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Impact of Ties in Stable Matching Problems

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Abstract

In Artificial Intelligence, agents refer to intelligent entities that are capable of preference-based decision making. Agents may interact with one another in multi-agent settings, where the decisions of one agent impacts the decisions of another. One real world application of a multi-agent system is online bidding, where agents interact with one another by placing bids to buy and sell resources. One type of multi-agent setting is when agents interact with each other and are ultimately arranged in pairs. Students and colleges represents agents paired in college admissions, Uber clients and drivers are agents paired in ride sharing, or applicants and job openings are agents paired in employment. From these examples comes the focus of this research on *two-sided matching*, where an agent can only be matched with another agent from the partner set. Preferences of agents may contain ties.

A matching is considered stable if there does not exist a pair such that both sides of the pair prefer the other to their current matching. The Deferred Acceptance (DA) Algorithm was proposed to always guarantee a stable matching. Given a preference profile, the Deferred Acceptance algorithm finds a matching for two sided matching markets by proceeding in rounds in which one side, the proposing side, "proposes" to the other side, the proposed-to side, who chooses to accept or reject based on their own preferences. When weak preferences are present, a tie-breaking rule, a priority ordering of agents, is used to make preferences strict with respect to the tie-breaking rule. The Deferred Acceptance algorithm is not strategy-proof for the "proposed-to side", meaning agents can strategically misreport their preferences to get matched with an agent that they strictly prefer more to their match under true preferences. This concept is known as strategic manipulation. Traditional research on two-sided strategic manipulation has focused on strict preferences. This thesis looks at how a manipulator may misreport preferences to become strictly better off in the context of weak preferences. A tie-breaking rule is shown to make: (1) a fixed woman, and all women, better off when partner man preferences are weak, and (2) a fixed man, and all men, better off when partner woman preferences are weak. Using the tie-breaking rule with weak preferences showed that the manipulator is strictly better off when the proposing side tie-breaking rule is known to the manipulator. When ties are used with strategic manipulation, the manipulator is more frequently strictly better off, but at the expense of a lower expected improvement, and making some women strictly worse off. This thesis looks at variations of the DA algorithm that were proposed for dealing with ties in preference lists and studies their strategic manipulation and new notions of stability that arise with these new algorithms.

Table of Contents

List of Figures

List of Tables

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Chapter 1

Introduction

1.1 *Intersection of Artificial Intelligence and Economics*

Artificial Intelligence (AI) is a rapidly growing area within computer science in which computers mimic the intelligence of the human mind. In artificial intelligence, an agent is an intelligent entity that wishes to optimize its objectives, and it interacts within the environment (Shoham and Leyton-Brown, 2009). Agents can be people, firms, schools, hospitals, and the list continues. Most work looks at AI within the context of a single agent, where an agent must learn to behave in an environment that they are unfamiliar with. An agent has preferences, which is the agent's desired ranking over potential alternatives and outcomes based on their own utility (Watson, 2008). An example of a preference is that worker X prefers company A to company B because of company A's higher salary and close proximity to where worker X currently lives, and worker X also prefers company B to company C due to healthcare benefits associated with company B. With this, the preferences of worker X are such that worker X prefers company B over company C, and company A over both company B and company C.

In the economic theory, multi-agent systems enable collective decision making by providing rules and guidelines that respect agent's preferences. Since agents are intelligent entities, they will behave rationally. Rationality is when agents make decisions that yield the best possible outcome given their available actions, preferences, and beliefs (Russell and Norvig, 2009). Going back to the previous example, consider worker X has to accept a job offer. Based on this, worker

X is rational and would accept a job from company A, as it prefers this company the most.

Multi-agent systems introduce an additional complication to this environment, as it introduces multiple agents who also interact with one another. If the worker-company example is extended further to introduce more workers, assume the company A has already hired some other worker Y. The worker X would not be able to accept a position with company A and would have to accept an offer from a company that is less desirable to this worker. In a multi-agent setting, decision-making is affected, as the agent can only make a selection from what is available, while still satisfying their preferences. A matching is introduced when there is an environment where agents want to be matched to other agents. There are many real life applications to matching: a worker needs to be matched to a job, a student needs to be matched to a school, a driver needs to be assigned to a bus, a college student needs to be matched to a residence hall, an Uber driver needs to be matched to Uber clients, and the list continues.

1.2 *Two Sided Matching Markets*

Matching looks at the question of who is allocated what. Consider this example: there are multiple candidates applying to companies where candidates have preferences with respect to companies and companies have preferences over candidates. A matching would refer to pairs of candidates and companies. This is an example of a two-sided matching market. A market is considered to be two-sided if there are two sets of agents, and an agent can only be matched with an agent in the opposing set. In the example, the two sets of agents are companies and candidates, where companies and candidates can only be matched with one another. A two-sided matching can be one-to-one, one-to-many, many-to-one, or many-to-many. A one-to-one matching refers to the idea that exactly one agent on side A must be mapped to exactly one agent on side B. A one-to-many matching indicates one agent on side A must be assigned to one agent on side B, but the agent on side B can map to multiple agents on side A. A many-to-one matching indicates an agent on side A can map to multiple agents on side B, but only one agent on side B must match to one agent on side A. A many-to-many matching is when an agent on side A can map to multiple agents on side B and vice versa. A key question in multi-agent systems is how to develop algorithms to find such matchings efficiently.

1.3 *The Stable Matching Problem*

The stable matching problem addresses two sided matching markets and was studied by two mathematical economists David Gale and Lloyd Shapley in 1962 (Gale and Shapley, 1962). The stable matching problem is the problem of having two disjoint sets of agents of equal size, in which their elements both have an ordering of preferences for the elements on the other side. Each agent has a preference list, which refers to their respective strict ordering of the agents on the opposing side. A preference profile contains the preference lists of all agents. With respect to these preferences, the stable matching problem seeks to find a matching. A matching between both sides is defined as a bijection from the elements of one side to the elements of the other side. The stable matching problems are key components of several applications such as roommate matching (one-to-one), college admissions (one-to-many), and residency matching (one-to-many). A blocking pair exists if there is a pair such that both sides of the pair prefer the other to their current matching. A matching is considered stable if there are no blocking pairs in the matching. Unstable matchings are undesirable, as they introduce the concept of envy, where agents who belong to a blocking pair envy the agent who was paired with the partner they prefer to their current matching.

Consider the example below in Figure 1.1, where A, B, C, D are agents on one side and E, F, G, H are the agents on the other side. The preferences for both agents are shown in the figure. In the current matching, agents C and G prefer each other to their current partners. Hence, C and G are a blocking pair, and this matching is unstable. Agent C envies agent B for being matched with agent G and agent G envies agent H for being matched with agent C. Unstable matchings are undesirable, as agents become envious of one another, making the matching unfair.

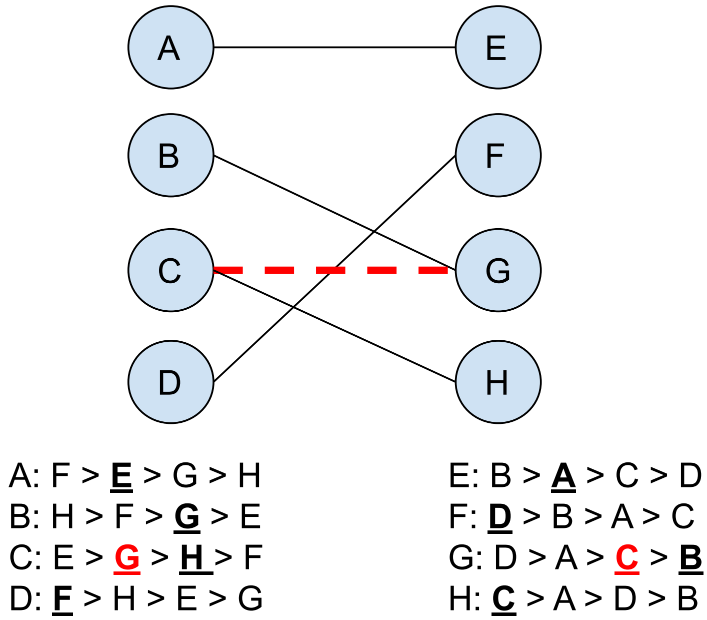


Figure 1.1: The figure above shows an unstable matching with agents A, B, C, D on one side and E, F, G, H on the other side. Since C strictly prefers G to its current partner and G strictly prefers C to its current partner, there is a blocking pair and the matching is unstable.

Gale and Shapley proposed the Deferred Acceptance (DA) algorithm (Gale and Shapley, 1962), which always yields a stable solution to the stable matching problem. In the DA algorithm, one side is known as the proposing side, commonly referred to as men, where men "propose" to women with respect to their preferences [1]. A woman accepts the proposal if she does not have a partner, and if she has a partner, she accepts the proposal from the man she prefers more with respect to its preferences. This process is repeated until all agents are matched to a partner. The algorithm will always output a stable matching (Gale and Shapley, 1962). Additionally, the algorithm is optimal for the proposing side, meaning that the proposing side is matched to their best stable partner (Gale and Shapley, 1962). The algorithm is pessimal for the proposed to side, meaning that the proposed to side is matched with their worst stable partner (McVitie and Wilson, 1971).

[1]Disclaimer: Throughout this thesis, I adopt the common terminology "man" and "woman" to indicate two sides of the market. This is common terminology used in literature in this field and the idea of men and women be matched to each other for marriage does not refer to the beliefs of the author or reviewers in any way.

1.4 *Strategic Manipulation*

The DA algorithm is not strategy-proof for women (Roth, 1982), meaning women can manipulate their preferences to get matched with a better partner. The woman who misreports is known as the manipulator. The DA algorithm is strategy-proof for men, as the proposing side cannot misreport their preferences to get strictly better off (Dubins and Freedman, 1981).

1.5 *Weak Preferences*

Consider students A and B who are applying to college X, college Y, and college Z. Student A prefers college X over college Y for financial reasons and college Z over college Y for location, but student A has no strict preference between attending college X and college Z. Student A is indifferent between college X and college Z, which introduces the concept of ties, also known as weak preferences. A tie, or weak preferences, is when an agent is indifferent between two or more alternatives. Student B strictly prefers college Z over college Y and college X over both college Z and college Y. When student A is matched with one of its top choices of college X, it takes a seat away from student B who would no longer be able to be matched with its first choice. This example introduces the importance of ties, as resolving the indifference has the ability to change the matching, and can consequently strictly improve particular agents or make agents strictly worse off (student B, in this example). In the setting of weak preferences, both sides of agents can have ties, or ties can appear on only one side of agents. Furthermore, only part of the preferences can be weak or agents can have total indifference, where they are indifferent between all agents.

New notions of stability arise when weak preferences are present. When ties are broken arbitrarily then passed into the DA algorithm, the resulting matching is guaranteed to be weakly stable. Weak stability is defined as a matching that does not contain a couple each of whom strictly prefers each other to its current partner in the matching. In addition to weak stability, there exist two stronger notions of stability. The weaker of the two is strongly-stable, which is defined as a matching that contains no pair (x, y) such that x strictly prefers y to its current partner, and y

either strictly prefers x to its partner or is indifferent between them (Irving, 1989). Super-stablity is the strongest notion of stability, and it is defined as a matching that contains no couple each of whom strictly prefers the other to his partner or is indifferent between them (Irving, 1989). A super-stable matching will always be strongly stable and weakly stable. A strongly stable matching will always also be weakly stable, but not always super-stable. A weakly stable matching is not guaranteed to be strongly stable or weakly stable. A super-stable or strongly-stable matching are not guaranteed to exist. In a case where a man is completely indifferent between all women, no super-stable matching exists (Irving, 1989).

1.6 *Tie-Breaking*

Because the standard DA algorithm assumes that the preferences of men and women are strict, a tie-breaking rule is used to arbitrarily break any ties. Two of the most common ways to break ties are through a single tie-breaking rule and multiple tie-breaking rule. Single tie-breaking (STB) is when all ties are broken according to a single given tie-breaking rule. Multiple tie-breaking (MTB) is when each agent can have its own tie-breaking rule that is used to break ties in that agent's preference list.

1.7 *Research Questions*

The main objective of this thesis is focusing on the impact of ties on a matching and impact of ties on strategic manipulation and answering the following research questions:

- RQ1: How does tie-breaking impact matching outcomes?

Tie-breaking has the ability to change the matching, but it is unclear of the extent of which it affects all agents on each side, a fixed agent on each side, and how tie-breaking can impact matching when either both sides have weak preferences, or when one side has weak preferences.

- RQ2: When weak preferences are made strict, what is the impact of the tie-breaking rule being unknown to the manipulator?

When preferences are made strict using a tie-breaking rule, the manipulator can compute the optimal manipulation strategy if she knows the tie-breaking rules. When the tie-breaking rule is unknown, it is not clear if the manipulator can still get strictly better off when she misreports its preferences.

- RQ3: For all tie-breaking rules, how can a manipulator compute its optimal strategy?

In the strict setting, a manipulator can get strictly better off for a given manipulation strategy. With ties, there has to exist a manipulation strategy that allows a manipulator to be better off for some tie-breaking rule. It remains an open question if a manipulator can stay the same or get strictly better off for all tie-breaking rules.

- RQ4: Given a super-stable or strongly stable matching, can a manipulator manipulate the outcome to make itself better off while preserving its respective notion of stability?

Because the new notions of stability are not known to be strategy-proof, this thesis will investigate whether a manipulator can successfully get better off by self manipulation with matchings that are either super-stable or strongly stable.

Chapter 2

Model

2.1 *Problem Setup*

An instance of the Stable Matching problem is denoted by $(M, W \succeq)$, where M is a set of men, W is a set of women where M and W contain the same number of agents n, and $\succeq$ is a preference list that contains the preference rankings for all the agents. The preference list of a man $m \in M$ can be denoted as $\succeq_m$, where $\succeq_m$ is a weak ordering over all w such that $w \in W$ and vice versa for the preference lists of women. A weak ordering indicates ties are present in the preference list. A bucket b in a weak preference list refers to the class of all the agents that an agent on the opposing side is indifferent to.

The notation $\succeq$ denotes $w_1 \succ_m w_2$ or $w_1 =_m w_1$. $w_1 \succ_m w_2$ indicates that m strictly prefers w_1 to w_2, and $w_1 =_m w_1$ indicates that is indifferent between w_1 and w_2. Throughout this paper, I will also use the notation $m : (w_1\ w_2)$ and $w_1 \sim_m w_2$ to denote that m is indifferent between w_1 and w_2. The notation $\succeq_{-w}$ denotes the preference lists of all men and women except w. Hence, $\succeq = \{\succeq_{-w}, \succeq_w\}$.

The goal of the Stable Matching Problem is to find a matching between both sides of agents. A matching, denoted as μ, is a function $M \cup W \rightarrow W \cup M$, where $\mu(m) \in W$ for all $m \in M$, $\mu(w) \in M$ for all $w \in W$, and $\mu(m) = w \iff \mu(w) = m$.

A blocking pair exists in μ if there exists a couple (m, w) such that m strictly prefers w to his current partner $w \succ_m \mu(m)$ and w strictly prefers m to its current partner $m \succ_w \mu(w)$ with respect to $\succ$. When preferences are strict, a matching is stable if there are no blocking pairs with respect to $\succ$. S_μ refers to the set of all matchings that are weakly stable with respect to $\succ$.

Consider an instance of the stable marriage problem below in Table 2.1. There are four men and four women where each man has weak preferences over the women and each woman has strict preferences over the men. When preferences contain ties, like how they do for men in the example below, it is critical in deciding how to break the ties. This introduces the concept of tie-breaking

m_1:	$(w_4\ w_1)$	w_2	w_3	w_1:	m_2	m_3	m_1	m_4
m_2:	$(w_4\ w_3)$	w_1	w_2	w_2:	m_4	m_2	m_1	m_3
m_3:	$(w_2\ w_1)$	w_4	w_3	w_3:	m_1	m_3	m_4	m_2
m_4:	$(w_3\ w_1)$	w_4	w_2	w_4:	m_3	m_2	m_4	m_1

Table 2.1: The above table shows an instance of the stable marriage problem, where there are the same number of men and women. The preference profiles are shown, where each man has weak preferences over the women, and each woman has strict preferences over the men.

2.2 *Tie-Breaking*

Consider a preference profile with ties $\succeq$. A tie-breaking rule is a strict ordering of agents that is used to break any ties of $\succeq$. The use of a tie-breaking rule generates a strict ordering of preferences, denoted as $\succ$. A tie-breaking rule can also be interpreted as a priority list that tells us how to break the ties, where the tie-breaking rule is either decided apriori or randomly generated.

A single tie-breaking rule (STB) uses an ordering of agents that is used to break any ties with respect to the ordering of the agents in the STB. Consider the weak men preferences in Table 5.1, and the ties in these men preferences are broken using the STB rule $[w_1 \succ w_2 \succ w_3 \succ w_4]$.

Weak Preferences	Strict Preferences
$m_1 : (w_4\ w_1) \succ w_2 \succ w_3$	$w_1 \succ w_4 \succ w_2 \succ w_3$
$m_2 : (w_4\ w_3) \succ w_1 \succ w_2$	$w_3 \succ w_4 \succ w_1 \succ w_2$
$m_3 : (w_2\ w_1) \succ w_4 \succ w_3$	$w_1 \succ w_2 \succ w_4 \succ w_3$
$m_4 : (w_3\ w_1) \succ w_4 \succ w_2$	$w_1 \succ w_3 \succ w_4 \succ w_2$

Table 2.2: Computing strict preferences of a set of weak man preference lists for single tie breaking rule $[w_1 \succ w_2 \succ w_3 \succ w_4]$. This tie-breaking rule breaks the tie $(w_4\ w_1)$ in m_1's preference list by making m_1 strictly prefer w_1 to w_4.

A multiple tie-breaking (MTB) is a set of tie-breaking rules, where each tie-breaking rule is used to break ties for the corresponding agent. Consider the weak men preferences from the previous example, and these preferences will become strict using the MTB rules shown in Table 2.3.

Weak Preferences	Tie-breaking Rule	Strict Preferences
$m_1 : (w_4\ w_1) \succ w_2 \succ w_3$	$w_2 \succ w_1 \succ w_3 \succ w_4$	$w_1 \succ w_4 \succ w_2 \succ w_3$
$m_2 : (w_4\ w_3) \succ w_1 \succ w_2$	$w_3 \succ w_1 \succ w_2 \succ w_4$	$w_4 \succ w_3 \succ w_1 \succ w_2$
$m_3 : (w_2\ w_1) \succ w_4 \succ w_3$	$w_2 \succ w_4 \succ w_3 \succ w_1$	$w_2 \succ w_1 \succ w_4 \succ w_3$
$m_4 : (w_3\ w_1) \succ w_4 \succ w_2$	$w_4 \succ w_3 \succ w_1 \succ w_2$	$w_3 \succ w_1 \succ w_4 \succ w_2$

Table 2.3: Computing strict preferences of a set of weak man preference lists for the multiple tie-breaking rules shown in table. The tie-breaking rule for m_1, $w_2 \succ w_1 \succ w_3 \succ w_4$, breaks the tie $(w_4\ w_1)$ in m_1's preference list by making m_1 strictly prefer w_1 to w_4. Meanwhile, the tie-breaking rule for m_4, $w_4 \succ w_3 \succ w_1 \succ w_2$, breaks the tie $(w_3\ w_1)$ in m_4's preference list by making m_4 strictly prefer w_3 to w_1.

2.3 *Stability with Weak Preferences*

When preferences contain ties, there are three variations of stability: weak stability, strong stability, and super stability.

A matching is weakly stable if there are no blocking pairs with respect to $\succ$. Consider the example below in Table 2.4 where both men and women have strict preferences and an unstable matching is denoted with *. The matching is unstable as it contains the underlined blocking pair (m_3, w_1). m_3 strictly prefers w_1 to his current match of w_2, and w_1 strictly prefers m_3 to its current match of m_1.

m_1:	w_1^*	w_4	w_2	w_3	w_1:	m_2	$\underline{m_3}$	m_1^*	m_4
m_2:	w_3	w_4^*	w_1	w_2	w_2:	m_4	m_2	m_1	m_3^*
m_3:	$\underline{w_1}$	w_2^*	w_4	w_3	w_3:	m_1	m_3	m_4^*	m_2
m_4:	w_1	w_3^*	w_4	w_2	w_4:	m_3	m_2^*	m_4	m_1

Table 2.4: The table above shows a matching, denoted with *, for the given preference lists that contains a blocking pair (m_3, w_1) to a weakly stable matching. The blocking pair is underlined.

A matching is strongly stable if there exists no couple (m, w) such that m strictly prefers

w to his current partner $w \succ_m \mu(m)$ and w strictly prefers m to its current partner or is indifferent between them $m \succeq_w \mu(w)$ with respect to $\succeq$. The definition of a blocking pair is analogous to that of weakly stable. Consider the example below in Table 2.5 where men have weak preferences and women have strict preferences and a matching that is not strongly stable is denoted with *. The matching is not strongly stable as it contains the underlined blocking pair (m_4, w_3). w_3 strictly prefers m_1 to its current match of m_2, and m_4 is indifferent between w_3 and his current match of w_1.

m_1:	$(w_4\ w_1)$	w_2^*	w_3	w_1:	m_2	m_3	m_1	m_4^*
m_2:	$(w_4\ w_3^*)$	w_1	w_2	w_2:	m_4	m_2	m_1^*	m_3
m_3:	$(w_2\ w_1)$	w_4^*	w_3	w_3:	m_1	m_3	$\underline{m_4}$	m_2^*
m_4:	$(\underline{w_3}\ w_1^*)$	w_4	w_2	w_4:	m_3^*	m_2	m_4	m_1

Table 2.5: The table above shows a matching, denoted with *, for the given preference lists that contains a blocking pair (m_4, w_3) to a strongly stable matching. The blocking pair is underlined.

A matching is super-stable if there exists no couple (m, w) such that m strictly prefers w to his current partner or is indifferent between them $w \succeq_m \mu(m)$ and w strictly prefers m to its current partner or is indifferent between them $m \succeq_w \mu(w)$ with respect to $\succeq$. The definition of a blocking pair is analogous to that of weakly stable. Consider the example below in Table 2.6 where both men and women have weak preferences and a matching that is not super stable is denoted with *. The matching is unstable as it contains the underlined blocking pair (m_4, w_4). m_4 is indifferent between w_4 and his current match of w_2, and w_4 is indifferent between m_4 and its current match of m_1.

m_1:	w_4^*	w_1	$(w_2\ w_3)$	w_1:	m_2	m_3^*	$(m_1\ m_4)$
m_2:	w_4	w_3^*	$(w_1\ w_2)$	w_2:	m_4^*	m_2	$(m_1\ m_3)$
m_3:	w_2	w_1^*	$(w_4\ w_3)$	w_3:	m_1	m_3	$(m_4\ m_2^*)$
m_4:	w_3	w_1	$(\underline{w_4}\ w_2^*)$	w_4:	m_3	m_2	$(\underline{m_4}\ m_1^*)$

Table 2.6: The table above shows a matching, denoted with *, for the given preference lists that contains a blocking pair (m_4, w_4) to a super stable matching. The blocking pair is underlined.

All super-stable matchings are strongly stable and weakly stable. All strongly stable matchings are weakly stable, but not necessarily super stable. A weakly stable matching is not guaranteed to be super stable or strongly stable. As shown in Figure 2.1, the set of all super-stable matchings are a subset of strongly-stable matchings and weakly stable matchings. Similarly, the set of all strongly stable matchings are a subset of weakly stable matchings.

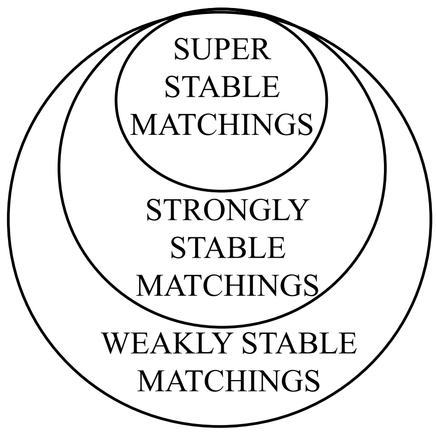


Figure 2.1: The figure above shows that the set of all super stable matchings belong to a subset of strongly stable matchings and weakly stable matchings and the set of all strongly stable matchings belong to a subset of weakly stable matchings.

2.4 *The Deferred Acceptance Algorithm*

The question of how to compute a stable matching arises. The Deferred Acceptance Algorithm, proposed by Gale and Shapley (1962), takes a strict preference profile $\succ$ as an input and outputs a stable matching μ, which is denoted as $\mu = DA(\succ)$.

Theorem 1 (Gale and Shapley, 1962) Consider a preference profile $\succ$ and let $\mu = DA(\succ)$. Then, $\mu \in S_\mu$. For any $\mu' \in S_\succ$, $\mu(m) \succeq_m \mu'(m)$ for all $m \in M$ and $\mu'(w) \succeq_w \mu(w)$ for all $w \in W$.

The psuedocode for the Deferred Acceptance algorithm is given below in algorithm 1. This algorithm runs in $O(n^2)$.

Algorithm 1 Computing a Stable Matching Through the Deferred Acceptance Algorithm

Input: Preference Profile $\succ$
Output: Stable Matching μ

Initialize all $m \in M$ and all $w \in W$ to be free
while $\exists m \in M$ who is free and has not proposed to every w **do**
  $w :=$ first woman on m's list who m has not proposed to
  if w is free **then**
    m proposes and becomes engaged to w
  else
    w is engaged to m'
    if $m' \succ_w m$ **then**
      m remains free
    else
      break the engagement (m', w)
      assign m' to be free
      m proposes and becomes engaged to w
    end if
  end if
end while
return μ

The DA algorithm is run on the example below in Table 2.7.

m_1:	w_1	w_4	w_2^*	w_3	w_1:	m_2	m_3^*	m_1	m_4
m_2:	w_3	w_4^*	w_1	w_2	w_2:	m_4	m_2	m_1^*	m_3
m_3:	w_1^*	w_2	w_4	w_3	w_3:	m_1	m_3	m_4^*	m_2
m_4:	w_1	w_3^*	w_4	w_2	w_4:	m_3	m_2^*	m_4	m_1

Table 2.7: The table above shows the stable matching denoted with *, which was produced by the Deferred Acceptance algorithm for the strict preference profile shown above.

The steps of obtaining the matching using the Deferred Acceptance algorithm are highlighted below, where men are the proposing side:

1. m_1 proposes to his first choice w_1. Since w_1 is single, she accepts.

2. m_2 proposes to his first choice w_3. Since w_3 is single, she accepts.

3. m_3 proposes to his first choice w_1. w_1 is already engaged to m_1. Observing w_1's preference list, m_3 is w_1's second choice, while m_1 is its third choice. m_3 and w_1 are now engaged. m_1 is now single.

4. m_1 proposes to his second choice w_4. Since w_4 is single, she accepts.

5. m_4 proposes to his first choice w_1. w_1 is already engaged to m_3. Observing w_1's preference list, m_3 is w_1's second choice, while m_4 is its fourth choice. m_3 and w_1 remain engaged.

6. m_4 proposes to his second choice w_3. w_3 is already engaged to m_2. Observing w_3's preference list, m_2 is w_3's fourth choice, while m_4 is its third choice. m_4 and w_3 are now engaged. m_2 is now single.

7. m_2 proposes to his second choice w_4. w_4 is already engaged to m_1. Observing w_4's preference list, m_1 is w_4's fourth choice, while m_2 is its second choice. m_2 and w_4 are now engaged. m_1 is now single.

8. m_1 proposes to his third choice w_2. Since w_2 is single, she accepts. All men and all women are assigned a partner so the algorithm terminates.

The resulting matching is denoted with *.

2.5 *The STRONG Algorithm*

The STRONG algorithm computes a strongly-stable matching if one exists in polynomial time. This algorithm proposed by Irving (1989) takes a weak preference profile $\succeq$ as an input and outputs a strongly stable matching μ_{STRONG}, which is denoted as $\mu_{STRONG} = STRONG(\succeq)$. The psuedocode for the STRONG algorithm is given below in algorithm 2. This algorithm runs in $O(n^4)$.

Algorithm 2 Computing a Strongly Stable Matching Through the STRONG Algorithm

Input: Preference Profile $\succeq$
Output: Strongly Stable Matching μ_{STRONG}

```
Initialize all m ∈ M and all w ∈ W to be free
while ∃m ∈ M who is free and has not proposed to every w do
    for each w at the head of m's list do
        m proposes and becomes engaged to w
        for e doach (strict successor m' of m on w's list
            if m' is engaged to w then
                break the engagement
            end if
            delete the pair (m', w)
        end for
    end for
    if engagement does not contain a perfect matching then
        Find the critical set Z of men
        for each w engaged to a m in Z do
            for each m at the tail of w's list do
                delete the pair (m, w)
            end for
        end for
    end if
end while
if all m ∈ M and all w ∈ W are engaged in one-to-one matching then
    matching is super strongly stable matching and return μ_STRONG
else
    no strongly stable matching exists
end if
```

The STRONG algorithm is run on the example below in Table 2.8 where men have weak preferences and women have strict preferences.

m_1:	$(w_4\ w_1)$	w_2^*	w_3	w_1:	m_2	m_3^*	m_1	m_4
m_2:	$(w_4^*\ w_3)$	w_1	w_2	w_2:	m_4	m_2	m_1^*	m_3
m_3:	$(w_2\ w_1^*)$	w_4	w_3	w_3:	m_1	m_3	m_4^*	m_2
m_4:	$(w_3^*\ w_1)$	w_4	w_2	w_4:	m_3	m_2^*	m_4	m_1

Table 2.8: The table above shows the strongly stable matching denoted with *, which was produced by the STRONG algorithm for the preference profile shown above.

The steps of obtaining the strongly stable matching using the STRONG algorithm are highlighted below, where men are the proposing side:

1. m_1 simultaneously proposes to w_4 and w_1. The engaged couples are $[(m_1, w_4), (m_1, w_1)]$.

Both w_4 and w_1 delete the men on their preference list that succeed m_1, so w_1's preference list becomes $[m_2 \succ m_3 \succ m_1]$ and w_4's preference list remains the same, as m_1 is its least preferred man.

2. m_2 simultaneously proposes to w_4 and w_3. w_3 accepts, as she is unmatched. Both w_4 and w_3 delete the men on their preference list that succeed m_2, so w_4's preference list becomes $[m_3 \succ m_2]$. Break the engagement between m_1 and w_4. w_3's preference list remains the same, as m_2 is its least preferred man. The engaged couples are now $[(m_1, w_1), (m_2, w_3), (m_2, w_4)]$.

3. m_3 simultaneously proposes to w_2 and w_1. w_2 accepts, as she is unmatched. Both w_2 and w_1 delete the men on their preference list that succeed m_3, so w_1's preference list becomes $[m_2 \succ m_3]$. Break the engagement between m_1 and w_1. w_2's preference list remains the same, as m_3 is its least preferred man. w_1 has already been proposed to. The engaged couples are now $[(m_2, w_3), (m_2, w_4), (m_3, w_2), (m_3, w_1)]$.

4. m_1 is now single again. He proposes to his second choice w_2. w_2 deletes all men on its preference list that succeed m_1. Break the engagement between m_3 and w_2. The engaged couples are now $[(m_1, w_2), (m_2, w_3), (m_2, w_4), (m_3, w_1)]$.

5. m_4 simultaneously proposes to w_3 and w_1. w_1 has already deleted m_4 from its preference list. Hence, there is no engagement between m_4 and w_1. w_3 deletes all men on its preference list that succeed m_4. w_3's preference list becomes $[m_1 \succ m_3 \succ m_4]$. Break the engagement between m_2 and w_3. The engaged couples are now $[(m_1, w_2), (m_2, w_4), (m_3, w_1), (m_4, w_3)]$. Since all men and women are engaged one-to-one, there does exist a strongly stable matching and the algorithm terminates.

The resulting matching is denoted with * in the table above.

2.6 *The SUPER Algorithm*

The SUPER algorithm computes a super-stable matching if one exists in polynomial time. This algorithm proposed by Irving (1989) takes a weak preference profile $\succeq$ as an input and outputs a super stable matching μ_{SUPER}, which is denoted as $\mu_{SUPER} = SUPER(\succeq)$. It is important to note that a super stable matching will not always exist, and the algorithm will only return a strongly stable matching if one exists. The psuedocode for the SUPER algorithm is given below in algorithm 3. This algorithm runs in $O(n^2)$.

Algorithm 3 Computing a Super Stable Matching Through the SUPER Algorithm

Input: Preference Profile $\succeq$
Output: Super Stable Matching μ_{SUPER}

Initialize all $m \in M$ and all $w \in W$ to be free
while $\exists m \in M$ who is free and has not proposed to every w **do**
    for each w at the head of m's list **do**
        for e **do**ach (strict successor m' of m on w's list
            if m' is engaged to w **then**
                break the engagement
            end if
            delete the pair (m', w)
        end for
    end for
    for each w who is multiply engaged **do**
        for each m at the tail of w's list **do**
            delete the pair (m, w)
        end for
    end for
end while
if all $m \in M$ and all $w \in W$ are engaged in one-to-one matching **then**
    matching is super strongly stable matching and **return** μ_{STRONG}
else
    no strongly stable matching exists
end if

The SUPER algorithm is run on the example below in Table 2.9 where both men and women have weak preferences.

m_1:	w_1	w_3^*	w_4	$(w_2\ w_5)$	w_1:	m_5	m_2^*	m_1	$(m_4$	$m_3)$
m_2:	w_1^*	w_2	w_4	$(w_5\ w_3)$	w_2:	m_5	m_1	m_2	m_4^*	m_3
m_3:	w_1	w_2	w_5^*	$(w_3\ w_4)$	w_3:	m_1^*	m_3	m_4	$(m_2$	$m_5)$
m_4:	w_3	w_2^*	w_4	$(w_1\ w_5)$	w_4:	m_3	m_2	m_1	m_4	m_5^*
m_5:	w_4^*	w_1	w_5	$(w_2\ w_3)$	w_5:	m_3^*	m_2	m_5	$(m_2$	$m_1)$

Table 2.9: The table above shows the super stable matching denoted with *, which was produced by the SUPER algorithm for the preference profile shown above.

The steps of obtaining the super stable matching using the SUPER algorithm are highlighted below, where men are the proposing side:

1. m_1 proposes to w_1. Since w_1 is unmatched, she accepts. w_1 deletes all men who succeed m_1 from its preference list. w_1's preference list becomes $[m_5 \succ m_2 \succ m_1]$. The engaged couples are now $[(m_1, w_1)]$.

2. m_2 proposes to w_1. w_1 deletes all men who succeed m_2 on its preference list. w_1's preference list becomes $[m_5 \succ m_2]$. Break the engagement between m_1 and w_1. The engaged couples are now $[(m_2, w_1)]$.

3. m_1 is now single. He proposes to his second choice w_3. Since w_3 is unmatched, she accepts. w_3 deletes all men who succeed m_1 on its preference list. its preference list becomes $[m_1]$The engaged couples are now $[(m_1, w_3), (m_2, w_1)]$.

4. m_3 proposes to w_1. w_1. w_1 has already deleted m_3 from its preference list. Hence, there is no engagement between m_3 and w_1. The engaged couples remain the same $[(m_1, w_3), (m_2, w_1)]$.

5. Since m_3 is still single, he proposes to his second choice w_2. w_2 has already deleted m_3 from its preference list. Hence, there is no engagement between m_3 and w_2. The engaged couples remain the same $[(m_1, w_3), (m_2, w_1)]$.

6. Since m_3 is still single, he proposes to his third choice w_5. Since w_5 is unmatched, she accepts. w_5 deletes all the men who succeed m_3 on its preference list. its preference list becomes $[m_1 \succ m_3]$. The engaged couples are now $[(m_1, w_3), (m_2, w_1), (m_3, w_5)]$.

7. m_4 proposes to his first choice w_3. w_3 has already deleted m_4 from its preference list. Hence, there is no engagement between m_3 and w_1. The engaged couples remain the same $[(m_1, w_3), (m_2, w_1), (m_3, w_5)]$.

8. m_4 proposes to his second choice w_2. Since w_2 is unmatched, she accepts. w_2 deletes the men who succeed m_4 on its preference list. w_2's preference list becomes $[m_5 \succ m_1 \succ m_2 \succ m_4]$. The engaged couples are now $[(m_1, w_3), (m_2, w_1), (m_3, w_5), (m_4, w_2)]$.

9. m_5 proposes to his first choice w_4. Since w_4 is unmatched, she accepts. w_4 deletes the men who succeed m_5 so w_4's preference list stays the same since m_5 is at the tail of its preference list. The engaged couples are now $[(m_1, w_3), (m_2, w_1), (m_3, w_5), (m_4, w_2), (m_5, w4)]$. Since all men and women are engaged one-to-one, there does exist a super stable matching and the algorithm terminates.

The resulting matching is denoted with *.

2.7 *Manipulation by Permutation*

As discussed previously, the DA algorithm is not strategy-proof for women, signifying they can misreport their preferences to get matched to a partner they prefer to their true partner (Dubins and Freedman, 1981; Roth, 1982). Since this thesis looks at the DA algorithm with men proposing, the DA algorithm makes truth-telling a dominant strategy for the proposing side (Dubins and Freedman, 1981; Roth, 1982). Consider strict preference profiles $\succ$ and $\succ'$ where they differ only on the preferences of w and the matchings obtained are $\mu = DA(\succ)$ and $\mu' = DA(\succ')$. This matching is considered to manipulable if $\mu'(w) \succeq_w \mu(w)$, where w is referred to as the manipulator. For instance, a woman w's true preference is $[m_2 \succ m_3 \succ m_1]$ but she reports a permutation of this: $[m_2 \succ m_1 \succ m_3]$. This example is manipulable if w is matched with its third true choice m_1 when she is truthful but matched with its first true choice m_2 when she reports the permutated list.

An example of manipulation in a setting with strict preferences is shown below in Table 2.10. The true matching is denoted by *. Under the true preferences, w_1 is matched with its second choice m_1. However, w_1 can do better by misreporting its preferences such that $[m_3 \succ m_2 \succ m_1]$, she is matched with its first choice m_3. The manipulator w_1 is able to strictly improve as a result of this manipulation, while also also making w_2 better off. The matching obtained as a result of this manipulation is denoted with an underline.

m_1:	w_1^*	$\underline{w_2}$	w_3	w_1:	$\underline{m_3}$	m_1^*	m_2
m_2:	w_1	$\underline{w_3^*}$	w_2	w_2:	m_2	$\underline{m_1}$	m_3^*
m_3:	w_2^*	$\underline{w_1}$	w_3	w_3:	m_3	m_1	$\underline{m_2^*}$

Table 2.10: Manipulation by permutation in the strict setting where w_1 is the manipulator, the original matching is denoted with *, and the manipulated matching is denoted with an underline.

2.8 *Optimal Manipulation*

Manipulation by permutation is possible, as shown in the previous section. Consider a preference profile $\succeq$. An optimal manipulation of the DA algorithm by a manipulator w with a preference profile $\succeq_w$ refers to a modified preference list of the manipulator that is made strict using a tie-breaking rule $\succ'_w$, where $\mu'(w) \succ_w \mu(w)$ and $\mu'(w) \succeq_w \mu''(w)$ for any other preference list $\succeq''$ and all possible tie-breaking rules, where $\mu' = DA(\succ_{-w}, \succ'_w)$ and $\mu'' = DA(\succ_{-w}, \succ''_w)$. The psuedocode for computing the optimal manipulation strategy algorithm is given below. The algorithm is split into two parts: finding an optimal partner given in algorithm 4 and permutating the preference list to be matched with the optimal partner in algorithm 5.

Algorithm 4 Computing an Optimal Partner

Input: Preference Profile $\succ$, and a manipulator w
Output: Women Optimal Partner m'

```
run DA(≻), except reject all proposals made to w. w and some man m stay single at the end
among all the men who proposed to w, take note of the best man m′ with respect to ≻_w
return m′
```

Algorithm 5 Permutating the Preference List to be Matched with the Optimal Partner

Input: Preference Profile $\succ$, and a manipulator w
Output: Women Optimal Partner m'

$DA(\succ_w, \succ_{-w})$ to get the men-optimal partner for w, m
w moves a man, m_j, who proposed to its to the front of its list $\succ'_w$ to get matched to m_j under the DA algorithm
repeat until all potential partners of w are exhausted, and N is the set of all potential partners for w
the optimal manipulation strategy is when w moves most preferred man in N to the top of its list
return $\succ'_w$

The optimal manipulation algorithm is run on the example below in Table 2.11 where both men and women have strict preferences.

m_1:	$\underline{w_2^*}$	w_3	w_4	w_5	w_1	w_1:	m_1	m_2	$\underline{m_3}$	m_5^*	m_4
m_2:	$\underline{w_3^*}$	w_4	w_5	w_1	w_2	w_2:	m_2	$\underline{m_1^*}$	m_4	m_5	m_3
m_3:	$\underline{w_5^*}$	$\underline{w_1}$	w_4	w_2	w_3	w_3:	m_3	$\underline{m_2^*}$	m_5	m_1	m_4
m_4:	w_3	w_1	w_2	$\underline{w_4^*}$	w_5	w_4:	$\underline{m_4^*}$	m_5	m_1	m_2	m_3
m_5:	w_1^*	$\underline{w_5}$	w_2	w_3	w_4	w_5:	$\underline{m_5}$	m_1	m_2	m_3^*	m_4

Table 2.11: The table above shows the matching when w_1 misreports using its optimal manipulation, which is denoted with an underline, compared to the DA matching denoted with *.

The steps of obtaining the optimal manipulation using the algorithms above are highlighted below, where men are the proposing side and the manipulator w is w_1:

1. Run the DA algorithm with the true preference list for w_1, $\succ_w$. its men optimal partner is m_5, and m_4 is the only other man who proposes to her. To get matched with m_5, $\succ_w := [m_1 \succ m_2 \succ m_3 \succ m_5 \succ m_4]$.

2. m_4 is moved to the head of w_1's preference list, $\succ'_w := [m_4 \succ m_1 \succ m_2 \succ m_3 \succ m_5]$. m_5 is exhausted, and m_4 is a potential partner.

3. m_4 is not exhausted, the DA algorithm is run with the modified preference list from the previous step $DA(\succ'_w, \succ'_{-w})$. m_3 rises as a new potential partner and m_3 is moved to the head of w_1's preference list, $\succ'_w := [m_3 \succ m_4 \succ m_1 \succ m_2 \succ m_5]$. m_4 is exhausted after this.

4. The algorithm is run with the new preference list for w_1 as m_3 is not exhausted yet and no new potential partner is found. The algorithm terminates.

The resulting matching is denoted with an underline, while the original DA matching is denoted with *.

Chapter 3

Literature Review of Related Work

This section looks at existing literature that has already looked at either strategic manipulation or weak preferences.

The earliest and most extensive studies of manipulation on the DA algorithm focus on using manipulation by truncation, where a manipulator can misreport preferences by shortening preference lists (Gale and Sotomayor, 1985; Roth and Rothblum, 1999; Coles and Shorrer, 2014; Jaramillo et al., 2014). Manipulation by permutation is another type of commonly studied manipulation (Teo et al., 2001; Vaish and Garg, 2017). Teo et al. (2001) presents a polynomial algorithm for computing the optimal permutation manipulation strategy for a woman. Vaish and Garg (2017) shows that the optimal strategy is inconspicuous. Some more recent studies look at manipulation through an accomplice, where an agent on the proposing side works with a manipulator, who is an agent on the proposed to side, to make the manipulator strictly better off while potentially making the accomplice strictly worse off (Bendlin and Hosseini, 2019; Hosseini et al., 2021, 2020). Bendlin and Hosseini (2019) looked at how accomplice manipulation can be more advantageous than the optimal self-manipulation. Hosseini et al. (2020) establishes an algorithm to compute the optimal accomplice manipulation strategy in polynomial time. Hosseini et al. (2021) further extends the accomplice manipulation by developing polynomial algorithms for a two for one model, where an accomplice and manipulator work together to benefit the manipulator and a one for all model, where one accomplice on the proposing side misreports to benefit all agents on the proposed to side.

There is also literature that studies tie-breaking. P.A. and J. (2011) studied STB and MTB are equivalent under the model of matching students to schools where students have strict preferences and schools have ties (P.A. and J., 2011). Ashlagi et al. (2019) showed that using STB in the context of school matching allows for a constant fraction of students to be matched with their top choice, while MTB allows for a vanishing fraction of students to be matched with their top choice (Ashlagi et al., 2019). Ashlagi et al. (2019) also proved that truncation manipulation with MTB allows for students to get as desirable of a school as under STB. Ashlagi and Nikzad (2020) proposed that schools should use a STB to resolve ties when the market is partitioned into popular and non-popular schools because of the stochastic dominance of the rank distribution. Weak preferences introduce new notions of stability: weakly stable, strongly stable, and super stable (Irving, 1989). Irving (1989) has developed polynomial algorithms to find a super-stable matching and a strongly stable matching. Manlove (2002) discovered that strongly stable matchings form a distributive lattice (Manlove, 2002).

Chapter 4

Generating Weak Preferences

This chapter investigates different methods of generating weak preferences and determines which method is the best way of doing so to ensure preferences are still being generated uniformly at random. When preferences are strict, generating preferences is straightforward: generate a list of agents uniformly at random for each man and each woman. Generating weak preferences uniformly at random is not as straightforward, as the partition and the ordering of agents needs to be uniformly at random.

4.1 *Experimental Design*

In order to determine the most effective way of generating weak preferences, this thesis looks at three methods of doing so: Equal buckets, Unequal buckets, and Stirling number buckets (Stam, 1982), where a bucket refers to the class of all the agents that an agent on the opposing side is indifferent to. Equal buckets generate strict preferences uniformly at random by randomly selecting a number k such that $1 \leq k \leq n$, which dictates how many agents will be in each bucket. Unequal buckets generate strict preferences uniformly at random and randomly selects a number k such that $1 \leq k \leq n$, where k is the number of agents placed in the first bucket. For the next bucket, a new value k' is selected such that $1 \leq k' \leq n-k$ where k' is the number of agents placed in the next bucket. A new value k is selected until the remaining agents are placed into a bucket. Stirling number buckets uses Stirling numbers to generate weak preferences uniformly at random.

Stirling number buckets use a bell number to randomly generate partitions for the preferences. Each of these weak preference cases results are compared against the strict setting.

In this experiment, weak preferences are made strict using a single tie-breaking rule that is generated uniformly at random. Strict preferences are passed into the DA algorithm to get the true matching. The strict preferences are passed into the manipulation algorithm to obtain the optimal manipulation strategy. The manipulator misreports its preferences using the optimal manipulation strategy. These misreported strict preferences are passed into the DA algorithm to get the manipulated matching.

4.2 *Experimental Results*

This experiment was conducted for 10,000 iterations for 4 to 20 agents on each side. For this experiment, the frequency of manipulation and the manipulator strict improvement were plotted.

Stirling Number Buckets have the highest frequency out of the weak preferences in which self manipulation is possible. In Figure 4.1, weak preferences generated using Stirling numbers is significantly more frequent than that of Equal buckets or Unequal buckets, and Stirling number buckets are closest to the strict preferences.

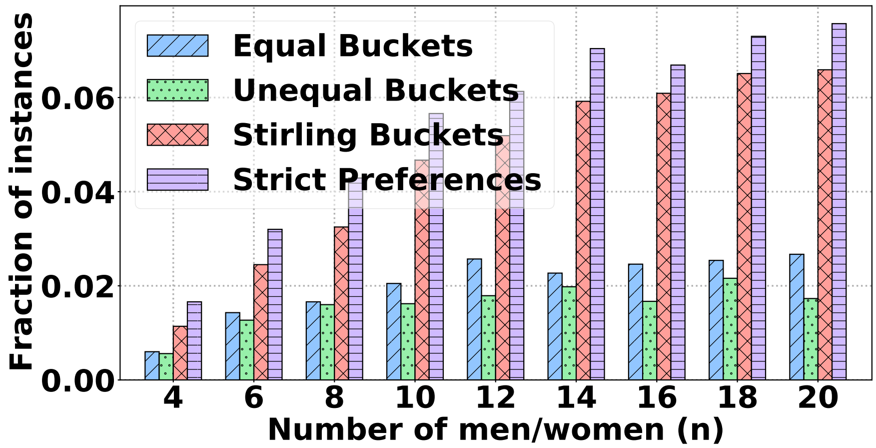


Figure 4.1: The figure above shows the fraction of instances that the woman manipulator can manipulate when weak preferences are generated using equal sized buckets, unequal sized buckets, Stirling number buckets and for when preferences are strict. All ties in weak preferences are broken using a single tie-breaking rule.

Not only do Stirling number buckets have the highest frequency in which a manipulator can successfully manipulate, but Stirling numbers also have the highest strict improvement. In Figure 4.2, Stirling number buckets always have a higher strict bucket improvement than that of unequal buckets. The graph also shows that Stirling number buckets have the same or higher strict bucket improvement as Equal buckets.

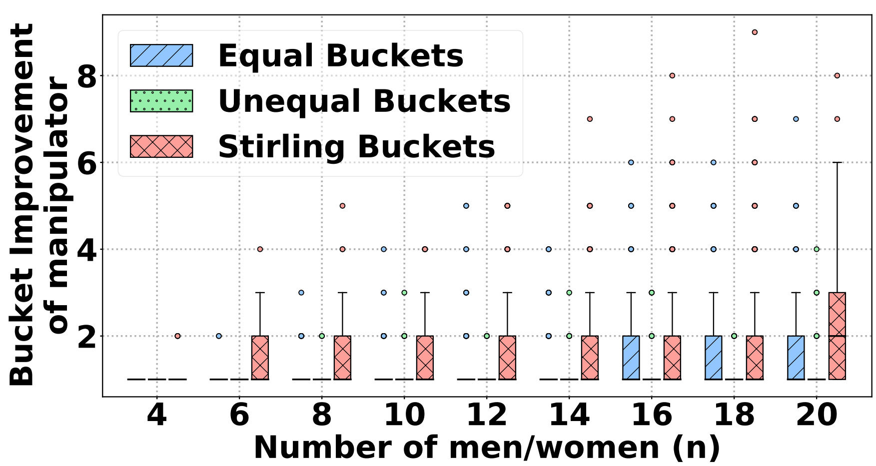


Figure 4.2: The figure above shows the strict bucket improvement of the new partner of the manipulator when weak preferences are generated using equal sized buckets, unequal sized buckets, and Stirling number buckets. All ties in weak preferences are broken using a single tie-breaking rule.

4.3 *Discussion*

The results indicate that Stirling number buckets generate weak preferences uniformly at random unlike equal buckets and unequal buckets. The following experiments use Stirling number buckets since they have the highest frequency, highest bucket improvement, and they are the best approximation to uniformly generated preferences.

Chapter 5

Impact of Tie-breaking

This section looks at the impact of tie-breaking using STB on a fixed man and fixed woman and the impact of tie-breaking on all men and all women through an example and through a series of experiments.

5.1 *Examples of Single Tie-Breaking*

When there ties are present in the preference list, the ordering of the tie-breaking rule is used to resolve any ties and make the preference list strict. Consider the preference profiles with weak preferences for both men and women in Table 5.1.

m_1:	$(\underline{w_1^*}\ w_6)$	$(w_3\ w_5)$	$(w_4\ w_2)$	w_1:	$(m_3\ m_6)$	$(\underline{m_1^*}\ m_4)$	$(m_5\ m_2)$
m_2:	$(w_6^*\ \underline{w_3})$	$(w_4\ w_2)$	$(w_5\ w_1)$	w_2:	$(m_1\ m_6)$	$(\underline{m_5}\ m_2)$	$(m_4^*\ m_3)$
m_3:	$(w_3^*\ \underline{w_5})$	$(w_1\ w_4)$	$(w_2\ w_6)$	w_3:	$(m_4\ m_3^*)$	$(m_5\ m_6)$	$(m_1\ \underline{m_2})$
m_4:	$(w_5\ \underline{w_6})$	$(w_2^*\ w_1)$	$(w_4\ w_3)$	w_4:	$(m_4\ m_5)$	$(\underline{m_6^*}\ m_1)$	$(m_3\ m_2)$
m_5:	$(w_1\ w_6)$	$(w_5^*\ \underline{w_2})$	$(w_4\ w_3)$	w_5:	$(m_2\ m_5^*)$	$(\underline{m_3}\ m_6)$	$(m_1\ m_4)$
m_6:	$(w_3\ \underline{w_4^*})$	$(w_6\ w_2)$	$(w_5\ w_1)$	w_6:	$(m_2^*\ \underline{m_4})$	$(m_3\ m_5)$	$(m_1\ m_6)$

Table 5.1: The table above shows how different tie-breaking rules ($[m_1 \succ m_5 \succ m_3 \succ m_2 \succ m_6 \succ m_4]$ denoted with * and $[m_4 \succ m_2 \succ m_6 \succ m_5 \succ m_3 \succ m_1]$ denoted with an underline) can affect the matching under the deferred acceptance algorithm when both men and women have weak preferences.

Consider two tie-breaking rules: rule TB1 is $[m_1 \succ m_5 \succ m_3 \succ m_2 \succ m_6 \succ m_4]$, where the matching corresponding to this tie-breaking rule is denoted by a * and rule TB2 is $[m_4 \succ m_2 \succ m_6 \succ m_5 \succ m_3 \succ m_1]$, where the matching is denoted with an underline. As a result of using TB1 over TB2: m_4 and w_2 are strictly worse off with both having a bucket difference of 1 ; w_3 and w_5 are strictly better off with bucket improvements of 2 and 1; m_2, m_3, m_5, and w_6 are weakly better off, as each of their new partner under TB2 is in the same bucket as their previous partner under TB1.

5.2 *Impact of Tie-breaking on One Man, One Woman*

This section looks at the impact of a tie-breaking rule on a fixed man and a fixed woman. The tie-breaking rule is critical for agents who could get strictly better off or strictly worse off through just the tie-breaking rule.

5.2.1 *Experimental Design*

There are $n!$ ways of breaking ties. The first tie-breaking rule is used to initially break all the ties in preferences and compute the bucket of the partner of all women and all men. The rest of the tie-breaking rules are iterated through to make preferences strict and compute the the new matching. With respect to the first tie-breaking rule, the improvement of a fixed man and a fixed woman is recorded.

For the impact of a tie-breaking rule on a fixed man and a fixed woman, these experiments look at three cases with weak preferences: men preferences contain ties, women preferences are strict; men preferences are strict, women preferences contain ties; both men and women preferences contain ties.

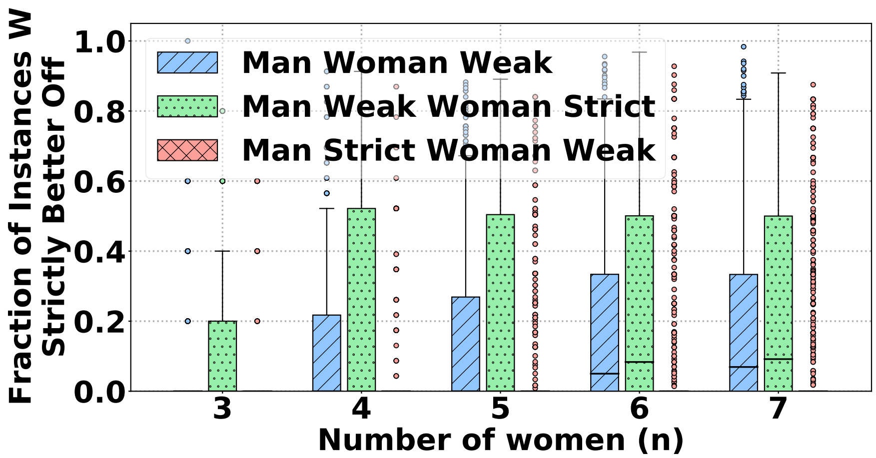
5.2.2 *Experimental Results*

This experiment was conducted for 2,500 iterations for 3 to 7 agents on each side. [1]

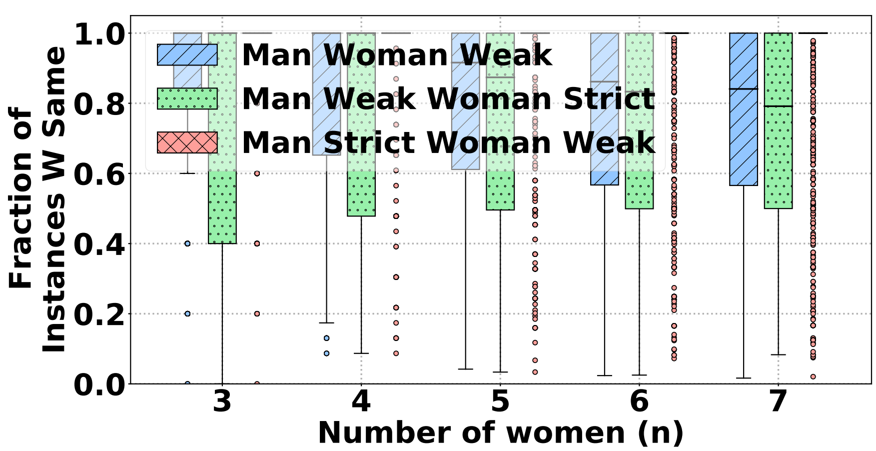
When man preferences contain ties, a fixed woman is able to strictly improve or stay in the same bucket as a result of tie-breaking rules. In Figure 5.1, when men preferences contain ties and woman preferences are strict, the fraction of instances that the fixed woman can strictly

[1] Because this experiment iterates through all possible tie-breaking rules, the experiment cannot be conducted for larger scale settings, say with 100 agents.

improve are highest. When woman preferences are strict, it is easier for the fixed woman to strictly improve, as the fixed woman only has to improve its partner by one in its preference list to be better off, whereas with weak preferences the fixed woman has to improve its partner by the number of agents that she is indifferent to in the bucket of its preference list to be better off. When man preferences are strict and woman preferences contain ties, the fixed woman stays the same for all instances, excluding the outlying data points. This suggests that a fixed woman has the highest potential to get better off when man preferences contain ties, especially when woman preferences are strict.



(a) Fraction of Instances Fixed W Strictly Improves for TB Rules



(b) Fraction of Instances Fixed W Stays Same for TB Rules

Figure 5.1: The graphs above show the fraction of instances a fixed woman is weakly better off as a result of all tie-breaking rules for when both sides of preferences are weak, men preferences contain ties and women preferences are strict, and men preferences are strict and women preferences contain ties.

Excluding the outlying data points, a fixed woman is never strictly worse off as a result of the tie-breaking rule regardless if man or woman preferences are strict or if they are both weak. As shown in Figure 5.2, the fraction of instances that the fixed woman is worse off is always zero regardless of the total number of women.

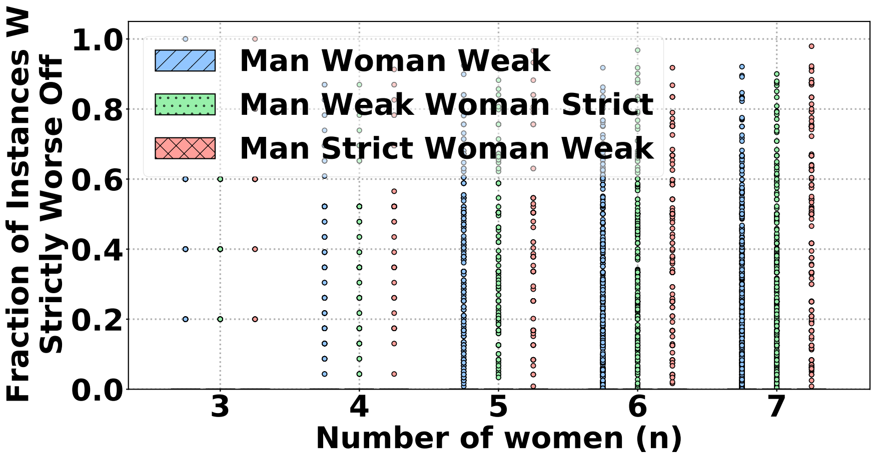
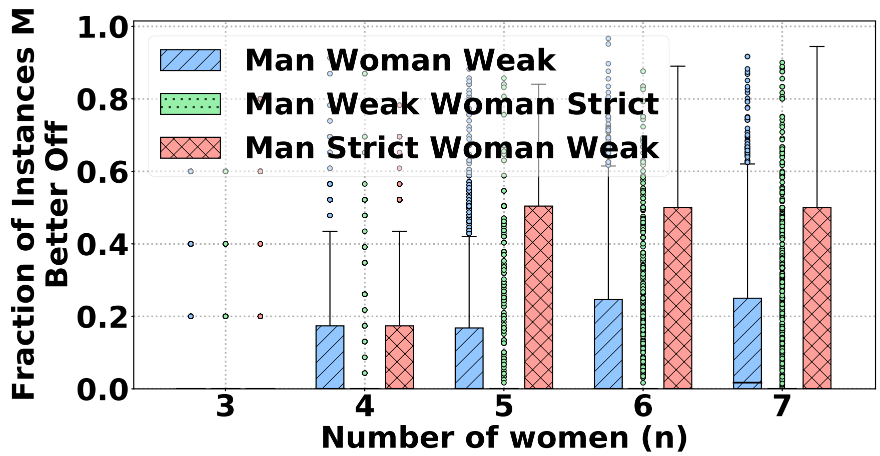
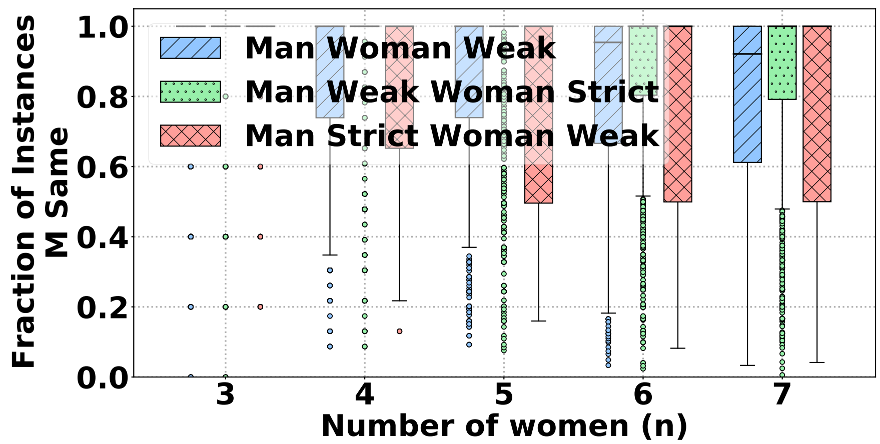


Figure 5.2: The graph above shows the fraction of instances a fixed woman is strictly worse off as a result of all tie-breaking rules for when both sides of preferences are weak, men preferences contain ties and women preferences are strict, and men preferences are strict and women preferences contain ties.

When woman preferences contain ties, a fixed man is able to strictly improve or stay in the same bucket. In Figure 5.3, when man preferences are strict and woman preferences contain ties, the fraction of instances that the fixed man can strictly improve are highest. When man preferences are strict, it is easier for the fixed man to strictly improve, as the fixed man only has to improve its partner by one in its preference list to be better off, whereas with weak preferences the fixed man has to improve its partner by the number of agents that he is indifferent to in the bucket of its preference list to be better off. When man preferences contain ties and woman preferences are strict, the fixed man mostly stays the same, but not always due to some outlying data that suggests the fixed man gets better off infrequently. This suggests that a fixed man has the highest potential to get better off when woman preferences contain ties, especially when man preferences are strict.



(a) Fraction of Instances Fixed M Strictly Improves for TB Rules



(b) Fraction of Instances Fixed M Stays Same for TB Rules

Figure 5.3: The graphs above show the fraction of instances a fixed man is weakly better off as a result of all tie-breaking rules for when both sides of preferences are weak, men preferences contain ties and women preferences are strict, and men preferences are strict and women preferences contain ties.

Excluding the outlying data points, a fixed man is never strictly worse off as a result of the tie-breaking rule regardless if man or woman preferences are strict or if they are both weak. As shown in Figure 5.4, the fraction of instances that the fixed woman is worse off is always zero regardless of the total number of women.

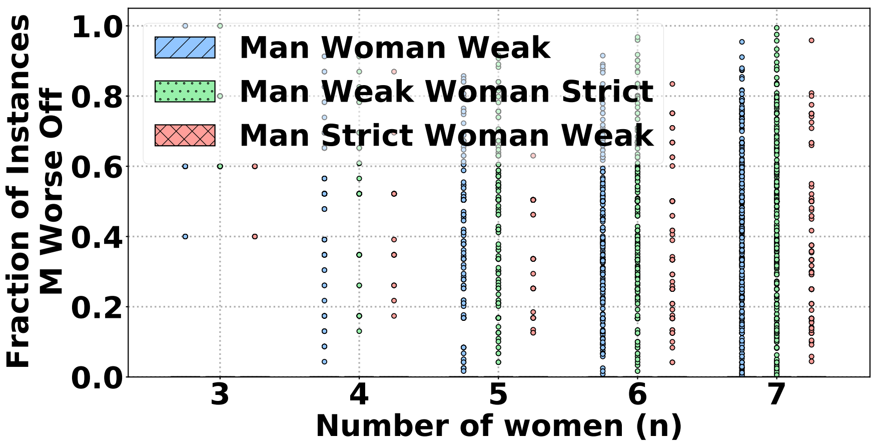


Figure 5.4: The graph above shows the fraction of instances a fixed man is strictly worse off as a result of all tie-breaking rules for when both sides of preferences are weak, men preferences contain ties and women preferences are strict, and men preferences are strict and women preferences contain ties.

5.2.3 *Discussion*

The fraction of instances where a fixed woman is weakly better off is higher when men preferences contain ties and vice versa for a fixed man. When the opposing side of the fixed agent has weak preferences, the fixed agent has a higher potential to improve, as the tie breaking rule has more of an impact on how the opposing side would behave. For example, when man preferences contain ties, the tie-breaking rule affects the proposal sequence of men. This allows a fixed woman to receive proposals from a man she would not have otherwise, potentially making its better off. On the other hand, when woman preferences contain ties, the tie-breaking rule has more of an impact of women rejecting or accepting proposals from men, which could allow a woman to accept a proposal from a man who she would not have otherwise. A set of preferences such that one side has strict preferences and the opposing side has weak preferences will always yield a better outcome for the strict side's fixed agent, as it is easier for the agent to strictly improve. These results indicate that tie-breaking alone can change the matching, making a fixed agent stay the same, or strictly better off with the exception of outlying data points. The overall findings for this experiment are:

- Ties on the proposing side are beneficial to the woman, whereas ties on the proposed-to side are not

- Ties on the proposed-to side are beneficial to the man, whereas ties on the proposing side are not

5.3 *Impact of Tie-breaking on All Men, All Women*

This section looks at the impact of a tie-breaking rule on all men and all woman. The tie-breaking rule is critical in seeing the holistic impact that it can have on the matching of all men and all women.

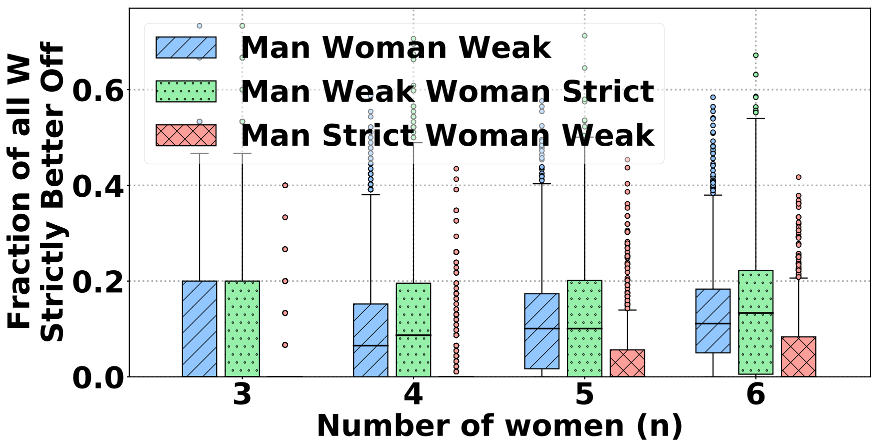
5.3.1 *Experimental Design*

In this experiment, I repeat the previous experiment, except keep track of the strict improvement for all women and all men.

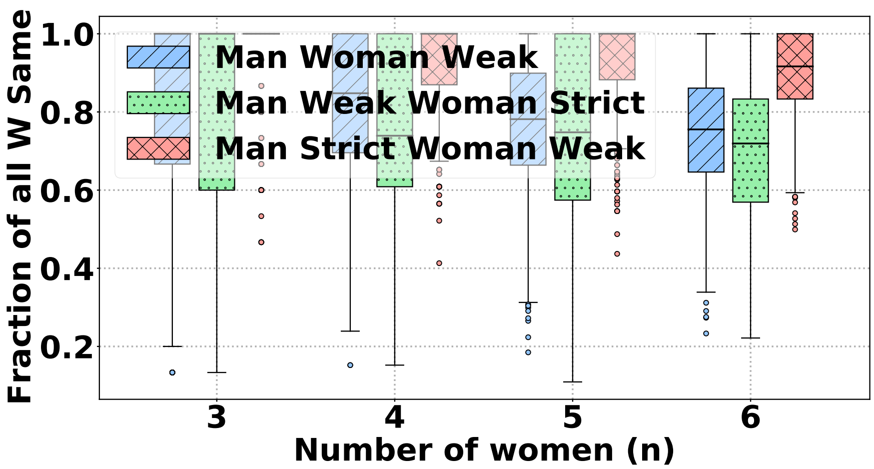
5.3.2 *Experimental Results*

This experiment was conducted for 2,500 iterations for 3 to 7 agents on each side.

Consistent with the last experiment, the highest fraction of all women are able to get strictly better off when man preferences contain ties because the tie-breaking rule impacts the proposal sequence of men. Unlike the previous experiments, some of the women get strictly better off when man preferences are strict and women preferences contain ties as shown in Figure 5.5.



(a) Fraction of Instances All W Strictly Improve for TB Rules



(b) Fraction of Instances All W Stay Same for TB Rules

Figure 5.5: The graphs above show the fraction of instances all women are weakly better off as a result of all tie-breaking rules for when both sides of preferences are weak, men preferences contain ties and women preferences are strict, and men preferences are strict and women preferences contain ties.

When the experiments look at all women instead of a fixed woman, some women get strictly worse off as a result of the tie-breaking rule, which is lowest when man preferences are strict and woman preferences contain ties. This is shown in Figure 5.6.

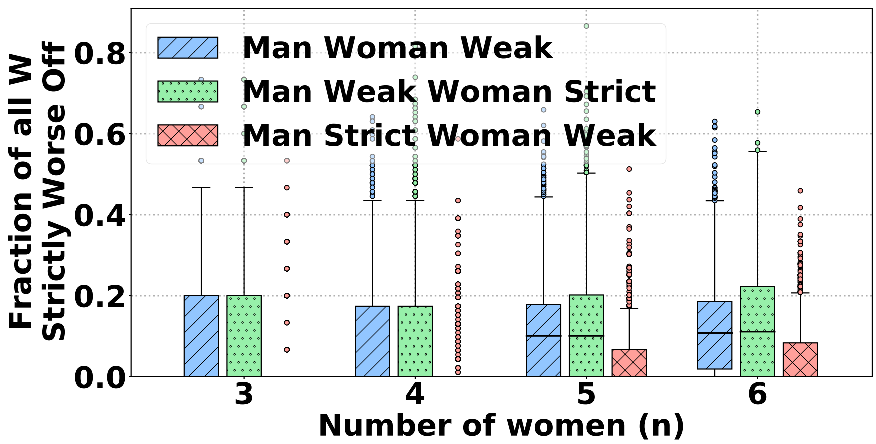
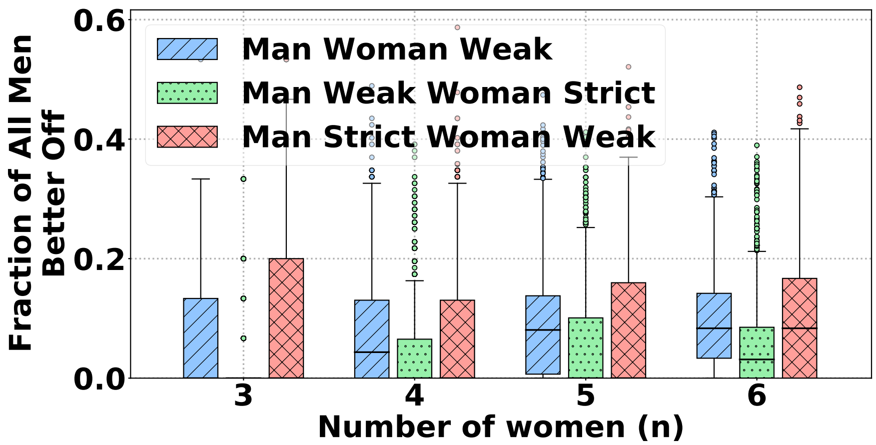
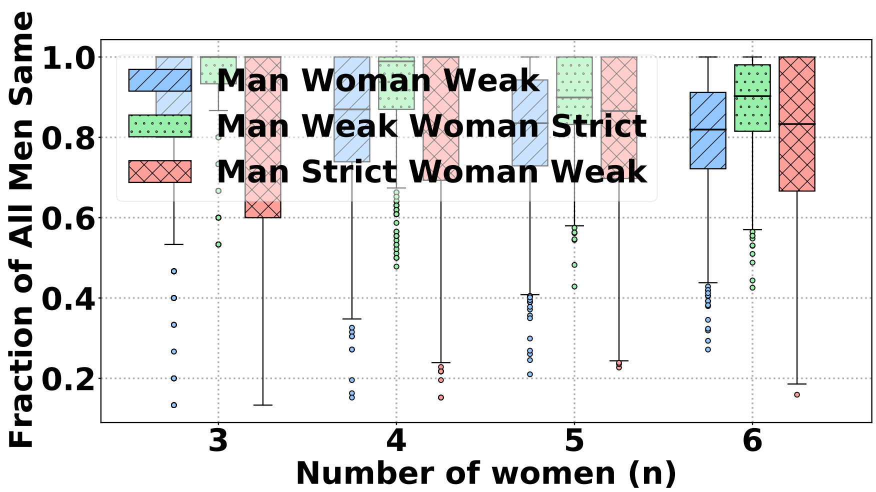


Figure 5.6: The graph above shows the fraction of instances all women are strictly worse off as a result of all tie-breaking rules for when both sides of preferences are weak, men preferences contain ties and women preferences are strict, and men preferences are strict and women preferences contain ties.

Consistent with the last experiment, the highest fraction of all men are able to get strictly better off when woman preferences contain ties because the tie-breaking rule impacts whether women will accept the proposal from a man. Unlike the previous experiments, some of the men get strictly better off when man preferences contain ties and women preferences are strict as shown in Figure 5.7.



(a) Fraction of Instances All M Strictly Improve for TB Rules



(b) Fraction of Instances All M Stay Same for TB Rules

Figure 5.7: The graphs above show the fraction of instances all men are weakly better off as a result of all tie-breaking rules for when both sides of preferences are weak, men preferences contain ties and women preferences are strict, and men preferences are strict and women preferences contain ties.

When the experiments look at all men instead of a fixed man, some men get strictly worse off as a result of the tie-breaking rule, which is lowest when man preferences contain ties and woman preferences are strict. This is shown in Figure 5.8.

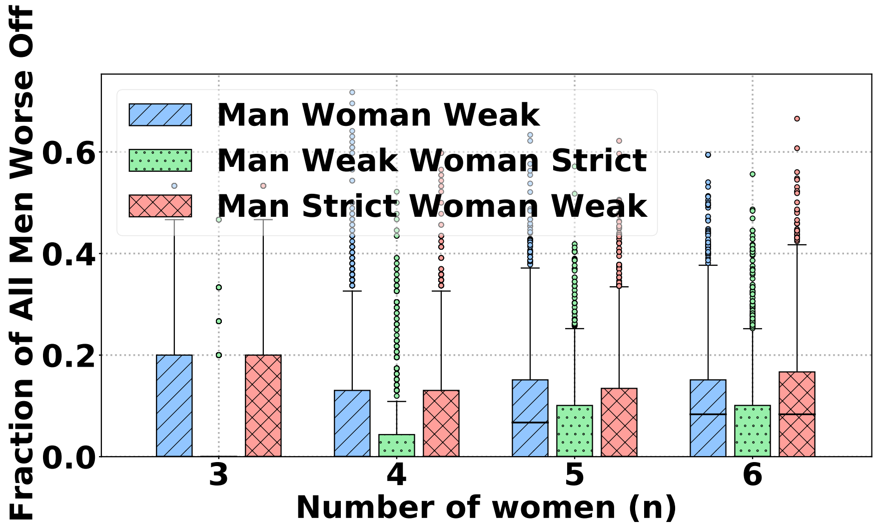


Figure 5.8: The graph above shows the fraction of instances all men are strictly worse off as a result of all tie-breaking rules for when both sides of preferences are weak, men preferences contain ties and women preferences are strict, and men preferences are strict and women preferences contain ties.

5.3.3 *Discussion*

When one gender has weak preferences and regardless if the opposite gender has strict or weak preferences, the fraction of all of opposite gender is more likely to be strictly better off, but also strictly worse off as a result of the tie-breaking rule. This suggests that a particular tie-breaking rule can make some agents strictly better off, but at the expense of making others strictly worse off. In the setting where a higher number of all agents strictly improve, a higher number of all agents are also strictly worse off. For example, when man preferences contain ties and women preferences are strict, the fraction of all women are more likely to get strictly better off but they are also more likely to get strictly worse off. The opposite applies for all men. Again, these results further confirm that tie-breaking itself has the potential to change the matching, which could benefit particular agents, while putting other agents at a disadvantage. The overall findings for this experiment are:

- Ties on the proposing side are beneficial to all women, whereas ties on the proposed-to side are not.

- Ties on the proposed-to side are beneficial to all men, whereas ties on the proposing side are not.

Chapter 6

When STB is Unknown to Manipulator

This chapter explores the impact of using a tie-breaking rule to make preferences strict and how knowing this tie-breaking rule can affect the manipulator's success in getting strictly better off.

6.1 *Experimental Design*

When the tie-breaking rule used to make preferences strict is unknown to the manipulator, the manipulator selects a new tie-breaking rule uniformly at random to make preferences strict. This experiment focuses on four cases:

- Woman tie-breaking rule known, man tie-breaking rule known: The optimal manipulation strategy is computed using the algorithm proposed by (Teo et al., 2001) where the strict preferences are passed as inputs.

- Woman tie-breaking rule unknown, man tie-breaking rule unknown: The optimal manipulation strategy was computed with the strict preferences that were generated using the manipulator's random tie-breaking rule.

- Woman tie-breaking rule known, man tie-breaking rule unknown: The optimal manipulation strategy is computed using the original strict preferences for women and the strict preferences that were generated using the manipulator's random tie-breaking rule for men.

- Woman tie-breaking rule unknown, man tie-breaking rule known: The optimal manipulation strategy is computed using the original strict preferences for men and the strict preferences that were generated using the manipulator's random tie-breaking rule for women.

In this experiment, frequency and bucket improvement were plotted.

6.2 *Experimental Results*

The experiments were conducted for 10,000 iterations for 4 to 20 agents on each side. Figure 6.1 shows that the women's tie-breaking rule does not matter:

- When both man and woman tie-breaking rules are known, the fraction of instances that a manipulator can misreport to strictly improve is the highest since this is equivalent to self manipulation under strict preferences.

- The case when both man and woman tie-breaking rules are unknown is essentially the same as the case when the women tie-breaking rule is known, men tie-breaking rule is unknown These two cases have the lowest fraction of instances that a manipulator can misreport to strictly improve.

- When the woman tie-breaking rule is unknown and man tie-breaking rule is known, the manipulator is more likely to strictly improve than either of the cases in which a women's tie-breaking rule is unknown.

The key finding of this experiment was that knowing the tie-breaking rule for men is much more important in successful manipulation than knowing the tie-breaking rule for women.

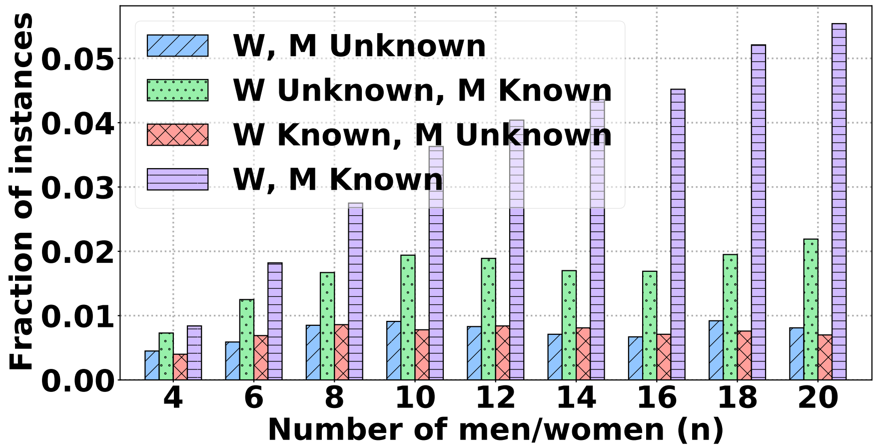


Figure 6.1: The graph above shows the fraction of instances that a manipulator can strictly improve as a result of misreporting its preferences for when both sides tie-breaking rule is unknown, women tie-breaking is unknown and man tie-breaking is known, women tie-breaking is known and man tie-breaking is unknown, and both sides tie-breaking is known.

Although knowing the tie-breaking rule for men is critical for manipulation, with respect to bucket improvement, all four cases are equivalent as shown in Figure 6.2. For the fraction of instances that a manipulator can manipulate, the strict bucket improvement for the manipulator is the same because preferences are weak on both sides. When the manipulator guesses the tie-breaking rule correctly, the match of the manipulator would be the same as when the tie-breaking rule is known on each side.

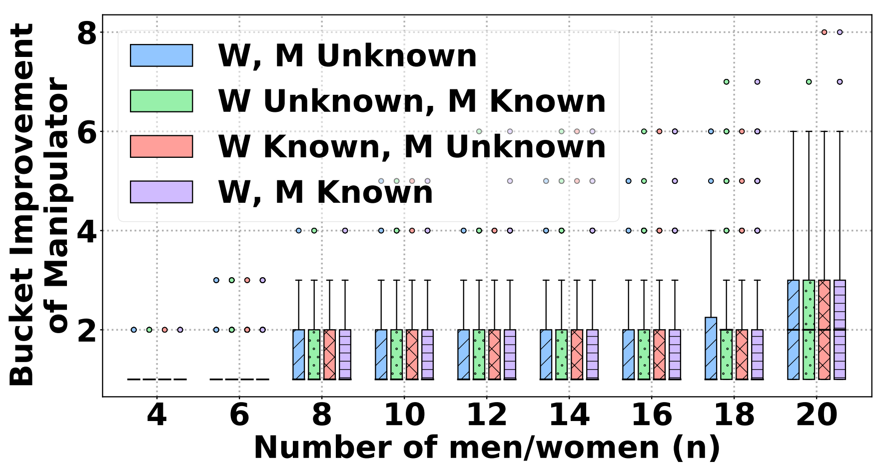


Figure 6.2: The graph above shows the strict improvement of the manipulator for when both sides tie-breaking rule is unknown, women tie-breaking is unknown and man tie-breaking is known, women tie-breaking is known and man tie-breaking is unknown, and both sides tie-breaking is known.

6.3 *Discussion*

It is most desirable for the manipulator to know the tie-breaking rule on both sides, as the fraction of instances that a manipulator can get strictly better off emulates that of strict preferences. When the man tie-breaking rule is known and the woman tie-breaking rule is unknown, the manipulator is more likely be able to manipulate due to the man tie-breaking rule allowing for men to propose in the same order. There is essentially no difference in the fraction of instances that a manipulator can get strictly better off when the tie-breaking rule is unknown on both sides and when the tie-breaking rule for man is unknown and the tie-breaking rule for woman is known. Since the preferences are weak, the bucket improvement is the same for every case of the tie-breaking rule being known/unknown to the manipulator as the manipulator can guess the tie-breaking rule correctly. The key findings for this experiment are:

- Knowing the man tie-breaking rule is critical in successful manipulation.
- With respect to bucket improvement, all four cases are equivalent when the tie-breaking rule is guessed correctly.

6.4 *Critical Open Theoretical Questions*

Our experimental analysis gave rise to several interesting open questions:

- Question 1: How can these experiments be conducted for MTB?
- Question 2: What is the best manipulation strategy when the tie-breaking rule is not known on either side?
- Question 3: What is the best manipulation strategy when the tie-breaking rule is not known on both sides?

Chapter 7

Impact of Tie-breaking on Optimal Manipulation Strategy

This chapter looks at strategic manipulation in the setting of weak preferences. Because there could be multiple tie-breaking rules that affects the matching under the DA algorithm, the best manipulation strategy is used for a fixed woman manipulator in expectation for all tie-breaking rules.

7.1 *Computing All Possible Manipulation Strategies*

When preferences are weak, the optimal manipulation algorithm cannot be used to obtain the optimal permutation manipulation. The question of how to generate all possible manipulation strategy rises. Consider S_w, the set of preference lists that can be obtained from w's true preference list by moving two men to the top two positions: $S_w := \{(m_i,\, m_j,\, \succ_w \setminus\{m_i, m_j\}) : m_i, m_j \in M$. Hosseini et al. (2021) proposed the proposition below that for any arbitrary misreport by the manipulator w, there exists a list in S_w that creates the same matching for all agents. This proposition reduces the time complexity of computing all possible manipulation strategies from $O(n!)$ to $O(n^2)$. In the following experiment, this method is used to generate all possible manipulation strategies when preferences contain ties.

Proposition 1 (Hosseini et al., 2021) Let $\succ$ be a profile and let $\succ'_w$ be any misreport for a fixed woman w. Then, there exists a list $\succ''_w \in S_w := \{(m_i,\, m_j,\, \succ_w \setminus\{m_i, m_j\}) : m_i, m_j \in M\}$ that achieves the same matching, i.e., $\mu'' = \mu'$, where $\mu' := DA(\succ'_{-w}, \succ'_w)$ and $\mu'' := DA(\succ''_{-w}, \succ''_w)$.

7.2 *Experimental Design*

In this experiment, a manipulator is fixed w_1. This experiment iterates through optimal manipulation strategies for w_1, while iterating through all possible tie-breaking strategies. There are n^2 optimal strategies by proposition 1 and $n!$ tie-breaking rules. If the total improvement for a particular strategy is greater than zero for all tie-breaking rules, the manipulator can strategically manipulate for this instance. This experiment plots the fraction of instances that the manipulator can strategically manipulate, the expected manipulator strict improvement, and the fraction of all women strictly better off, worse off, and who stay the same all for when the manipulator strictly improves. These experiments were run for three cases: Man Preferences Strict, Woman Preferences Strict; Man Preferences Strict, Woman Preferences Weak; and Man Preferences Weak, Woman Preferences Strict

It is important to note strict preferences do not change as a result of the tie-breaking rule.

7.3 *Observations*

One of the open questions with this experiments is if there exists a manipulation strategy such that the manipulator would get strictly better off or stay the same for every tie-breaking rule. Consider the following example in Table 7.1 where men have weak preferences and women have strict preferences.

m_1:	$(\underline{w_5^*}\ w_3)$	$(w_4\ w_1)$	w_2	w_1:	$\underline{m_5}$	m_1	m_2	m_4^*	m_3
m_2:	$(w_3^*\ w_1)$	$(w_2\ \underline{w_4})$	w_5	w_2:	m_4	$\underline{m_3^*}$	m_5	m_1	m_2
m_3:	$(w_3\ w_5)$	$(\underline{w_2^*}\ w_1)$	w_4	w_3:	$\underline{m_4}$	m_2^*	m_1	m_3	m_5
m_4:	$(w_1^*\ w_4)$	$(w_2\ \underline{w_3})$	w_5	w_4:	m_1	$\underline{m_2}$	m_5^*	m_4	m_3
m_5:	$(w_4^*\ \underline{w_1})$	$(w_5\ w_3)$	w_2	w_5:	m_2	$\underline{m_1^*}$	m_4	m_5	m_3

Table 7.1: The table shows an example of an optimal manipulation strategy that makes the manipulator strictly better off, denoted with an underline, or the same, denoted with *, for all tie-breaking rules.

When w_1 misreports its preferences to be $[m_5 \succ m_1 \succ m_3 \succ m_2 \succ m_4]$, w_1 manipulates and is consequently strictly better off or the same for all tie-breaking rules. w_1 strictly improves by

3 ranks for ten of the tie-breaking rules and strictly improves by 2 ranks for five of the tie-breaking rules, which confirms that this misreport is the optimal manipulation strategy.

Consider the tie-breaking rule: $[w_5 \succ w_4 \succ w_3 \succ w_1 \succ w_2]$. The matching from the DA algorithm with the single tie-breaking rule is denoted with *. When this tie-breaking rule is used with the manipulation strategy, the consequent matching is denoted with an underline. The manipulator is originally matched to its fourth choice of m_4 under the DA algorithm, but improved by 3 and getting matched with its first choice of m_5 under the manipulation. Consider another tie-breaking rule: $[w_5 \succ w_4 \succ w_2 \succ w_3 \succ w_1]$. The matching from the DA algorithm with this single tie-breaking rule is the same as the matching denoted with *. When this tie-breaking rule is used with the manipulation strategy, the consequent matching is also the same as the matching denoted with *. For this tie-breaking rule, the manipulator's partner is the same under its true preferences and the manipulated preferences.

Another open question is whether there exists an optimal manipulation strategy such that a manipulator can get strictly worse off for some tie-breaking rules but stay strictly better off in expectation. Consider the example in Table 7.2 where men have weak preferences and women have strict preferences.

m_1:	(w_3)	$(\underline{w_1}\ w_4\ w_2^{*\dagger\ddagger})$	w_1:	$m_2^{\ddagger}$	m_3	$m_4^{*\dagger}$	$\underline{m_1}$
m_2:	$(\underline{w_4^{*\dagger}})$	$(w_2\ w_3\ w_1^{\ddagger})$	w_2:	$m_1^{*\dagger\ddagger}$	m_3	m_2	$\underline{m_4}$
m_3:	$(\underline{w_3^{*\dagger\ddagger}})$	$(w_2\ w_4\ w_1)$	w_3:	$\underline{m_3^{*\dagger\ddagger}}$	m_4	m_1	m_2
m_4:	$(w_1^{*\dagger})$	$(\underline{w_2}\ w_3\ w_4^{\ddagger})$	w_4:	$m_4^{\ddagger}$	$\underline{m_2^{*\dagger}}$	m_3	m_1

Table 7.2: The table shows an example of an optimal manipulation strategy that makes the manipulator strictly better off in expectation, but strictly worse off for some tie-breaking rules.

Under the weak preferences, the optimal self manipulation misreport for w_1 is $[m_2 \succ m_1 \succ m_3 \succ m_4]$. For every possible tie-breaking strategy, some tie-breaking rules make the manipulator strictly better off, strictly worse off, or stays the same. When $[w_3 \succ w_1 \succ w_2 \succ w_4]$ is used as the tie-breaking rule, the DA matching is denoted with *. When this tie-breaking rule is used with the optimal manipulation strategy, the manipulator w_1 is matched with m_1, which is a rank strictly worse than that of its true matching of m_4. The outcome of this tie-breaking rule and the manipulation strategy is indicated by an underline. Alternatively, When the tie-breaking rule $[w_3 \succ w_2 \succ w_1 \succ w_4]$ is used, the DA matching is the same as before denoted with *. When the manipulation strategy is used with the DA algorithm, the manipulator w_1 is matched with m_4, which is the same as its true matching under this same tie-breaking rule. The outcome of this tie-breaking rule is indicated by a dagger. When the tie-breaking rule $[w_3 \succ w_4 \succ w_1 \succ w_2]$ is used with the DA algorithm, the manipulator w_1 is matched with m_2, which is its best possible partner and a strict improvement of 2 from its true partner m_4. The outcome of this tie-breaking rule is indicated by a double dagger. There are four tie-breaking rules such that the manipulator is strictly worse off by 1 rank, twelve tie-breaking rules such that the manipulator stays the same, and eight tie-breaking rules such that the manipulator is strictly better off by 2 rank. Despite being worse off for some tie-breaking rules under the optimal strategy, the manipulator's total strict improvement is 12 ranks, indicating that its expected strict improvement is 0.33 ranks.

7.4 *Experimental Results*

The following experiments were conducted for 1500 iterations for 3 to 6 agents. When weak preferences are present, the tie-breaking rule in combination with the manipulation strategy increases the fraction of instances that a manipulator can strictly improve. When there are more than four agents on each side, Figure 7.1 shows the fraction of instances that a manipulator strictly improves is always greater in cases when weak preferences are present. The tie-breaking rule can offer a strict improvement to the manipulator by making a strategy that would not be optimal under strict preferences optimal when weak preferences are present. Weak preferences on either side are more desirable for increasing the frequency that a manipulator can improve through a misreport, as shown by Figure 7.1. This does not hold when there are three agents on each side, likely because there are significantly less possible orderings of the agents, signifying fewer optimal manipulation strategies and tie-breaking rules to change the matching. As the number of agents increases, the fraction of instances that a manipulator can strictly improve also increases for all three cases. Because the optimal strategy is when a manipulator gets better off in expectation, the cases with weak preferences have a higher fraction of instances that a manipulator can get better off in expectation than the case with strict preferences. The main finding of this plot is that the manipulator can strictly improve more frequently in expectation when ties are present.

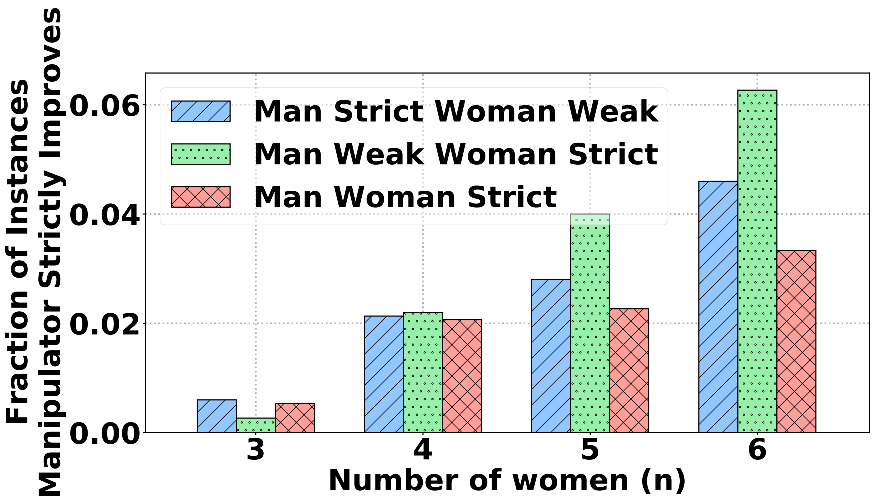


Figure 7.1: The graph shows the fraction of instances that a manipulator can misreport to get strictly better off in expectation for when men preferences are strict and women preferences are weak, men preferences are weak and women preferences are strict, and both men and women preferences are strict.

Weak preferences do not guarantee a high expected strict improvement and rather strict preferences allow for the manipulator to have the highest expected strict improvement. The highest expected improvement of the manipulator is possible when men and women preferences are strict, followed by the case where man preferences are strict and women preferences are weak, and finally when man preferences are weak and women preferences are strict as shown by Figure 7.2. The expected improvement will always be highest when both sides are strict as the tie-breaking rule has no impact on strict preferences. As the number of agents increases, the expected strict improvement of the manipulator increases for all three cases of preference profiles.

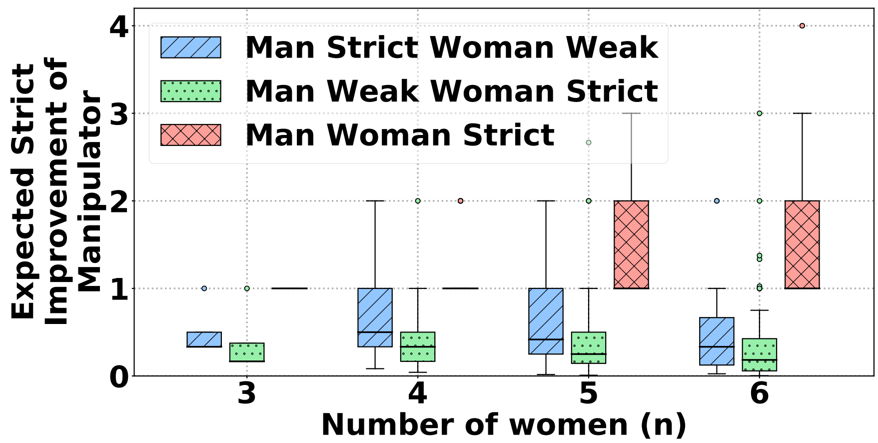
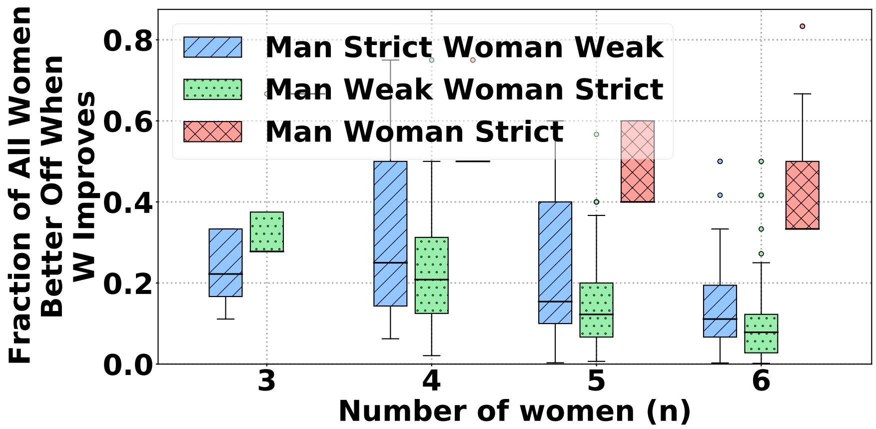
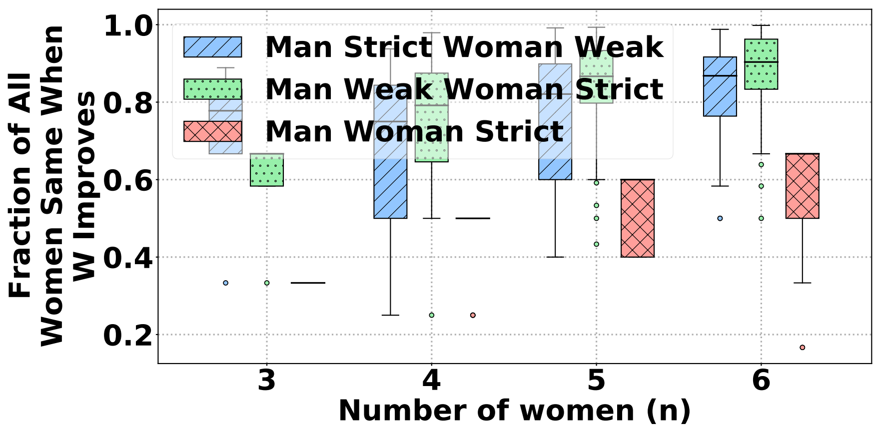


Figure 7.2: The graph shows the average strict improvement of the manipulator for when men preferences are strict and women preferences are weak, men preferences are weak and women preferences are strict, and both men and women preferences are strict.

When a manipulator strictly improves as a result of an optimal manipulation strategy and weak preferences are present on either the side of men or women, the rest of the woman are most likely to stay matched with a partner that they are indifferent to than their original partner as shown by Figure 7.3. When both man and woman preferences are strict, the most of the woman are more likely to get strictly better off by being matched with a more preferable partner and if not, then they are guaranteed to be weakly better off by being matched to the same partner. When man preferences are strict and woman preferences are weak, the rest of the woman are more likely to get strictly better off than in the case when man preferences are weak and woman preferences are strict. Similarly, when man preferences are weak and woman preferences are strict, all women are more likely to stay matched with a partner that they prefer equally to their original partner.



(a) Fraction of All Women Strictly Improve when W Improves



(b) Fraction of All Women Stay Same when W Improves

Figure 7.3: The graphs above show the fraction of all women who weakly improve when the manipulator strictly improves for when men preferences are strict and women preferences are weak, men preferences are weak and women preferences are strict, and both men and women preferences are strict.

When weak preferences are weak present on either side and the manipulator is strictly better off as a result of the manipulation strategy for all tie-breaking rules, the rest of the woman can get strictly worse off as shown by Figure 7.4. When both sides are strict and the manipulator strictly improves, the rest of the women never get strictly worse off. The fraction of woman who are worse off is significantly less than that of the fraction of woman who are better off or the same when weak preferences are present on either side. This indicates that a manipulation strategy that makes a manipulator better off on expectation can come at the expense of occasionally making other women strictly worse off.

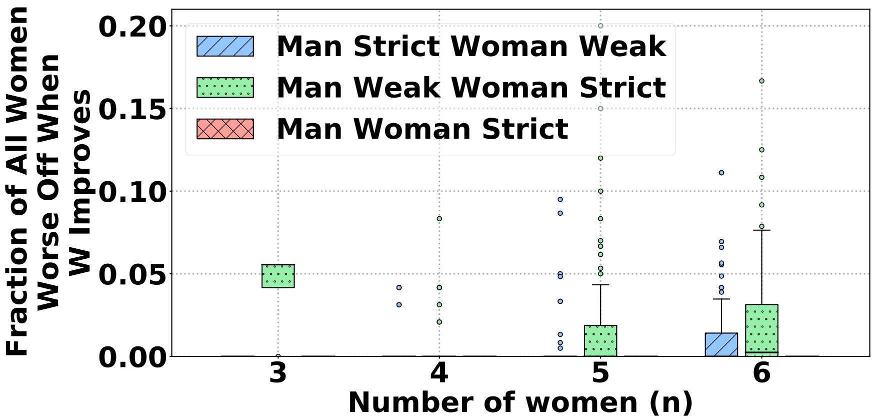


Figure 7.4: The graph above shows the fraction of instances a fixed woman is strictly worse off when the manipulator strictly improves for when men preferences are strict and women preferences are weak, men preferences are weak and women preferences are strict, and both men and women preferences are strict.

7.5 *Discussion*

When man preferences are weak and woman preferences are strict, the fraction of instances that a manipulator can get strictly better off is highest. This comes at the expense of some of the women getting strictly worse off and most of the women staying the same.

When a manipulator gets strictly better off when man preferences are strict and woman preferences are weak, the fraction of all women who are strictly worse off is significantly less than when man preferences are weak and woman preferences are strict. As noted above, the fraction of instances that a manipulator can strictly improve is also less than when man preferences are weak and woman preferences are strict, so the frequency of manipulation comes with a trade-off of less of the fraction of women strictly worse off as a result of the self-manipulation.

Introducing weak preferences in either woman preferences or men preferences can be more advantageous than having strict preferences since it increases the fraction of instances that a manipulator can get strictly better off; however, it comes at the following expenses: the expected improvement is not as high as when preferences are strict on both sides, the fraction of all women strictly better off as a result of the manipulation is not as high, and some of the women can get strictly worse off as a result of the manipulation.

A higher expected bucket improvement when both sets of preferences are strict reciprocates the idea that with strict preferences the manipulator only needs to improve the partner by one to be better off, but in settings with weak preferences, the manipulator has to improve its partner by the number of agents that she is indifferent between to be strictly better off.

7.6 *Critical Open Theoretical Questions*

This experiment shows that an optimal manipulation strategy does exist when preferences are weak. This raises the following open question:

- Question 1: How can the optimal manipulation strategy be computed when preferences are weak?
- Question 2: Is the optimal manipulation strategy different when men preferences are strict and women preferences are weak compared to when men preferences are weak and women preferences are strict?

Chapter 8

Theoretical Observations

This chapter looks at how a manipulator can misreport its preferences to get matched to a better partner in a strongly stable matching under the execution of STRONG and a super stable matching under the execution of SUPER.

The previous experiments were conducted using STB and the DA algorithm. There are two other algorithms that can also be used to find a matching when the preference profiles are weak. The algorithm SUPER (Irving, 1989) provides a super-stable matching in $O(n^2)$. The algorithm STRONG (Irving, 1989) provides a super-stable matching in $O(n^4)$. This chapter studies manipulation strategies under these algorithms.

8.1 *Strategic Manipulation of the Strongly Stable Algorithm*

Table 8.1 shows a strongly stable matching, in which a manipulator can weaken its preferences to get better off, while maintaining the strong stability of the matching. Consider the following preference profiles with weak preferences for men and strict preferences for women.

m_1:	$(w_4\ w_1)$	w_2^*	$\underline{w_3}$	w_1:	m_2	$\underline{m_3^*}$	m_1	m_4
m_2:	$(\underline{w_4^*}\ w_3)$	w_1	w_2	w_2:	$\underline{m_4}$	m_2	m_1^*	m_3
m_3:	$(w_2\ \underline{w_1^*})$	w_4	w_3	w_3:	$\underline{m_1}$	m_3	m_4^*	m_2
m_4:	$(w_3^*\ w_1)$	$(w_4$	$\underline{w_2})$	w_4:	m_3	$\underline{m_2^*}$	m_4	m_1

Table 8.1: The table shows an example of strategic manipulation under the strongly stable algorithm, where m_2 weakens its preferences to get matched with m_4 (denoted with an underline) instead of m_1 (denoted with *).

When the strongly-stable algorithm is run, the resulting matching μ is denoted with *. Under the strongly-stable algorithm, w_2 is matched with its third true choice of m_1. However, w_2 can get strictly better off when she misreports its preference list to be $[m_4 \succ m_2 \succ (m_1\ m_3)]$. w_2 is able to strictly improve by getting matched to its first choice m_4. The manipulated matching μ' is denoted by an underline. μ' remains strongly-stable. In this example, all the other women are either strictly better off or the same as a result of the manipulation.

8.2 *Strategic Manipulation of the Super-Stable Algorithm*

Table 8.2 shows an example of a super stable matching, in which a manipulator can weaken its preferences to get better off, while maintaining the super stability of the matching. Consider the following preference profiles with strict preferences for men and weak preferences for women.

m_1:	w_1	$\underline{w_3^*}$	w_4	w_2	w_5	w_1:	$\underline{m_5}$	m_2^*	m_1	$(m_4\ \ m_3)$
m_2:	w_1^*	$\underline{w_2}$	w_4	w_5	w_3	w_2:	m_5	m_1	$\underline{m_2}$	$m_4^*\ \ m_3$
m_3:	w_1	w_2	$\underline{w_5^*}$	w_3	w_4	w_3:	$\underline{m_1^*}$	m_3	m_4	$(m_2\ \ m_5)$
m_4:	w_3	w_2^*	$\underline{w_4}$	w_1	w_5	w_4:	m_3	m_2	m_1	$\underline{m_4}\ \ m_5^*$
m_5:	w_4^*	$\underline{w_1}$	w_5	w_2	w_3	w_5:	$\underline{m_3^*}$	m_2	m_5	$(m_2\ \ m_3)$

Table 8.2: The table shows an example of strategic manipulation under the super stable algorithm, where m_2 weakens its preferences to get matched with m_2 (denoted with an underline) instead of m_4 (denoted with *).

When the super-stable algorithm is run, the resulting matching μ is denoted by *. Under the super-stable algorithm, w_2 is matched with its fourth true choice of m_4. However, w_2 can get strictly better off by misreporting its preference list to be $[m_5 \succ m_1 \succ m_2 \succ (m_4\ m_3)]$. w_2 is able to strictly improve by getting matched to its third choice m_2. The manipulated matching μ' is denoted by an underline. The manipulated matching remains super-stable. In this example of a super-stable matching, all other women are either strictly better off or the same as a result of the manipulation.

8.3 *Manipulation Through Weakening Preferences*

Both examples discussed earlier show that a manipulator can get matched to a partner she strictly prefers to its true partner when she weakens its preference list. This raises the question if a manipulator can weaken its preferences to get strictly better off when preferences are strict. When preferences are strict, SUPER and STRONG return matchings that are equivalent to that of the DA algorithm. Given an instance of strict preferences, this chapter shows two examples below where a manipulator can weaken its preferences to get strictly better off with the SUPER algorithm and the STRONG algorithm. Consider Table 8.3 with strict preferences for both men and women where we use SUPER to obtain a matching.

m_1:	$\underline{w_1}$	w_2^*	w_3	w_1:	m_3^*	$\underline{m_1}$	m_2
m_2:	w_1	$\underline{w_3^*}$	w_2	w_2:	m_2	m_1^*	$\underline{m_3}$
m_3:	$\underline{w_2}$	w_1^*	w_3	w_3:	m_3	m_1	$\underline{m_2^*}$

Table 8.3: The table above shows the stable matching denoted with an underline, which was produced by the SUPER algorithm for the strict preference profile shown above. When w_1 misreports its preferences to be $[m_3 \succ (m_1\ m_2)]$, the matching is denoted with *.

When we run the super-stable algorithm on the strict preferences above, the resulting matching is denoted by an underline. Under the super-stable algorithm, w_1 is matched with its second true choice m_1. When w_1 manipulates its preference list to be weak $[m_3 \succ (m_1\ m_2)]$, w_1 is able to strictly improve by getting matched to its first true choice m_3. The weak manipulated is denoted with *. In this example of a super-stable matching, all other women are either strictly better off or the same as a result of the manipulation. Since super-stable is a stronger notion of stability than strongly stable, this also applies to a strict matching under the STRONG algorithm.

Consider Table 8.4 with strict preferences for both men and women where we use the STRONG algorithm to obtain a matching.

m_1:	w_3	w_4	w_5	$\underline{w_6^*}$	w_1	w_2	w_1:	$\underline{m_4}$	m_2	m_1	m_5^*	m_3	m_6
m_2:	w_5	$\underline{w_3^*}$	w_1	w_2	w_4	w_6	w_2:	m_5	m_1	m_4	$\underline{m_3^*}$	m_2	m_6
m_3:	w_1	w_6	w_3	w_4	w_5	$\underline{w_2^*}$	w_3:	m_6	$\underline{m_2^*}$	m_5	m_3	m_4	m_1
m_4:	w_5^*	$\underline{w_1}$	w_2	w_3	w_6	w_4	w_4:	$\underline{m_6^*}$	m_3	m_2	m_1	m_5	m_4
m_5:	w_1^*	w_3	$\underline{w_5}$	w_4	w_6	w_2	w_5:	$\underline{m_5}$	m_4^*	m_3	m_2	m_6	m_1
m_6:	$\underline{w_4^*}$	w_3	w_6	w_5	w_1	w_2	w_6:	m_5	m_4	m_6	m_2	$\underline{m_1^*}$	m_3

Table 8.4: The STRONG algorithm produces the matching denoted with * with these strict preferences. When w_1 misreports its preferences to be $[(m_4\ m_2) \succ (m_1\ m_5\ m_3\ m_6)]$, she can get strictly better off and the manipulated matching denoted with an underline for some tie-breaking rules. The underlined matching is strongly stable but not super-stable.

When we run the strongly-stable algorithm on the strict preferences above, the resulting matching is denoted with *. Under the strongly-stable algorithm, w_1 is matched with its fourth true choice m_5. When w_1 manipulates its preference list to be weak $[(m_4\ m_2) \succ (m_1\ m_5\ m_3\ m_6)]$, w_1 is able to strictly improve by getting matched to its first true choice m_4. The manipulated matching is denoted with an underline. The manipulated matching is strongly stable but not super-stable. In this example of a strongly-stable matching, all other women are either strictly better off or the same as a result of the manipulation.

8.4 *Open Algorithmic Questions*

These examples illustrate that ties can be used as means for manipulation under the STRONG and SUPER algorithms. A manipulator can weaken its preferences to get strictly better off, while preserving its stability. It is not always the case that the stability is preserved. Furthermore, a manipulator can weaken its preferences as a manipulation strategy for a setting in which both men and women preferences are strict under the SUPER algorithm and the DA algorithm with tie-breaking. This rises the following open questions:

- Question 1: Can an optimal manipulation strategy that preserves stability be computed for a strongly stable matching?

- Question 2: Can an optimal manipulation strategy that preserves stability be computed for a super stable matching?

- Question 3: Does weakening preferences under the SUPER algorithm always guarantee the same partner as manipulation by permutation under the DA algorithm?

Chapter 9

Conclusion and Further Research

This chapter will discuss my main results, limitations of my experiments, and further directions I can take to expand this thesis.

9.1 *Main Results*

Introducing ties itself has the ability to change a matching through the use of tie-breaking rules. The experimental analysis presented in this thesis showed that a tie-breaking rule can make a fixed woman and all women, better off when man preferences are weak, and a fixed man and all men better off when woman preferences are weak. When ties are present in the context of strategic manipulation, experimental analysis showed that the manipulator is strictly better off when the proposing side tie-breaking rule is known to the manipulator. When the experiments in this thesis looked at the impact of an optimal manipulation strategy for all tie-breaking, the manipulator is more frequently strictly better off, but it comes at the expense of a lower expected strict improvement, and making some women strictly worse off. Furthermore, ties introduce three variations of stability. When studying the algorithms STRONG and SUPER, the theoretical observations revealed that both these algorithms are not strategy-proof, as the "proposed to" side can weaken its preferences to get strictly better off while conserving the respective stability. It is not always the case that a manipulator can weaken its preferences and get strictly better off.

9.2 *Limitations*

This thesis posed some limitations. Many of the experiments iterate through all possible tie-breaking rules. Since tie-breaking rules are possible permutations of the agents, there are $n!$ possible tie-breaking rules. Hence, the runtime of iterating through all possible agents is $O(n!)$ where n is the number of men or women. In experiments that iterated through all possible tie-breaking rules, the maximum number of agents that the experiment could be run for was 7 agents. The experiment ran in Chapter 5 iterated through all optimal manipulation strategies and all possible tie-breaking rules, which restricted this experiment to have a maximum of 6 agents.

9.3 *Future Research*

The experiments from Chapter 6 can be repeated under MTB. With MTB, it is likely that the man tie-breaking rule still plays a critical role in successful manipulation. However, the fraction of instances a manipulator would be able to get strictly better off is less, as the manipulator has to guess the tie-breaking rule correctly for every agent in MTB, instead of just once as done in the Chapter 6 experiments with STB. In Chapter 7, the manipulator can get better off in expectation for an optimal manipulation strategy for all tie-breaking rules. This raises the question of how to actually compute the optimal strategy when preferences contain ties. This question can be a direction for future work for the following cases: man preferences weak, woman preferences strict; man preferences strict, woman preferences weak; and man and woman preferences weak. Comparing the ways in which each of the optimal strategies is computed for each case would also be interesting to study. In Chapter 9, the SUPER and STRONG were found to be manipulatable by weakening preferences. How to computing the optimal strategy under these algorithms and ensuring that the stability is preserved is also a direction of future work. It is unclear if an optimal strategy that always preserves stability always exists for SUPER and STRONG. Another question raised as a direction for future work is if weakening preferences under SUPER guarantees the same partner as manipulation by permutation.

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- Worked with Professor Dr. Hadi Hosseini on extensions of the stable matching problem with weak preferences and strategic manipulation

Software Engineering Intern | Instagram — May 2021 – Aug 2021
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- Created distributed data pipeline for filtering notification recipients using Python and SQL
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Software Intern | Pfizer Inc. — Jun 2019 - Aug 2019
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