$, the number of diameters of $f(M)$ is at least

$$\frac{1}{2}(B_{\mathbb{Z}_2}^2 + (n-1)B_{\mathbb{Z}_2}),$$

where $\mathbb{Z}_2$ denotes the field of two elements.

Theorem 2.2 is sharp in several cases, including spheres S^n, products of spheres $S^n \times S^m$, and orientable surfaces of genus $g \geq 1$. See [Pus98] for further discussion of sharpness.

2.2 Diameters of embedded submanifolds

To illustrate the novelty of the techniques involved in proving Theorem 2.2, we briefly sketch a proof of Theorem 2.1 using a more naive approach. Let M be a closed n-dimensional manifold and $f : M \to \mathbb{R}^{n+k}$ an embedding. Define a smooth function $D_f : M \times M \to \mathbb{R}$ by $D_f(x,y) = \|f(x) - f(y)\|^2$. The "genericity" in Theorem 2.1 will be taken in this section to mean that D_f is Morse–Bott on $M \times M$.

Lemma 2.3. There is a bijection between the set of critical points (x, y) of D_f with nonzero critical value and the set of pairs of diameters $([f(x), f(y)], [f(y), f(x)])$. □

Let D denote the number of diameters of $f(M)$ and let $B_{\mathbb{F}}$ denote the sum of Betti numbers $\sum_i H_i(M;\mathbb{F})$. A straightforward calculation shows that $M \subset M \times M$ is a critical submanifold for D_f, where we identify M with its image under the diagonal inclusion $\Delta : M \to M \times M$ defined by $p \mapsto (p, p)$. Moreover, since f is an embedding, all critical points of D_f with critical value zero must lie on the diagonal. In light of this observation and Lemma 2.3, we can apply Corollary 1.24 to D_f to obtain

$$\begin{aligned}\sum_{S\subset \mathrm{Crit}(D_f)} \sum_i \dim H_i(S;\mathbb{F}) &\geq \#\{\text{critical points with nonzero critical value}\} + \sum_i \dim H_i(M;\mathbb{F}) \\ &= 2D + B_{\mathbb{F}} \\ &\geq \sum_i H_i(M\times M;\mathbb{F}).\end{aligned}$$

The Künneth formula implies $\sum_i H_i(M \times M; \mathbb{F}) = B_{\mathbb{F}}^2$. This proves Theorem 2.1. □

Remark 2.4. It is possible to obtain Theorem 2.1 without applying Morse–Bott theory. Following e.g. Laudenbach [Lau11], one can extend Theorem 1.13 and its corollaries to manifolds with boundary for homology with coefficients in $\mathbb{R}$. Let $\varepsilon > 0$ be small enough that the closed tubular neighborhood $T := D_f^{-1}([0,\varepsilon])$ of $M \subset M \times M$ contains no critical points of D_f with nonzero critical value. Then, letting $\mathrm{Int}(T)$ denote the interior of T, we have

$$H_i(M\times M \setminus \mathrm{Int}(T), \partial T;\mathbb{R}) \cong H_i(M\times M, T;\mathbb{R}) \cong H_i(M\times M, M;\mathbb{R}),$$

where the first isomorphism follows from excision and the second follows from contracting T to M. The analogue of Corollary 1.16 for manifolds with boundary applies to the pair $(M \times M \setminus \mathrm{Int}(T), \partial T)$. On the other hand, for all i we have

$$\dim H_i(M\times M, M;\mathbb{R}) = \dim H_i(M\times M;\mathbb{R}) - \dim H_i(M;\mathbb{R}), \tag{2.1}$$

from which Theorem 2.1 follows immediately. Formula (2.1), valid for homology with coefficients in any field, follows from the fact that the map $M \times M \to M$ given by $(p, q) \mapsto p$ is a left inverse for the diagonal map $\Delta : M \to M \times M$. Combining this with the long exact sequence in relative homology, we conclude that

$$0 \to H_i(M;\mathbb{R}) \xrightarrow{\Delta_*} H_i(M\times M;\mathbb{R}) \to H_i(M\times M, M;\mathbb{R}) \to 0$$

is exact. □

2.3 Diameters of immersed submanifolds

Theorem 2.1 amounts to a straightforward application of Morse–Bott theory to the function $D_f(x, y) = \|f(x) - f(y)\|^2$ (see Section 2.2). Neither this approach nor the approach outlined in Remark 2.1 are suitable for proving Theorem 2.2. We instead follow [Pus97]: given a "generic" immersion $f : M \to \mathbb{R}^{n+k}$, we construct a manifold M' and a function $F : M' \to \mathbb{R}$ such that certain critical points of F correspond to diameters of $f(M)$ and such that F is invariant under a free action of $\mathbb{Z}_2$ on M'. After computing the homology of the quotient $M'/\mathbb{Z}_2$, we can apply Morse–Bott theory to the function induced by F on $M'/\mathbb{Z}_2$ to conclude Theorem 2.2. Roughly speaking, the improvement of Theorem 2.2 over Theorem 2.1 and [TW72] is a consequence of two things. First, as we will see, the homology of the quotient $M'/\mathbb{Z}_2$ is much more complicated than that of the pair $(M \times M, M)$. Second, there is no obvious D_f-invariant free $\mathbb{Z}_2$-action on $M \times M$.

We now fill in the details. Let M be an n-dimensional closed manifold and let $f : M \to \mathbb{R}^{n+k}$ be an immersion. Fix an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n+k}$ and let S^{n+k-1} be the unit sphere centered at the origin in $\mathbb{R}^{n+k}$. Define a map $F : S^{n+k-1} \times M \times M \to \mathbb{R}$ by

$$F(v, x, y) = \langle v, f(x) - f(y) \rangle. \tag{2.2}$$

Definition 2.5. An immersion $f : M \to \mathbb{R}^{n+k}$ of an n-dimensional closed manifold M is *generic* if the self-intersections in $f(M)$ are transverse and the associated function $F : S^{n+k-1} \times M \times M \to \mathbb{R}$, as defined in (2.2), is Morse–Bott.

We will henceforth fix a generic immersion $f : M \to \mathbb{R}^{n+k}$ and denote by F the corresponding function on $S^{n+k-1} \times M \times M$ defined in (2.2). The following two lemmas establish the relevance of F to the problem of counting diameters.

Lemma 2.6. Suppose (v', x', y') is a critical point of F such that $x' \neq y'$. Then $f(x') \neq f(y')$, the segment $[f(x'), f(y')]$ is a diameter of $f(M)$, and $F(v', x', y') \neq 0$.

Proof. Differentiating F with respect to x at (v', x', y'), we get

$$\begin{aligned} 0 &= \partial_x F(v', x', y') \\ &= \langle \partial_x \mathrm{Id}_{S^{n+k-1}}(v'), f(x') - f(y') \rangle + \langle v', \partial_x f(x') - \partial_x f(y') \rangle \\ &= \langle v', df_{x'} \rangle, \end{aligned}$$

so $v' \perp df_{x'}(T_{x'}M)$. An identical calculation shows that $v' \perp df_{y'}(T_{y'}M)$. Since self-intersections in $f(M)$ are assumed to be transverse, it follows that $f(x') \neq f(y')$. On the other hand, since every $v \in S^{n+k-1} \subset \mathbb{R}^{n+k}$ is normal to itself, differentiating F with respect to v at (v', x', y') demonstrates that $(f(x') - f(y')) \perp T_{v'}S^{n+k-1}$, or equivalently that v' is parallel to $f(x') - f(y')$. It therefore follows from our previous observations about v' that $[f(x'), f(y')]$ is a diameter. Moreover, since $f(x') - f(y') \neq 0$, we have $F(v', x', y') \neq 0$. □

Lemma 2.7. Suppose $[f(x), f(y)]$ is a diameter of $f(M)$. Then

$$\left(\pm \frac{f(x) - f(y)}{\|f(x) - f(y)\|}, x, y \right), \left(\pm \frac{f(x) - f(y)}{\|f(x) - f(y)\|}, y, x \right)$$

are critical points of F.

Proof. Using the calculations in the proof of Lemma 2.6, we get

$$dF_{(v,x,y)} = \langle \mathrm{Id}_{T_v S^{n+k-1}}, f(x) - f(y) \rangle + \langle v, d_x f - d_y f \rangle. \tag{2.3}$$

Set $v = \frac{f(x)-f(y)}{\|f(x)-f(y)\|}$ in (2.3). If $[f(x), f(y)]$ is a diameter then v is perpendicular to $df_x(T_xM)$ and $df_y(T_yM)$, so the right-hand side in (2.3) vanishes. The same argument shows that the other three points in Lemma 2.7 are critical. □

Clearly the points in Lemma 2.7 have nonzero critical values, so Lemmas 2.6 and 2.7 together provide a bijection between diameters $[f(x), f(y)]$ of $f(M)$ and quadruples of critical points of F with nonzero critical value. To apply Morse–Bott theory, it therefore remains only to study the critical points with zero critical value. To do this, we projectivize. Let $\mathbb{Z}_2$ act on $S^{n+k-1} \times M \times M$ by antipodal involution on the first coordinate and transposition on the second and third: $(v, x, y) \mapsto (-v, y, x)$. This is a free action that leaves F invariant, so the quotient $(S^{n+k-1} \times M \times M)/\mathbb{Z}_2$ is a smooth manifold and F descends to a smooth function $\widetilde{F} : (S^{n+k-1} \times M \times M)/\mathbb{Z}_2 \to \mathbb{R}$. In this setting, we have the following:

Lemma 2.8. The critical points of $\widetilde{F}$ with zero critical value form a nondegenerate critical submanifold for $\widetilde{F}$ that is diffeomorphic to the projectivization $\mathbb{P}(NM)$ of the normal bundle NM of M.

Proof. Let K denote the set of critical points of F with zero critical value. Lemma 2.6 implies that

$$K \subset S^{n+k-1} \times M \subset S^{n+k-1} \times M \times M,$$

where the first M denotes the diagonal in $M \times M$. We thus have a 2-to-1 map from K to the set $\widetilde{K}$ of critical points of $\widetilde{F}$ with zero critical value. Fix $v \in S^{n+k-1}$ and $x \in M$. For $\eta \in T_v S^{n+k-1}$ and $\xi_1, \xi_2 \in T_x M$, formula (2.3) gives

$$dF_{(v,x,x)}(\eta, \xi_1, \xi_2) = \langle v, df_x(\xi_1 - \xi_2) \rangle.$$

In light of the argument in the proof of Lemma 2.6, we conclude that (v, x, x) is a critical point of F if and only if $v \perp df_x(T_x M)$. Hence K is diffeomorphic to the unit normal bundle of $f(M)$. Passing to the quotient by the $\mathbb{Z}_2$-action, we get a diffeomorphism $\widetilde{K} \cong \mathbb{P}(NM)$.

A tedious but straightforward computation in local coordinates shows that $\mathbb{P}(NM)$ is nondegenerate for $\widetilde{F}$. □

Recall that we are trying to apply Morse–Bott theory to $\widetilde{F}$. In light of Lemma 2.8, we therefore want to study the homology of $\mathbb{P}(NM)$. The following general lemma does the trick:

Lemma 2.9. Let $E \to B$ be a fiber bundle with fiber $\mathbb{P}(\mathbb{R}^k)$. Then

$$\sum_i \dim H_i(E; \mathbb{Z}_2) \leq k \sum_i \dim H_i(B; \mathbb{Z}_2).$$

Proof. We study $H_i(E; \mathbb{Z}_2)$ using the Serre spectral sequence. Recall that $H_i(\mathbb{P}(\mathbb{R}^k); \mathbb{Z}_2) = \mathbb{Z}_2$ for all $0 \leq i \leq k-1$. Using this fact and analyzing the E_2 page, we see that $P_E(1) = \sum_i \dim H_i(E; \mathbb{Z}_2)$ is maximized when E is trivial. On the other hand, the Künneth formula yields

$$\sum_i \dim H_i(B \times \mathbb{P}(\mathbb{R}^k); \mathbb{Z}_2) = P_{B \times \mathbb{P}(\mathbb{R}^k)}(1) = P_B(1) P_{\mathbb{P}(\mathbb{R}^k)}(1) = k \sum_i \dim H_i(B; \mathbb{Z}_2),$$

and we are done. □

The last ingredient needed to apply Morse–Bott theory is the homology of $(S^{n+k-1} \times M \times M)/\mathbb{Z}_2$.

Lemma 2.10. Let M be a closed manifold and let $\mathbb{Z}_2$ act on $S^N \times M \times M$ by $(v, x, y) \mapsto (-v, y, x)$. Write $B_{\mathbb{Z}_2} = \sum_i \dim H_i(M; \mathbb{Z}_2)$. Then

$$\sum_i \dim H_i((S^N \times M \times M)/\mathbb{Z}_2; \mathbb{Z}_2) = B_{\mathbb{Z}_2}^2 + N B_{\mathbb{Z}_2}.$$

Proof. Let $(\alpha^i)_i$ be a cell decomposition of M, which we know exists by Corollary 1.12. Let $\sigma^0, \widehat{\sigma}^0, \ldots, \sigma^N, \widehat{\sigma}^N$ be the standard cell decomposition of S^N, where $\sigma^i, \widehat{\sigma}^i$ are i-cells that are interchanged under antipodal involution on S^N. Then $S^N \times M \times M$ decomposes into cells of the form $\sigma^i \times \alpha^j \times \alpha^k$ and $\widehat{\sigma}^i \times \alpha^j \times \alpha^k$. The map $(v, x, y) \mapsto (-v, y, x)$ is cellular, since it carries a cell $\sigma^i \times \alpha^j \times \alpha^k$ to the cell $\widehat{\sigma}^i \times \alpha^k \times \alpha^j$. The cell decomposition of $S^N \times M \times M$ therefore induces a cell decomposition of $(S^N \times M \times M)/\mathbb{Z}_2$.

Let ∂ be the boundary map on $C_*(M)$, let ∂_1 be the boundary map on $C_*(M \times M)$, and let ∂_2 be the boundary map on $C_*((S^N \times M \times M)/\mathbb{Z}_2)$, where $C_*(\cdot)$ denotes the chain complex formed by the direct sum of the cellular chain groups with coefficients in $\mathbb{Z}_2$. Set $\sigma^{-1} = 0$. Let ∂_2' be a lift of ∂_2 to $C_*(S^N \times M \times M)$ such that

$$\partial_2'(\sigma^i \times \alpha^j \times \alpha^k) = \sigma^i \times (\partial_1(\alpha^j \times \alpha^k)) + \sigma^{i-1} \times (\alpha^j \times \alpha^k) + \widehat{\sigma}^{i-1} \times (\alpha^j \times \alpha^k).$$

Passing to the quotient, we get

$$\partial_2(\sigma^i \times \alpha^j \times \alpha^k) = \sigma^i \times (\partial_1(\alpha^j \times \alpha^k)) + \sigma^{i-1} \times (\alpha^j \times \alpha^k + \alpha^k \times \alpha^j),$$

where by abuse of notation we formally identify cells of $S^N \times M \times M$ with those of $(S^N \times M \times M)/\mathbb{Z}_2$.

We consider two cases. First, suppose $\partial = 0$. It follows that $\partial_1 = 0$. We therefore have

$$\partial_2(\sigma^i \times \alpha^j \times \alpha^k) = \sigma^{i-1} \times s(\alpha^j \times \alpha^k),$$

where we define $s(\alpha^j \times \alpha^k) := \alpha^j \times \alpha^k + \alpha^k \times \alpha^j$. The dimensions of the cells in S^N therefore induce a new grading of $C_*((S^N \times M \times M)/\mathbb{Z}_2)$ given by

$$\begin{aligned} C_*((S^N \times M \times M)/\mathbb{Z}_2) &\cong \bigoplus_{i=0}^{N} \{\text{chains in } C_*((S^N \times M \times M)/\mathbb{Z}_2) \text{ containing } \sigma^i\} \\ &\cong \bigoplus_{i=0}^{N} C_*(M \times M). \end{aligned}$$

This becomes a chain complex when we identify ∂_2 with s by mapping an element α of the i-th copy of $C_*(M \times M)$ to $s(\alpha)$ in the $(i-1)$-st copy of $C_*(M \times M)$. Clearly this chain complex is isomorphic to $C_*((S^N \times M \times M)/\mathbb{Z}_2)$, so it suffices to compute its homology. By the Künneth formula, $\dim C_*(M \times M) = B_{\mathbb{Z}_2}^2$. Our assumption that $\partial = 0$ implies that the cell decomposition $(\alpha^i)_i$ consists of exactly $B_{\mathbb{Z}_2}$ cells. Therefore, the image under s of the N-th copy of $C_*(M \times M)$ is the subspace of the $(N-1)$-st copy with basis $\{s(\alpha^i \times \alpha^j) : 0 \leq i < j \leq B_{\mathbb{Z}_2}\}$. Hence

$$\dim \operatorname{Im}(s) = \frac{1}{2}(B_{\mathbb{Z}_2}^2 - B_{\mathbb{Z}_2}), \text{ and therefore } \dim \operatorname{Ker}(s) = \frac{1}{2}(B_{\mathbb{Z}_2}^2 + B_{\mathbb{Z}_2}).$$

The same holds for all but the 0-th copy of $C_*(M \times M)$, where we of course have $\mathrm{Ker}(s) = C_*(M \times M)$. We conclude that the homology H_i of our new chain complex satisfies

$$\dim H_i = \begin{cases} \frac{1}{2}(B_{\mathbb{Z}_2}^2 + B_{\mathbb{Z}_2}), & i = 0, i = N \\ B_{\mathbb{Z}_2}, & 0 < i < N. \end{cases}$$

The desired formula follows immediately.

Before considering the case $\partial \neq 0$, we make note of a general fact: any chain complex $(C_\bullet, d)$ over a field admits a decomposition into the sum of a chain complex with trivial boundary maps and chain groups equal to the homology groups $H_\bullet$ of $(C_\bullet, d)$, and a sum of exact chain complexes that each consist of precisely two chain groups. Indeed, let $\mathrm{Ker}(d_i)'$ denote a complementary subspace to $\mathrm{Ker}(d_i)$ in C_i. Then for all i we have

$$C_i \cong H_i \oplus \mathrm{Im}(d_{i+1}) \oplus \mathrm{Ker}(d_i)', \tag{2.4}$$

and

$$0 \to \mathrm{Ker}(d_i)' \xrightarrow{d_i|_{\mathrm{Ker}(d_i)'}} \mathrm{Im}(d_i) \to 0$$

is clearly exact.

Now suppose $\partial \neq 0$. We construct a manifold M' such that $C_*((S^N \times M \times M)/\mathbb{Z}_2)$ is isomorphic to $C_*((S^N \times M' \times M')/\mathbb{Z}_2)$ and such that the homology of the latter complex satisfies the conclusion of the lemma. First, let M'' be a disjoint union of spheres, where for each i there are $\dim H_i(M; \mathbb{Z}_2)$ spheres of dimension i. In symbols,

$$M'' = \bigsqcup_{i=1}^{\dim M} \left(\bigsqcup_{j=1}^{\dim H_i(M;\mathbb{Z}_2)} S^i \right). \tag{2.5}$$

Decompose $C_*(M)$ as in (2.4). To each two-term exact complex $\mathrm{Ker}(\partial_i)' \oplus \mathrm{Im}(\partial_i)$ we associate $\dim(\mathrm{Ker}(\partial_i)')$ copies of the i-disk D^i. Let M' be the wedge product of M'' and the disks corresponding to the complexes $\mathrm{Ker}(\partial_i)' \oplus \mathrm{Im}(\partial_i)$. Now by construction, each i-sphere S^i in $M'' \subset M'$ corresponds to a generator of $H_i(M)$, each i-disk D^i corresponds to a generator of $\mathrm{Ker}(\partial_i)'$, and their boundaries ∂D^i correspond to the generators of $\mathrm{Im}(\partial_i)$. We conclude that

$$C_*((S^N \times M' \times M')/\mathbb{Z}_2) \cong C_*((S^N \times M \times M)/\mathbb{Z}_2).$$

On the other hand, $M'' \subset M'$ is a strong deformation retract for M', and it follows that

$$H_*((S^N \times M' \times M')/\mathbb{Z}_2) \cong H_*((S^N \times M'' \times M'')/\mathbb{Z}_2).$$

The cellular chain complex $C_*(M'')$ evidently has $\partial = 0$, so we have successfully reduced to the previous case. The lemma is proven. □

Proof of Theorem 2.2. Let D denote the number of diameters of $f(M)$. Lemmas 2.6 and 2.7 imply that $2D$ is bounded below by the number of critical points of $\widetilde{F}$ with nonzero critical value. Any Morse–Bott function is $\mathbb{Z}_2$-orientable, so we can apply Corollary 1.24 to $\widetilde{F}$. Combining this with Lemma 2.8, we get

$$2D + \sum_i \dim H_i(\mathbb{P}(NM); \mathbb{Z}_2) \geq P_{\widetilde{F}}(1) \geq \sum_i \dim H_i((S^{n+k-1} \times M \times M)/\mathbb{Z}_2; \mathbb{Z}_2). \tag{2.6}$$

As before, write $B_{\mathbb{Z}_2} = \sum_i \dim H_i(M;\mathbb{Z}_2)$. Applying Lemma 2.9 to the left-hand side of (2.6) and Lemma 2.10 to the right-hand side yields

$$2D + kB_{\mathbb{Z}_2} \geq B_{\mathbb{Z}_2}^2 + (n+k-1)B_{\mathbb{Z}_2}.$$

Rearranging gives

$$D \geq \frac{1}{2}(B_{\mathbb{Z}_2}^2 + (n-1)B_{\mathbb{Z}_2}),$$

which completes the proof. □

Bibliography

[Bot82] Raoul Bott. Lectures on Morse theory, old and new. *Bull. Amer. Math. Soc. (N.S.)*, 7(2):331–358, 1982.

[CLOT03] Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré. *Lusternik–Schnirelmann category*, volume 103 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[Flo89] Andreas Floer. Symplectic fixed points and holomorphic spheres. *Communications in Mathematical Physics*, 120(4):575 – 611, 1989.

[Hut02] Michael Hutchings. Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves), December 2002.

[Lau11] François Laudenbach. A Morse complex on manifolds with boundary. *Geometriae Dedicata*, 153(1):47–57, 2011.

[Mil63] J. Milnor. *Morse theory*. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963. Based on lecture notes by M. Spivak and R. Wells.

[Mis95] Konstantin Mischaikow. Conley index theory. In *Dynamical systems (Montecatini Terme, 1994)*, volume 1609 of *Lecture Notes in Math.*, pages 119–207. Springer, Berlin, 1995.

[Mis99] Konstantin Mischaikow. The Conley index theory: a brief introduction. In *Conley index theory (Warsaw, 1997)*, volume 47 of *Banach Center Publ.*, pages 9–19. Polish Acad. Sci. Inst. Math., Warsaw, 1999.

[Nic11] Liviu Nicolaescu. *An invitation to Morse theory*. Universitext. Springer, New York, second edition, 2011.

[Pus97] Petr E. Pushkar'. Generalization of the Chekanov theorem: diameters of immersed manifolds and wave fronts, 1997.

[Pus98] Petr E. Pushkar'. Diameters of immersed manifolds and of wave fronts. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 326(2):201–205, 1998.

[TW72] Floris Takens and James White. Morse theory of double normals of immersions. *Indiana Univ. Math. J.*, 21:11–17, 1971/72.

[Web06] Joa Weber. The Morse–Witten complex via dynamical systems. *Expositiones Mathematicae*, 24(2):127–159, 2006.

[Wit82] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geometry*, 17(4):661–692 (1983), 1982.

Victor Ginsburg

Education

Pennsylvania State University Schreyer Honors College University Park, PA
Bachelor of Science in Mathematics *Expected Graduation May 2022*

Research Experience

Summer@ICERM Providence, RI
Participant *Jun 2021 - Aug 2021*

- Investigated properties of dilation surfaces, a generalization of the translation surfaces commonly studied in Teichmüller dynamics. Used SageMath to generate extensive computational data.

Polymath REU Online
Participant *Jun 2020 - Aug 2020*

- Studied a generalization of the alternating sign matrix conjecture, a well-studied problem in combinatorics. Learned about applications of statistical mechanics to combinatorics.

Kent State University Mathematics REU Kent, OH
Participant *Jun 2019 - Jul 2019*

- Conducted research on linear preserver problems, an active research area in matrix theory. Co-wrote a paper.

Papers

- Cui C., Ginsburg V., Kirgios V., Lin V., Wang J. *Periodic trajectories and saddle connections on dilation surfaces.* In preparation.
- Ginsburg V., Julius H., Velasquez R. *On maps preserving Lie products equal to a rank-one nilpotent.* Linear Algebra and its Applications (2020).

Presentations and Posters

- *The ball-box theorem and the Hausdorff dimension of a subriemannian manifold.* Dynamics student seminar. Pennsylvania State University. February 24, 2022.
- *Lyapunov exponents of products of random matrices.* Dynamics student seminar. Pennsylvania State University. November 18, 2021.
- *The Hopf argument.* Online. November 6, 2021.
- *Closed geodesics and saddle connections on dilation surfaces.* Summer@ICERM Final Presentation Session. ICERM. August 5, 2021
- *Topological entropy.* Entropies in dynamics and algebra workshop. Pennsylvania State University. July 10, 2021.
- *On maps preserving Lie products equal to a rank-one nilpotent.* MAA General Contributed Paper Session on Other Topics at the 2020 Joint Mathematics Meeting. Denver, CO. January 17, 2020.
- *On maps preserving Lie products equal to a rank-one nilpotent.* Undergraduate Mathematics Symposium Poster Session. University of Illinois at Chicago. November 2, 2019.
- *Bifurcation theory.* Mathematical Outing for Undergraduates. Pennsylvania State University. April 20, 2019.

Work Experience

Penn State Learning University Park, PA
200-Level Math Tutor *Aug 2020 - Dec 2021*

- Tutored undergraduate students in introductory linear algebra, multivariable calculus, and differential equations
- Completed pedagogical training tailored for peer tutoring
- Received overwhelmingly positive feedback from students in 2020-2021 academic year

Penn State Mathematics Department University Park, PA
Math 311W/311M Grader (Professor: Svetlana Katok) *Aug 2020 - Dec 2020*

- Graded homework assignments for honors and non-honors sections of an introduction to mathematical proofs course

Additional Experience and Awards

- Steven and Sherry McCrystal Mathematics Award (2021)
- The President Sparks Award (2020)
- President's Freshman Award (2019)
- Academic Excellence Scholarship (2018-present)
- Working knowledge of Python and SageMath