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Odd Grassmannian Permutations Avoiding Patterns of Size Three

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Abstract

The set of Grassmannian permutations is described as the set of permutations having at most one descent. In this thesis, we study restrictions of pattern-avoidance and parity for these permutations. Specifically, we study the odd Grassmannian permutations that avoid patterns of size three. We derive recurrence relations and generating functions for such permutations, as well as connections to other combinatorial objects such as symmetric convex polygons, Dyck paths, and multigraphs.

Acknowledgements

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I would also like to thank everyone who has been a part of my educational journey, big and small. Meeting others passionate about education and mathematics is invigorating, invaluable, and unforgettable.

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Introduction

A permutation, π, of $[n] = \{1, \ldots, n\}$ is a bijective mapping from $[n]$ to itself. We use one-line notation, $\pi = \pi_1, \pi_2, \ldots, \pi_n$, where π_i is the output of element i through π [2]. There are many restrictions that can be put on a permutation including pattern avoidance, parity, etc. Within a permutation, π contains a pattern, σ, if there is a subsequence of π with the same relative order as σ. So, a permutation avoids a pattern if there is no such subsequence. The permutations that avoid σ are denoted by $\mathrm{Av}(\sigma)$ [2]. For example, the permutation 12534 contains a 132-pattern as 153 and 154, but it does not contain any 321-pattern. We also discuss restrictions on parity. An inversion of a permutation can be defined as an instance of a 21-pattern. The parity of a permutation, even or odd, is then determined by the parity of the number of inversions in the permutation.

In general, the study of patterns in words and permutations can be traced back to the late 18th century while the modern study of patterns and permutations is contributed to papers by Rotem, Rogers, Knuth in the 1970s and 80s. In 1985, Simion and Schmidt published a paper on the study of permutation patterns, and since then, the area of study has expanded. With applications in other areas of mathematics, computer science, physics, and biology, the argument for studying patterns in various permutations remains compelling [2].

In this paper, we study the set of Grassmannian permutations, $\mathscr{G}_n$, which have at most one descent (a consecutive i, j where $\pi(i) > \pi(j)$). Specifically, we study those with restrictions of parity and pattern-avoidance. We denote Grassmannian permutations of size n that avoid the pattern σ as $\mathscr{G}_n(\sigma)$, so the odd and even avoiders are $\mathscr{G}_n^{odd}(\sigma)$ and $\mathscr{G}_n^{even}(\sigma)$, respectively. We rely on methods of enumeration including recurrence relations and generating functions. In the proofs of generating functions, we use power series and treat them formally as we are not concerned with convergence. They are used solely as a counting structure and presented as another representation that encodes the counting sequence. We also produce connections to other combinatorial objects such as Dyck paths, multigraphs, and symmetric pentagons. Further combinatorial interpretations of the various sequences studied in this thesis can be found at The On-Line Encyclopedia of Integer Sequences [4].

In the first chapter, we review some previous results for pattern-avoiding Grassmannian permutations obtained in [1]. In addition, we derive generating functions and present a map to describe the reverse complement of a Grassmannian Dyck path.

The second chapter contains the main results where we look at odd Grassmannian permutations that avoid patterns of size 3. The two sections, $\mathscr{G}_n^{odd}(132)$ and $\mathscr{G}_n^{odd}(312)$, are similar. For each section, we prove a recurrence relation and derive generating functions for both odd and even pattern-avoiding permutations. We also connect the permutations to another set of combinatorial objects and present a Dyck path interpretation for each of the equivalent patterns.

Grassmannian Permutations

The set of Grassmannian permutations, $\mathscr{G}_n$, is the subset of vexillary permutations (2143-avoiders) that have at most one descent. For $\pi \in \mathscr{G}_n$, the reverse complement[1], π^{rc}, is also in $\mathscr{G}_n$ since the reverse complement maintains the number of descents. It is also important to note that the reverse complement maintains the number of inversions, so if π is odd, π^{rc} is also odd.

We first review previous results for pattern-avoiding Grassmannian permutations from [1]. The following theorem generalizes enumeration of pattern-avoidance in Grassmanians for patterns with exactly one descent.

Theorem 1. *If $k \geq 3$ and $\sigma \in S_k$ with $\operatorname{des}(\sigma) = 1$, then*

$$|\mathscr{G}_n(\sigma)| = 1 + \sum_{j=3}^{k} \binom{n}{j-1} \textit{ for } n \in \mathbb{N}.$$

Specifically for patterns of size three, Theorem 1 results in:

Corollary 1.1. *For $\sigma \in \{132, 213, 231, 312\}$ and $n \in \mathbb{N}$, we have*

$$|\mathscr{G}_n(\sigma)| = 1 + \binom{n}{2}.$$

From this formula, we derive the generating function. First, we prove the following lemma that will be used in the proof of Prop. 3 and later proofs.

Lemma 2. *The generating function for the sequence given by $a_0 = a_1 = 0$ and $a_n = \binom{n}{2}$ for $n \geq 2$, satisfies*

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} \binom{n}{2} x^n = \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n = \frac{x^2}{(1-x)^3}.$$

Proof. From the known power series,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$$\frac{d}{dx} \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=2}^{\infty} n(n-1) x^{n-2} = \frac{2}{(1-x)^3}.$$

[1] For a permutation, $\pi = \pi_1 \dots \pi_n$, the reverse of π is $\pi^r = \pi_n \dots \pi_1$. The complement is $\pi^c = \pi'_1 \dots \pi'_n$ where $\pi'_i = (n+1) - \pi_i$. So, the reverse complement is $\pi^{rc} = \pi'_n \dots \pi'_1$ [2].

Multiplying by $\frac{x^2}{2}$,

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n = \frac{x^2}{(1-x)^3} \,.$$

Therefore, since $n = 0$ and $n = 1$ implies the terms are zero,

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n = \frac{x^2}{(1-x)^3} \,.$$

□

Proposition 3. *For $\sigma \in \{132, 213, 231, 312\}$, if $f(x) = \sum_{n=1}^{\infty} |\mathscr{G}_n(\sigma)|\, x^n$, then $|\mathscr{G}_n(\sigma)|$ is counted by the generating function,*

$$f(x) = \frac{x - x^2 + x^3}{(1-x)^3} = x + 2x^2 + 4x^3 + 7x^4 + 11x^5 + 16x^6 + \cdots \,.$$

Proof. We proceed to derive the generating function from the formula, $a_1 = 1$ and $a_n = \binom{n}{2} + 1$ for $n \geq 2$. If we let $a_0 = 0$, then

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} \binom{n}{2} x^n + \left(\left(\sum_{n=0}^{\infty} 1 x^n \right) - 1 \right) \\ &= \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n + \left(\left(\sum_{n=0}^{\infty} x^n \right) - 1 \right) \,. \end{aligned}$$

By Lemma 2,

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n &= \frac{x^2}{(1-x)^3} + \left(\frac{1}{1-x} - 1 \right) \\ &= \frac{x^2 + (1-x)^2 - (1-x)^3}{(1-x)^3} \\ &= \frac{x^2 + 1 - 2x + x^2 - (1-x)(1-2x+x^2)}{(1-x)^3} \\ &= \frac{2x^2 + 1 - 2x - 1 + 2x - x^2 + x - 2x^2 + x^3}{(1-x)^3} \\ &= \frac{x - x^2 + x^3}{(1-x)^3} \,. \end{aligned}$$

□

As introduced in [1], there is a map that allows us to connect Grassmannian permutations with a specific set of Dyck paths[2]. The map

$$\varphi : \mathrm{Dyck}(n) \to S_n(321) \tag{1}$$

where Dyck(n) are the Dyck paths of semilength n is described as:

[2]A Dyck path of semilength n is a path from $(0, 0)$ to (n, n) that never passes below the x-axis using only up and down steps.

- From left to right, number the down-steps of the Dyck path with $[n]$ in increasing order.
- At each peak UD, label the up-step with the number already assigned to its paired down-step.
- Going through the ascents from left to right, label the remaining up-steps from bottom to top on each ascent in a greedy fashion.
- The resulting labeling gives a 321-avoiding permutation on $[n]$.

An example of a permutation and its Dyck path as defined by φ is shown in Fig. 1. With the map, the following two propositions are presented.

Proposition 4. *The set $\mathscr{G}_n$ of Grassmannian permutations on $[n]$ is in bijection with the set of Dyck paths of semilength n having at most one long ascent.*

Proposition 5. *The set $\mathscr{G}_n^{odd}$ is in bijection with the set of Grassmannian Dyck paths of semilength n having an odd number of peaks at even height. Moreover, the elements of $\mathscr{G}_n^{even}$ correspond to Grassmannian Dyck paths with an even number of peaks at even height.*

We can also connect the Grassmannian Dyck path of a permutation, P_σ, with the Dyck path of its reverse-complement, $\mathrm{P}_{\sigma^{rc}}$. We consider the map

$$\psi : \mathrm{Dyck}(n) \to \mathrm{Dyck}(n) : \mathrm{P}_\sigma \mapsto \mathrm{P}_{\sigma^{rc}} \tag{2}$$

which is defined as follows:

- Begin with the labelled Dyck path, P_σ, from φ.
- Identify the LR (left-to-right) maxima[3] of σ, $m_1, \ldots, m_k, m_{k+1}, \ldots m_{k+j}$ where m_k is the element of the descent and $0 \leq j \leq n-k$. If $m_k = n$, then $m_{k+1}, \ldots m_{k+j}$ is empty, and if there is no descent, every element is a LR-maximum which implies the permutation is the identity.
- The complement is drawn as follows:
$$\mathrm{P}_{\sigma^c} = \mathsf{U}^{m_1}\mathsf{D}\mathsf{U}^{m_2-m_1}\mathsf{D}\ldots\mathsf{U}^{m_{k-1}-m_{k-2}}\mathsf{D}\mathsf{U}^{m_k-m_{k-1}}\mathsf{D}^{m_k-k+1}(\mathsf{UD})^j.$$
- From left to right, label the up-steps in decreasing order. At each peak, label the down-step with the number assigned to its corresponding up-step. Label the remaining down-steps in a decreasing greedy fashion.
- The path is then reversed which results in
$$\mathrm{P}_{\sigma^{rc}} = (\mathsf{UD})^j\mathsf{U}^{m_k-k+1}\mathsf{D}^{m_k-m_{k-1}}\mathsf{U}\mathsf{D}^{m_{k-1}-m_{k-2}}\ldots\mathsf{U}\mathsf{D}^{m_2-m_1}\mathsf{U}\mathsf{D}^{m_1}.$$

[3] An element of a permutation, π_i, is called a LR-maxima if for every $j < i$, $\pi_j < \pi_i$.

Observe that the labelled Dyck path, P_{σ^c} is the path of σ^c by construction. For each LR-maxima in the Grassmannian, m_i, we label the down-step corresponding to the m_i up-step in the new Dyck path, which is labelled $n-(m_i-1)$ by the process described above. This is exactly the complementary element of m_i in the same position as m_i. Once the $k+j$ LR-maxima are paired with their complementary element, there are $n-k-j$ remaining labels which must be the complementary elements of those that are not LR-maxima in the Grassmannian. Otherwise, they already would have been paired. Since the elements in the Grassmannian are increasing, the remaining elements in the complement are labelled in decreasing order from left to right. While P_{σ^c} is not a Grassmannian Dyck path, it does have exactly one long descent with labels on the down-steps. So, when reversed, the path has at most one long ascent with the labels on the up-steps, equivalent to the labelling in the map, φ.

For example in Figure 1, there is the Dyck path of $124579368|10 \in \mathscr{G}_{10}$ as well of the path of its complement and reverse complement through ψ.

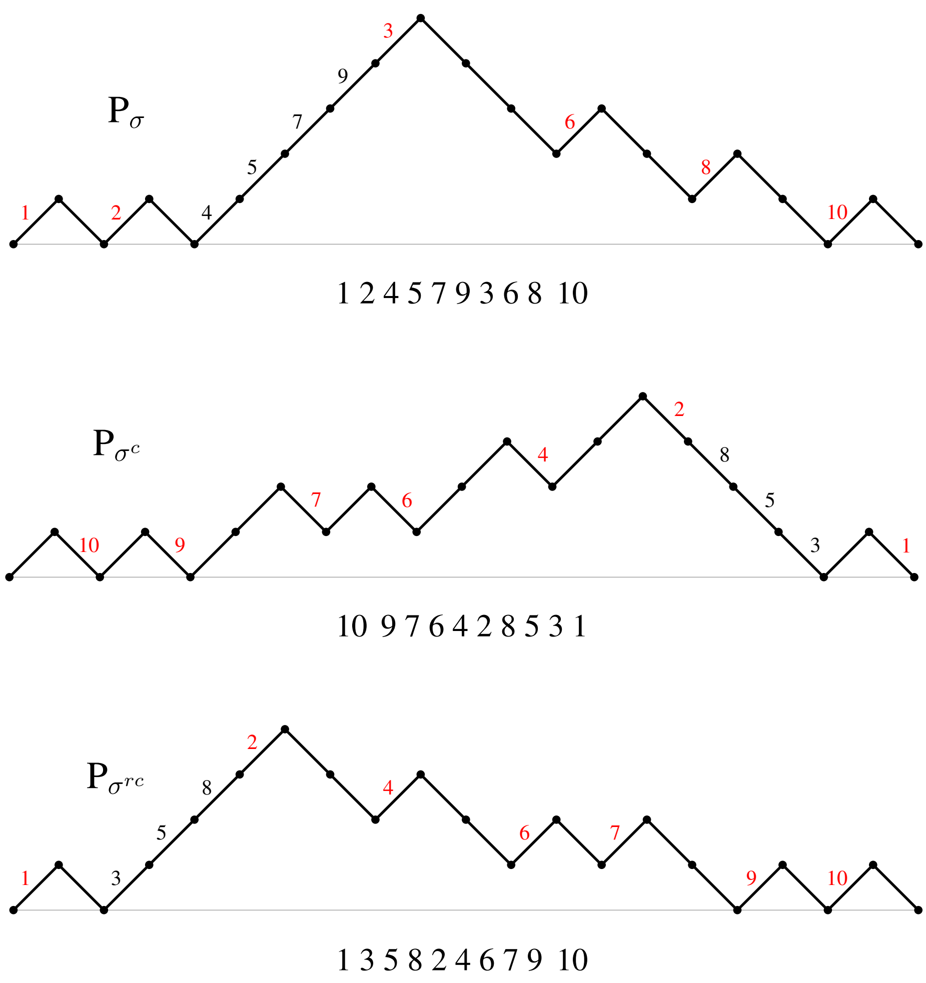


Figure 1: P_{σ}, P_{σ^c}, and $\mathrm{P}_{\sigma^{rc}}$ where $\sigma = 124579368|10 \in \mathscr{G}_{10}$

Pattern-Avoiding Odd Grassmannian Permutations

For the odd Grassmannian permutations that avoid patterns of size 3, there are are four subsets of objects to study. The class $\mathscr{G}_n^{odd}(321)$ is trivial as the Grassmannian permutations naturally avoid a 321-pattern (at most one descent). In other words, $\mathscr{G}_n^{odd}(321) = \mathscr{G}_n^{odd}$. The class $\mathscr{G}_n^{odd}(123)$ is finite:

Theorem 6. *The odd Grassmannian permutations that avoid 123 are finite and counted by the sequence, 0, 1, 2, 1, 0, 0, . . .*

The permutations of $\mathscr{G}_n^{odd}(123)$ and their graphs are given in Fig. 2. It is easy to see that once you get to a permutation of size four, there's no place to insert another element without creating a 123-pattern or an additional descent.

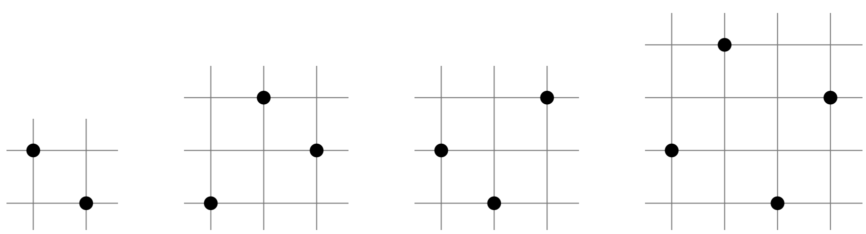


Figure 2: Odd Grassmannians that avoid 123

The remaining patterns are split into two equivalence classes since the reverse-complement maintains the odd Grassmannian property. The first is the class of odd Grassmannians avoiding 132 (or 213) and the other is the class of odd Grassmannians avoiding 312 (or 231).

Odd Grassmannian avoiding 132 (or 213)

First, we review the shape of a Grassmannian permutation that avoids 132. Excluding the identity, an element of $\mathscr{G}_n(132)$ must be of the form shown in Figure 3, where a line segment represents a sequence of consecutive entries. In other words, the permutation can be written as

$$\underbrace{i+1,\ldots,m,}_{\tau_1}\ \underbrace{1,\ldots,i,}_{\tau_2}\ \underbrace{m+1,\ldots,n}_{\tau_3} \tag{3}$$

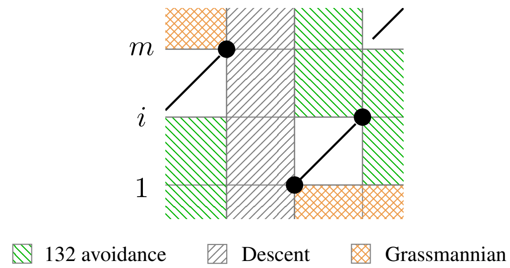


Figure 3: Grassmannians that avoids 132

where $\tau_3 = \varnothing$ if $m = n$. If the elements of τ_1, τ_2, or τ_3 were not consecutive, there would necessarily be a 132-pattern or an additional descent.

We now proceed with the enumeration of the odd Grassmannian permutations that avoid 132.

Theorem 7. *The odd Grassmannian permutations of size n that avoid 132, $\mathscr{G}_n^{odd}(132)$, are counted by*

$$a_0 = 0,\ a_1 = 1, \quad \text{and} \quad a_n = \begin{cases} \binom{\frac{n}{2}+1}{2}, & \text{if } n \text{ is even} \\ a_{n-1}, & \text{if } n \text{ is odd} \end{cases} \quad \text{for } n \geq 2,$$

which generates the sequence, $0, 1, 1, 3, 3, 6, 6, 10, 10, 15, \ldots$

Proof. As an element of $\mathscr{G}_n(132)$ must be of the form shown in Eq. 3, in order for the permutation to be odd, $|\tau_1| \cdot |\tau_2|$ must be odd. Therefore, m must be even and i odd, which implies the descent must be at an odd position.

If n is even, suppose $n = 2k$. We can construct these permutations by placing the k even numbers in the k odd positions with the restriction that $s > t$ where $s \in \{2, 4, \ldots, 2k\}$ and $t \in \{1, 3, \ldots, 2k-1\}$. Otherwise, there are not enough elements to satisfy the consecutive ascent beforehand. Thus, there are k permutations with the descent in the first position, $k-1$ in the third position, etc., so there are $k + (k-1) + \cdots + 2 + 1 = \frac{k(k+1)}{2} = \frac{\frac{n}{2}(\frac{n}{2}+1)}{2} = \binom{\frac{n}{2}+1}{2}$ even-sized permutations.

If $n = 2k+1$, then we can create all the permutations of size $2k+1$ by appending $2k+1$ at the end of all the permutations of size $2k$. This is due to the fact that $2k+1$ cannot be placed in the descent and appending $2k+1$ at the end of the permutation does not affect the number of inversions. Therefore, $a_{2k+1} = a_{2k}$. $\square$

We can then derive the generating function for $\mathscr{G}_n^{odd}(132)$ as well as the one for $\mathscr{G}_n^{even}(132)$ using Prop. 3.

Theorem 8. *Equivalent to Thm. 7, the odd Grassmannian permutations that avoid* 132 *are counted by the generating function*

$$f(x) = \frac{x^2 + x^3}{(1-x^2)^3} = x^2 + x^3 + 3x^4 + 3x^5 + 6x^6 + 6x^7 + \cdots.$$

Proof. We proceed to derive the generating function from the recurrence relation in Thm. 7. From Lemma 2, we know

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n = \frac{x^2}{(1-x)^3}$$

and can replace $n = n+1$, so

$$\sum_{n=1}^{\infty} \frac{(n+1)n}{2} x^{n+1} = \frac{x^2}{(1-x)^3}\,.$$

Thus, as $a_0 = 0$ and dividing by x,

$$\sum_{n=0}^{\infty} \frac{(n+1)n}{2} x^{n} = \frac{x}{(1-x)^3}\,.$$

Therefore, from the recurrence relation,

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} \frac{n(n+1)}{2} x^{2n} + \sum_{n=0}^{\infty} \frac{n(n+1)}{2} x^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{n(n+1)}{2} x^{2n} + x\left(\sum_{n=0}^{\infty} \frac{n(n+1)}{2} x^{2n}\right) \\
&= \frac{x^2}{(1-x^2)^3} + x\left(\frac{x^2}{(1-x^2)^3}\right) \\
&= \frac{x^2+x^3}{(1-x^2)^3}.
\end{aligned}$$

□

Theorem 9. *The even Grassmannian permutations that avoid 132, $\mathscr{G}_n^{even}(132)$, are counted by the generating function*

$$h(x) = \frac{x+x^4+x^5}{(1-x)(1-x^2)^2} = x + x^2 + 3x^3 + 4x^4 + 8x^5 + 10x^6 + \cdots\,.$$

Proof. The generating function for the even Grassmannian permutations that avoid 132 is determined by using the generating functions from Prop. 3 and Thm. 8. Subtracting the odd permutations that avoid 132 from all Grassmannian permutations that avoid 132, we are left with the generating function of the even,

$$\begin{aligned}
h(x) &= \frac{x - x^2 + x^3}{(1-x)^3} - \frac{x^2 + x^3}{(1-x^2)^3} \\
&= \frac{(1+x)^3(x - x^2 + x^3) - x^2 - x^3}{(1-x)^3(1+x)^3} \\
&= \frac{x(1+x)((1+x)^2(1-x+x^2) - x)}{(1-x)^3(1+x)^3} \\
&= \frac{x((1+2x+x^2)(1-x+x^2) - x)}{(1-x)^3(1+x)^2} \\
&= \frac{x((1-x+x^2+2x-2x^2+2x^3+x^2-x^3+x^4) - x)}{(1-x)^3(1+x)^2} \\
&= \frac{x(1+x^3+x^4)}{(1-x)(1-x^2)^2} \\
&= \frac{x+x^4+x^5}{(1-x)(1-x^2)^2}\,.
\end{aligned}$$

□

The elements of $\mathscr{G}_n^{odd}(132)$ can be connected to other combinatorial objects counted by the same sequence. In particular, we connect them with distinct symmetric pentagons in a regular $(n+3)$-gon.

Theorem 10. *The odd Grassmannian permutations of size n that avoid 132 are in one-to-one correspondence with distinct, symmetric, convex pentagons in a regular $(n+3)$-gon.*

Proof. Given a symmetric pentagon in a regular $(n+3)$-gon, we can create an odd, 132-avoiding, Grassmannian permutation through the following map:

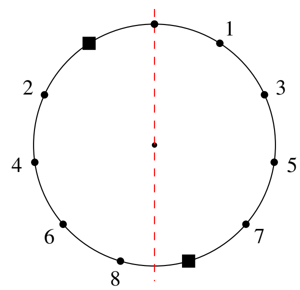
(i) Represent the $(n+3)$-gon as a unit circle with equidistant $(n+3)$ vertices, whose line of symmetry is $x = 0$.

(ii) Label the vertices left of the line of symmetry counter-clockwise with even numbers, excluding the first vertex, and label the right vertices clockwise with odd numbers, excluding the last vertex.

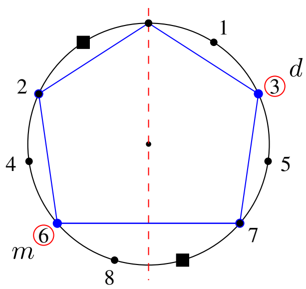
(iii) Draw the symmetric pentagon with fixed vertex $(0,1)$. The first vertex of the pentagon on the right determines the position of the descent, d, and the second vertex on the left, m, determines the number that makes the descent with 1.

(iv) The remaining values are filled around the descent to create a permutation of size n of the form $(i{+}1), \ldots, m, 1, \ldots, i, m{+}1, \ldots, n$ where $((i+1), \ldots, m)$, $(1, \ldots, i)$, and $(m+1, \ldots, n)$ are each segments filled with consecutive elements.

For example, in the following 11-gon,



we are given a pentagon and can identify m and d.



Then, we create a permutation with the descent, 61, in the third position resulting in 45612378. This permutation is precisely as described in Thm. 7 and will therefore be Grassmannian, odd, and avoid 132.

Conversely, given an odd, Grassmannian permutation of size n that avoids 132, we can draw an $(n+3)$-gon as described in steps (i) and (ii) above. Then, we can identify, m and d in the permutation, which must be even and odd respectively. With fixed vertex $(0,1)$, m on the left, and d on the right as vertices, we can create a convex pentagon with line of symmetry $x = 0$, since $m > d$ for every permutation in $\mathscr{G}^{odd}(132)$, cf. proof of Thm. 7. □

Remark. The argument to directly count the distinct, symmetric pentagons in an n-gon is clearly described by the recurrence relation in Thm. 7 shifted. If n is odd, let $n = 2k+3$. With one element on the axis of symmetry, there are $k+1$ elements on each side and two are chosen as vertices. Thus, there are $\binom{k+1}{2}$ such pentagons. It is also clear for the even case $(n+1)$, the number of pentagons will be equivalent since the one additional vertex will be on the line of symmetry and cannot be used to create any more distinct pentagons.

There is also an interpretation for $\mathscr{G}_n^{odd}(132)$ in terms of Dyck paths.

Theorem 11. *The odd Grassmannian permutations that avoid 132 are in one-to-one correspondence with odd Grassmannian Dyck paths (having an odd number of peaks at even height with at most one long ascent) that begin with a long ascent and have exactly one long descent.*

Proof. Using the map φ in Eqn. 1, it is clear that if there is an identity path, UD, before the long ascent, then the corresponding permutation must have a 132-pattern. Therefore, the Dyck path must begin with the long ascent.

Suppose there are two long descents. So, there is at least one peak between them with label j. There are also at least two descents, one from each long descent, whose labels occur on the initial long ascent, say i and k where $i < j < k$. In the permutation, they would occur as $\ldots, i, \ldots, k, \ldots, j, \ldots$ where ikj is a 132-pattern. □

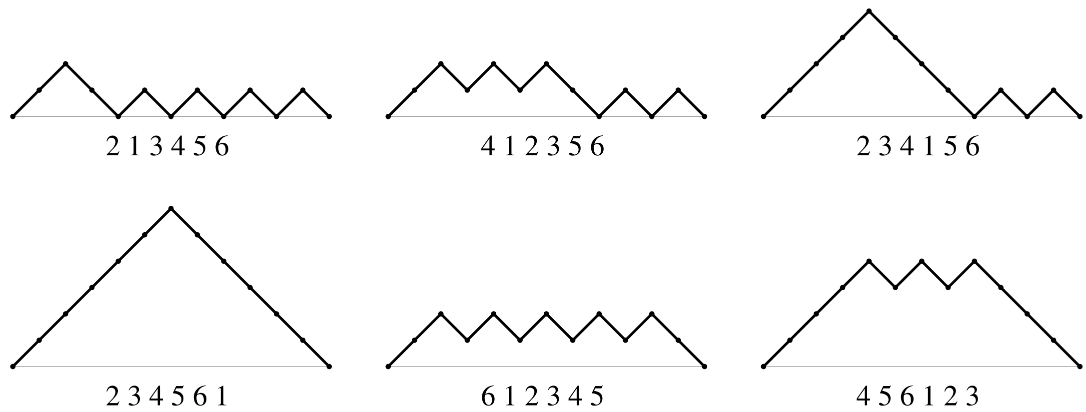


Figure 4: Odd Grassmannians that avoid 132 of size 6 and their Dyck paths

The Dyck paths of the odd Grassmannian permutations of size 6 that avoid 132 are drawn in Fig. 4. There is a similar theorem for the odd Grassmannian permutations that avoid 213.

Theorem 12. *The odd Grassmannian permutations that avoid* 213 *are in one-to-one correspondence with odd Grassmannian Dyck paths (having an odd number of peaks at even height with at most one long ascent) that have exactly one long ascent and one long descent, with no identity peak (*UD*) after the long descent.*

Proof. Rather than prove this directly, this can be seen through the reverse complement map, ψ, described previously in Eqn. 2. We know the odd permutations that avoid 132 must be of the form shown in Eqn. 3:

$$\underbrace{i+1,\ldots,m,}_{\tau_1}\ \underbrace{1,\ldots,i,}_{\tau_2}\ \underbrace{m+1,\ldots,n}_{\tau_3}$$

where m is even, i is odd, and $\tau_3 = \varnothing$ if $m = n$. The LR-maxima of the permutation are the elements of τ_1 and τ_3. So, after applying the map, ψ, we get the complement,

$$\mathsf{U}^{i+1}\mathsf{D}(\mathsf{UD})^{m-(i+1)-1}\mathsf{U}\mathsf{D}^{m-(m-i)+1}(\mathsf{UD})^{n-m}$$

which is then reversed and simplified to get the reverse-complement:

$$(\mathsf{UD})^{n-m}\mathsf{U}^{i+1}\mathsf{D}(\mathsf{UD})^{m-i-2}\mathsf{U}\mathsf{D}^{i+1}.$$

Clearly, the resulting Dyck path has exactly one long ascent and one long descent, and there are no identity peaks after the long descent. It is also an odd Grassmannian Dyck path as there are $m-i$ peaks at even height $i+1$, and the corresponding permutations will avoid 213 from the map. □

The Dyck paths of the odd Grassmannian permutations of size 6 that avoid 213 are drawn in Fig. 5. Through the map ψ, they correspond to each of the graphs drawn in Fig. 4 respectively.

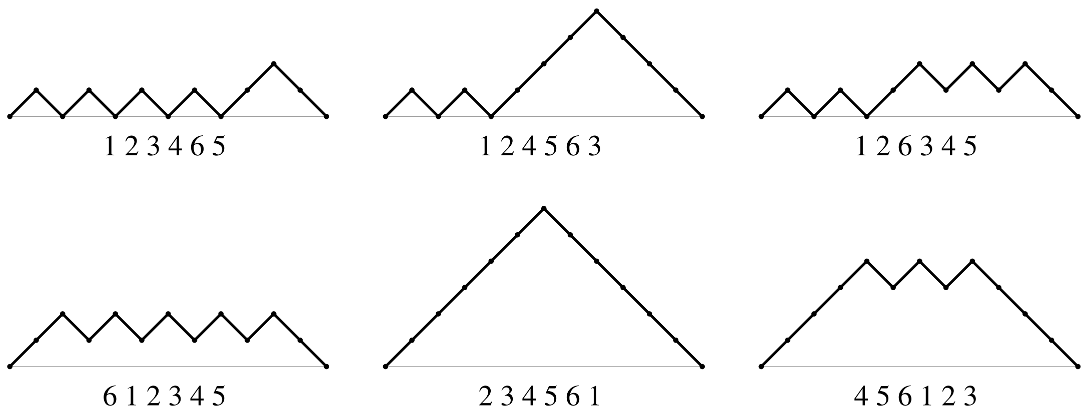


Figure 5: Odd Grassmannians that avoid 213 of size 6 and their Dyck paths

Odd Grassmannian avoiding 312 (or 231)

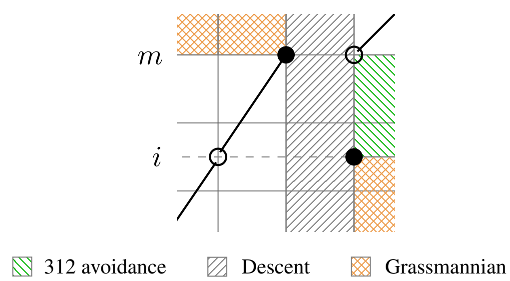


Figure 6: Grassmannians that avoids 312

Again, we begin with the shape of an odd Grassmannian permutation avoiding 312. Without the identity, we know such permutations are of the form shown in Fig. 6. In other words, the permutations can be written as

$$\underbrace{1,\ldots,i-1,}_{\tau_1}\ \underbrace{i+1,\ldots,m,}_{\tau_2}\ i,\ \underbrace{m+1,\ldots,n}_{\tau_3} \tag{4}$$

where $\tau_3 = \varnothing$ if $m = n$. If the elements of τ_1, τ_2, or τ_3 were not consecutive, there would necessarily be a 312-pattern or an additional descent.

We proceed with the enumeration of the odd Grassmannian permutations that avoid 312 through the following recurrence relation.

Theorem 13. *The odd Grassmannian permutations that avoid* 312, $\mathscr{G}_n^{odd}(312)$, *are counted by the recurrence*

$$a_0 = a_1 = 0, \quad \textit{and} \quad a_n = \binom{n}{2} - a_{n-1} \quad \textit{for } n \geq 2,$$

which generates the sequence, $0, 1, 2, 4, 6, 9, 12, 16, 20, 25, \ldots$

Proof. Let $A_n = \mathscr{G}_n^{odd}(312)$ and $B_n = \mathscr{G}_n^{even}(312)/\{id_n\}$, where id_n is the identity permutation of size n. Using the fact that all 312-avoiders are counted by $\binom{n}{2}+1$, we can describe them by the following equation

$$\binom{n}{2} + 1 = a_n + b_n + 1. \tag{5}$$

where a_n is the number of odd Grassmannian avoiders and b_n are the number of even avoiders excluding the identity, which is the additional 1. Since the permutations must be of the form shown in Eqn. 4, in order for the permutation to be odd, m and i must have opposite parity, so $|\tau_2|$ is odd.

We proceed to prove the recurrence relation by proving a bijection between A_{n-1} and B_n. The former set can be divided into two disjoint groups:

(i) The permutations that end in $n-1$, i.e. $\tau_3 \neq \varnothing$

(ii) The permutations that do not end in $n-1$, i.e. $\tau_3 = \varnothing$.

For the corresponding groups above, there are two separate processes to create an even Grassmannian permutation of size n that avoids 312.

(i) Move the element $m+1$ to the end of τ_2, so $m+1$ is the descent with i. Add n to the end of the permutation.

(ii) Insert n into the descent.

For the process described in (i), we maintain the shape of a 312-avoider. By moving $m+1$, $|\tau_2|$ is even, so the permutation is even. For (ii), τ_3 is empty, so the descent must occur in the position $n-2$. Therefore, when n is inserted into the descent, $|\tau_2|$ is even, and there cannot be a 312-pattern since there is only one element after n.

This map is reversible since the disjoint groups are also identifiable in B_n as the permutations that end in n and those that do not end in n. For the permutations that end in n, you remove n and move the top element of the descent to the first position of τ_3. For the permutations that do not end in n, we remove n from the descent.

With the established bijection, $A_{n-1} \cong B_n$, we can replace b_n in Eqn. 5 with a_{n-1} and simplify to get the exact result in the theorem. □

Using the formula in Thm. 13, we derive the generating function for $\mathscr{G}_n^{odd}(312)$. Then, we derive the generating function for $\mathscr{G}_n^{even}(312)$ using Prop. 3 and the result from Thm. 14.

Theorem 14. *Equivalent to Thm. 13, the odd Grassmannian permutations that avoid* 312 *are counted by the generating function*

$$f(x) = \frac{x^2}{(1-x)^3(1+x)} = x^2 + 2x^3 + 4x^4 + 6x^5 + 9x^6 + 12x^7 + \cdots .$$

Proof. We proceed to derive the generating function from the recurrence relation, $a_n = \binom{n}{2} - a_{n-1}$ where $a_0 = a_1 = 0$.

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} \binom{n}{2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n \\ &= \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n - x\left(\sum_{n=1}^{\infty} a_{n-1} x^{n-1}\right) \\ &= \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n - x\left(\sum_{n=0}^{\infty} a_n x^n\right).\end{aligned}$$

So,

$$\sum_{n=0}^{\infty} a_n x^n + x\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n,$$

and by Lemma 2,

$$(1+x)\left(\sum_{n=0}^{\infty} a_n x^n\right) = \frac{x^2}{(1-x)^3}.$$

Therefore,

$$\sum_{n=0}^{\infty} a_n x^n = \frac{x^2}{(1-x)^3(1+x)}.$$

□

Theorem 15. *The even Grassmannian permutations that avoid 312 are counted by the generating function*

$$h(x) = \frac{x - x^2 + x^4}{(1-x)^3(1+x)} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 7x^6 + 10x^7 \cdots.$$

Proof. The generating function for the even Grassmannian permutations that avoid 312 is determined using the generating functions from Prop. 3 and Thm. 14.

$$\begin{aligned}h(x) &= \frac{x - x^2 + x^3}{(1-x)^3} - \frac{x^2}{(1-x)^3(1+x)} \\ &= \frac{(x - x^2 + x^3)(1+x) - x^2}{(1-x)^3(1+x)} \\ &= \frac{x - x^2 + x^3 + x^2 - x^3 + x^4 - x^2}{(1-x)^3(1+x)} \\ &= \frac{x - x^2 + x^4}{(1-x)^3(1+x)}.\end{aligned}$$

□

The elements of $\mathscr{G}_n^{odd}(312)$ can be connected with the multigraphs with loops[4] on two nodes having $n-2$ edges as they are counted by the same sequence.

Theorem 16. *The odd Grassmannian permutations of size n that avoid 312 are in one-to-one correspondence with the multigraphs with loops on two nodes having $n-2$ edges.*

Proof. To ensure there is no repetition in the multigraphs, we will always draw them with more loops on the left node. The map between the two sets is described as follows. We represent the permutation as a graph, similar to Fig. 6. With the two established elements that form the descent, m and i, the graph is vertically divided into three sections: between m and i, below i, and above m. The numbers of elements between m and i are translated as the number of loops evenly split between the two nodes in the multigraph. The number of elements below i represent the number of extra loops placed on the left node. Finally, the number of elements above m is the number of edges placed between the two nodes. An example of a permutation, its graph, and corresponding multigraph is given in Fig. 7, and the multigraphs for permutations of size 5 are given in Fig. 8.

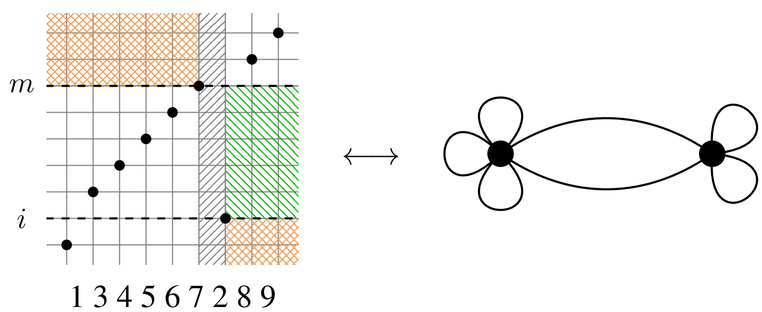


Figure 7: Graph of $134567289 \in \mathscr{G}_9^{odd}(312)$ and its corresponding multigraph

The reverse map is clear since the three groups are easily identifiable in the multigraphs as well. For example in Fig. 7, we identify the number of elements above m as the number of edges between the two nodes, the number of elements between m and i as two times the loops on the right node, and the number of elements below i as the number of loops on the left node minus the number on the right. With $n-2$ edges, we end up with $n-2$ elements dispersed between the three vertical sections of the graph. Drawing the graph of the permutation with the given numbers (including m and i), we determine m and i, and we create a permutation of size n that is odd, Grassmannian, and avoids 312. $\square$

There is also a Dyck path interpretation for $\mathscr{G}_n^{odd}(312)$, and using the map ψ for the reverse complement, we get a Dyck path interpretation for $\mathscr{G}_n^{odd}(231)$.

Theorem 17. *The odd Grassmannian permutations that avoid 312 are in one-to-one correspondence with odd Grassmannian Dyck paths (having an odd number of peaks at even height with at most one long ascent) that have no valleys above height zero.*

[4] A multigraph is a graph that can have multiple edges between the same vertices. If we include loops, then we can have multiple edges that connect a vertex to itself as well.

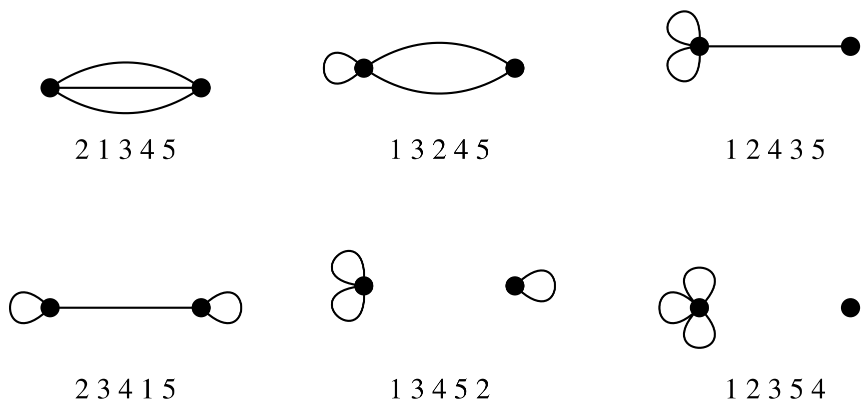


Figure 8: Odd Grassmannians that avoid 312 of size 5 and their multigraphs

Proof. Once again, we make use of φ from Eqn. 1. Suppose there is a valley at height greater than zero. Therefore, aside from the peak of the long ascent, labelled i, there is another peak, j, at height greater than or equal to 2. This implies there is another descent, k, after the peak whose label must occur on the long ascent. The occurrence of $\ldots, k, \ldots, i, \ldots, j, \ldots$ in the permutation is exactly a 312-pattern. □

The Dyck paths of the odd Grassmannian permutations of size 5 that avoid 312 are shown in Fig. 9.

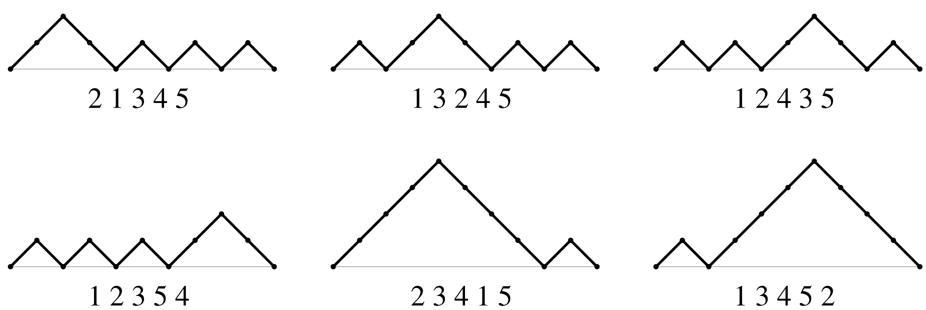


Figure 9: Odd Grassmannians that avoid 312 of size 5 and their Dyck paths

Theorem 18. *The odd Grassmannian permutations that avoid 231 are in one-to-one correspondence with odd Grassmannian Dyck paths (having an odd number of peaks at even height with at most one long ascent) that have no peaks above height two.*

Proof. Using the path reverse complement map, ψ, we describe the odd Dyck paths that avoid 231. From Eqn. 4, we know the odd permutations that avoid 312 must be of the form

$$\underbrace{1, \ldots, i-1,}_{\tau_1} \underbrace{i+1, \ldots, m,}_{\tau_2} i, \underbrace{m+1, \ldots, n}_{\tau_3}$$

where m and i are opposite parity, and $\tau_3 = \varnothing$ if $m = n$. The LR-maxima of the permutation are all the elements of the permutation except i. So, after applying the map, ψ, we get the complement,

$$(\mathsf{UD})^{i-1}\mathsf{U}^{(i+1)-(i-1)}\mathsf{D}(\mathsf{UD})^{m-(i+1)-1}\mathsf{U}^{m-(m-1)}\mathsf{D}^{m-(m-1)+1}(\mathsf{UD})^{n-m}$$

which is then reversed and simplified to get:

$$(\mathsf{UD})^{n-m}\mathsf{U}^2\mathsf{D}(\mathsf{UD})^{m-i-2}\mathsf{UD}^2(\mathsf{UD})^{i-1}.$$

The resulting paths will have no peaks above height two. They also maintain the odd Grassmannian Dyck path property since the number of peaks at even height are $m - i$, which must be odd, and the corresponding permutations will avoid 231. □

The Dyck paths of the odd Grassmannian permutations of size 5 that avoid 231 are shown in Fig. 10. They correspond directly to those in Fig. 9 using the map ψ.

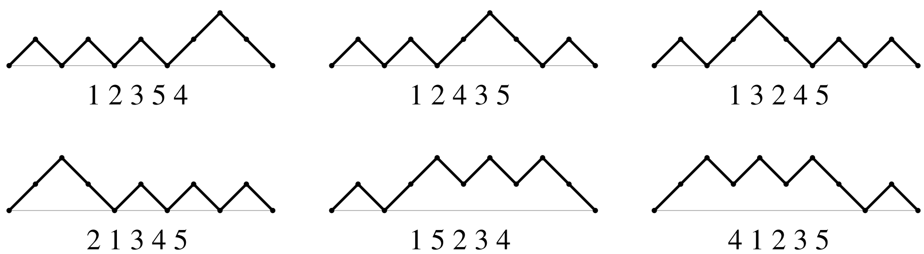


Figure 10: Odd Grassmannians that avoid 231 of size 5 and their Dyck paths

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Learning Resources Center Peer Tutor
Altoona, PA — August 2019 - Present
- Reviewed class materials with students
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- Assisted students in developing good study habits

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PUBLICATIONS

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PRESENTATIONS

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RESEARCH GRANTS

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	George P. Gasbarre Engineering Scholarship	2018-19
	Francis Mehall Memorial Scholarship	2018-19