THE PENNSYLVANIA STATE UNIVERSITY
SCHREYER HONORS COLLEGE

DEPARTMENT OF MATHEMATICS

SOLVING THE LADY IN THE LAKE PROBLEM AND ITS FASTEST OPTIMAL STRATEGY

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Summer 2022

A thesis
submitted in partial fulfillment
of the requirements
for baccalaureate degrees
in Mathematics
with honors in Area of Mathematics

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Abstract

The Lady in the Lake problem in differential game theory has been classically solved via the Isaacs minimax equation (for example, in [1] Chapter 8 and in [2]). The first half of this paper takes a different approach to give a proof without using the Isaacs Equation. First, we will set up an ODE model using a geometric construction in a straight-forward manner. Then, by using some nice tricks and proved intuitions, we can make the model a lot more workable, hence simplifying the problem from a 2D minimax problem to a 1D minimization problem. Finally, the game is naturally divided into two stages, and this paper solves for the optimal strategy of the lady in each of the two stages using calculus and change of variables. An expression for the terminal payoff is provided in the end as well. The second half of this paper develops a numerical algorithm to look for the fastest optimal strategy of this game, after proving that the optimal strategy is non-unique. The algorithm is a recursive nonlinear programming problem, which yields a numerical approximation of the lady's fastest optimal strategy in stage 1.

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Acknowledgements

To Professor Jessica Conway, who introduced to me the problem and generously offered her time and expertise to guide me through my exploration in this field. None of this could have happened without her dedication and patience for this project and myself.

To Professor Luen-Chau Li, who has always been so patient and understanding throughout not only my thesis process, but my undergraduate career as a whole. His kindness continues to encourage and inspire all his students, including me.

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Chapter 1

Introduction and Model Set Up

1.1 Introduction of the Game

The lady in the lake problem is a classic problem in differential game theory. The following offers a description of the game's set up.

A lady (E, "evader") is swimming in a circular pond with a maximum speed v_E. She can change the direction in which she swims instantaneously. A monster (P, "pursuer"), who cannot swim, and who wishes to intercept the lady when she reaches the shore, is on the side of the pond and can run along the perimeter with maximum speed v_P. He, also, can change his direction instantaneously. Furthermore, it is assumed that both E and P never get tired. E does not want to stay in the lake forever, though; she wishes eventually to come out without being caught by the monster. (On land, E can run faster than P.) E's goal is to maximize the payoff, which is the angular distance PE viewed from the center of the pond, at the time E reaches the shore. P obviously wants to minimize this pay-off. Additionally, we assume $v_E < v_P$ (since otherwise the problem is trivial). Can the lady escape the monster? If so, what is her best route such that she can maximize her distance with the monster?

1.2 Notations and set up

To set up a model for this problem, we will use the following notations. Refer to Figure 1.1 for a visual representation of the notations.

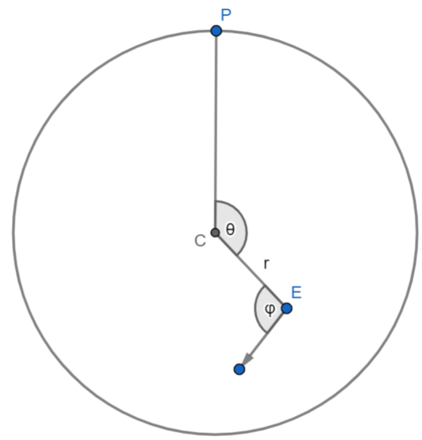

Figure 1.1: Notations

Define the distance between the lady's position and the center of the circular pond as r. Define the radius of the circular pond as constant R. Then, define the angle between CE and CP (where the lady sits and where the monster sits) as θ, in that orientation. Naturally, $-\pi < \theta \leq \pi$. We choose this branch because in reality, the further the lady can be with the monster is π away. This way, the relative position of the lady can always be uniquely described by the pair (r, θ) with respect to the monster. Since there is complete symmetry in the circular pond, we assume a starting position where the lady is at the center and the monster is directly above her.

Then we make a normalized version of the game with respect to the two maximum speeds. Define $v_2 := v_E/v_P < 1$. We claim that this current game is strategically equivalent to a game where the lady has maximum speed v_2 and the monster 1. It is easy to see why: multiplying a constant to both speeds doesn't impact the ratio, and hence the relatively advantage, between the two players. Thus, the final solution, in terms of the qualitative behavior of the optimal strategies and the value of the game, should be the same. Physically, multiplying the two speeds by the same constant means both players are speeding up or down at the same rate. We could also conceptualize that by thinking time itself slowing down or speeding up, which doesn't affect the strategies, only how fast they reach the end. Therefore, for the sake of argumentation, we will instead use v_2 and 1 for the speeds of the lady and the monster, respectively, from now on. To verify this change is indeed equivalent to the original problem, you can try to replace the two speeds by a constant multiple of them in the following model set up, and you will find that you arrive at the same model as this paper.

Next, we shall define the strategy profiles of the lady and the monster. Theoretically, each player at each instance chooses a direction (an angle) and a speed (a magnitude less than their caps). But in this continuous time game, we can assume that both players always go in constant maximum speed without loss of any significance. This is true because any slowing down in speed can be accounted for in terms of change in direction, without change in the time passed. We will omit the proof here and give a very simple example: going east at 0.5 miles/second for 2 seconds is essentially the same as going at a constant a 1 mile/second constant speed in a smooth S-shaped route to the same destination. Thus, we can view strategies like these as equivalent, which simplifies our model and makes sure that all paths can be represented by a smooth, equivalent path. Thus, since the players' decision-making only depends on each other's relatively position and not speed, we can simplify the strategy profiles of the two players to only their choices of directions.

Per the simplification above, denote the lady's choice of direction of movement as ϕ, which is defined as the angle between r and the lady's choice of movement as shown in Figure 1.1. We shall follow convention to let counter-clockwise be positive. In theory, ϕ is between $-\pi$ and π. On the other hand, the monster at each instance chooses whether to go clockwise or counter-clockwise. This means that the monster's strategy profile is always $(1/R, -1/R)$, which are his two choices of angular velocity.

1.3 The geometry of the model

In order to set up the equations we need, some geometric work is required. As shown in Figure 1.2, the lady is at position E with her instantaneous direction ϕ. Since every position of the lady is uniquely expressed in the pair (θ, r), we try to come up with a set of equations that describe (θ, r) using ϕ and t. Let the lady move for a time duration of dt towards the direction ϕ, and denote the change in angular distance as $d\theta$. Denote the lady's position after dt in the direction of ϕ as A. Then, consider the circle with radius AC centered on C, which is the dashed enclosing circle. Denote its intersection with the extension of the original radius CE as B. Finally, make a line l perpendicular to BC through the point A, and denote the intersection of l and BC as D. Without loss of generality, assume all angles involved are positive.

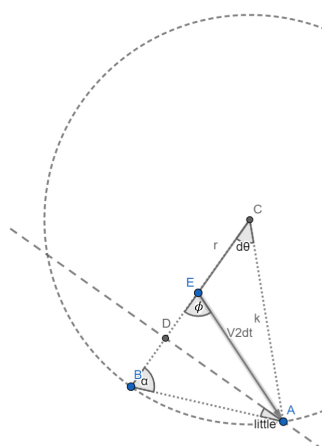

Figure 1.2: Geometry of the Model

First, notice that the overall circle is not the pond here. It is instead with radius from the center to the position that the lady would be in after time dt. Now, since the sum of the angles in a triangle is π, we have $\angle\alpha = (\pi - \angle d\theta)/2$. Thus, we then can use the sum of angles in the right triangle ABD to set up $\angle(little) = \pi/2 - \angle\alpha = \angle d\theta/2$.

We will use the fact that when $d\theta$ approaches 0, the arc length AB and the distance AB are roughly the same. Then, we have $k \times d\theta = AB$, where k is the radius of the overall circle. But notice that, by setting, $k = AC = BC$, thus, we can write out k as the sum of the three segments on BC. Namely, $k = CE + ED + DB = r + \cos(\phi) v_2 dt + \sin(little) AB$. But since $\angle little = d\theta/2$, as we let $d\theta$ approach 0, $\sin(little)$ also approaches 0, so we will drop out this term. Thus, plugging this into the original equation $k \times d\theta = AB$ we have

$$d\theta(r + \cos(\phi) v_2 dt) \approx AB \tag{1.1}$$

Now, again we use the fact that $\angle little$ goes to 0 as $d\theta$ goes to 0. This means that $\cos little$ goes to 1, which means AB and AD are roughly the same. But $AD = v_2 dt \sin(\phi)$ by trigonometry of the right triangle ADE. Thus, combining with the equation above we have

$$d\theta(r + \cos(\phi) v_2 dt) \approx v_2 dt \sin(\phi) \tag{1.2}$$

$$d\theta \approx \frac{v_2 dt \sin(\phi)}{r + v_2 dt \cos(\phi)} \tag{1.3}$$

$$\frac{d\theta}{dt} \approx \frac{v_2 \sin(\phi)}{r + \cos(\phi) v_2 dt} \tag{1.4}$$

Finally, by letting dt go to 0, the second term of the denominator drops out and we have just $\frac{d\theta}{dt} \approx \frac{v_2 \sin(\phi)}{r}$. But of course, this only considers the lady's movement. As for the monster, his strategy profile is $(1/R, -1/R)$. Thus, the combined rate of change of angular distance should be the sum of the lady's strategy and the monster's strategy, which means

$$\frac{d\theta}{dt} \approx \frac{v_2 \sin(\phi)}{r} \pm 1/R \tag{1.5}$$

But notice here that r is still a function of t, so we need to model r as well. But now it is relatively easy. Notice that the change in r, dr, is simply the difference between k and r. Thus we have $dr = k - r = r + \cos{(\phi)} v_2 dt + \sin{(little)} AB - r \approx \cos{(\phi)} v_2 dt$. Thus we have

$$\frac{dr}{dt} \approx v_2 \cos(\phi) \tag{1.6}$$

1.4 The trick to set up the model

Now that we can represent $\frac{d\theta}{dt}$ and $\frac{dr}{dt}$ in terms of ϕ, the obvious problem is the plus-minus sign in $\frac{d\theta}{dt}$. This plus-minus sign means that the current model cannot be expressed via a single constant function, as $\frac{d\theta}{dt}$ can change from having a plus sign to a minus sign at every instance. This prevents us from further analysis. In fact, in the classic proof included in the textbook of Basar and Olsder, 1982, the use of the Isaacs Equation was necessary to derive the result that the monster's optimal choice of sign is always the sign of $\theta(T)$.

To avoid this ambiguity, the Isaacs Equation is used classically. But in this paper we will instead use a trick to avoid this complication. Instead of letting θ to be the payoff, we will let $|\theta|$, the absolute value, to be the payoff. In this way, the lady is strictly trying to increase $|\theta|$ while the monster is strictly trying to decrease it. The rates at which they can alter the payoff remain the same, which are their maximum (and as we can assume per discussion before, constant) speeds. Thus, we have

$$\begin{aligned} \frac{d|\theta|}{dt} &= |v_2 \sin(\phi)/r| - 1/R && (1.7)\\ &= v_2 |\sin(\phi)|/r - 1/R && (1.8)\\ &= v_2 \sin(|\phi|)/r - 1/R && (1.9) \end{aligned}$$

for $-\pi < \phi \leq \pi$. By changing the payoff from θ to $|\theta|$, we have gotten rid of the plus-minus sign and now have a deterministic expression for $\frac{d|\theta|}{dt}$. Here we need to be careful because notice that

$\frac{d|\theta|}{dt}$ could be negative, which could potentially lead to a negative $|\theta|$ value given sufficient time, which doesn't make sense. In other words, the time derivative of $|\theta|$ is only before θ turns 0. At $\theta = 0$, when $\frac{d|\theta|}{dt} < 0$, this model breaks. Thus, we should make sure that our final solution does not involve anywhere θ turns 0.

Another potential problem is when $|\theta|$ might overshoot to beyond our hard limit π. But in the context of our analysis of this specific problem, divided into stage 1 and 2 in Chapter 2, this problem is avoided circumstantially (in stage 1, $\frac{d|\theta|}{dt}$ is always 0, while in stage 2 it is negative, so no overshoot is possible).

Further, since the cos function is even, we can write

$$\frac{dr}{dt} = v_2 \cos(\phi) \tag{1.10}$$

$$= v_2 \cos(|\phi|) \tag{1.11}$$

so that we have the same variable $|\phi|$ in both equations. For clarification, $|\phi|$ here stands not for the functional norm, but an absolute value, i.e. $|\phi| \coloneqq sign(\phi) \times \phi$

1.5 The final model

Thus, now we can establish the model:

$$\frac{d|\theta|}{dt} = v_2 \sin(|\phi|)/r - 1/R \tag{1.12}$$

$$\frac{dr}{dt} = v_2 \cos(|\phi|) \tag{1.13}$$

Since our initial condition is $r_0 = 0$ (as the lady starts at the center) and $\theta_0 = \pi$, we can write $|\theta|$ as

$$|\theta| = \int_0^\infty (\frac{v_2 \sin(|\phi|)}{r} - 1/R)dt + \theta_0 \tag{1.14}$$

$$= \int_0^\infty (\frac{v_2 \sin(|\phi|)}{\int_0^t v_2 \cos(|\phi|)d\tilde{t}} - 1/R)dt + \pi \tag{1.15}$$

where θ is the angular distance, r is the lady's radius, and ϕ is lady's choice of movement. With this model, we are trying to find the lady's best strategy ϕ^* as well as the conditions on which she can successfully escape.

1.6 Another important simplification

By definition, the lady's instantaneous strategy ϕ is chosen between $-\pi$ and π. Such is the strategy space of the lady. However, by defining a more refined "relevant" Nash Equilibrium strategies, we can make an important simplification to the problem.

Definition 1. *A non-relevant Nash Equilibrium strategy (in the context of this game) is an optimal strategy that returns the value of the game strictly slower than another optimal strategy.*
A relevant Nash Equilibrium strategy is an optimal strategy that is not non-relevant.
The non-relevant Nash Equilibrium strategy space is a subspace of the Nash Equilibrium strategy space that contains all the non-relevant optimal strategies of the lady.
The relevant Nash Equilibrium strategy space is a subspace of the Nash Equilibrium strategy space that contains all the relevant optimal strategies of the lady.

This definition is intuitive because while the lady cares about her final distance to the monster, she naturally also wants to spend less time in the water if possible, so that she could preserve more stamina for when she gets ashore. Further, the nature of this definition guarantees that if the Nash

Equilibrium space of the lady is not empty, then the relevant Nash Equilibrium space is also non-empty. Thus, if we are looking for an optimal strategy for the lady, we might as well first shuffle out all the obviously non-relevant ones.

Theorem 1. *The relevant Nash Equilibrium strategy space of the lady is a subset of* $(-\pi/2, \pi/2)$.

Proof. We will give the proof analytically. Define the lady "turning back" as picking a ϕ value that is outside the $(-\pi/2, \pi/2)$ range. In reality, this means she moves closer or maintains the same distance instead of moving away from the center. Suppose the lady chooses strategy ϕ_1 to go from point a to a point b inside the pond which involves "turning-back". Then, there always exists another strategy ϕ_2 which allows the lady to go from a to point b in shorter distance, and hence, shorter time, and such ϕ_2 involves no "turning-back". This is true because we can always pick ϕ_2, for example, to be the straight line between the origin and a.

Now, suppose ϕ_1 is a relevant Nash Equilibrium strategy. This means that when the monster chooses optimally, say, using strategy γ, then the payoff of all other strategies of the lady are less than or equal to the payoff under ϕ_1. In other words, we must have $|\theta(\phi, \gamma)| \leq |\theta(\phi_1, \gamma)|$ for any ϕ in the lady's strategy space. However, we can construct a strategy for the lady that pays almost strictly better. Suppose that it takes the lady T time to reach point b via strategy ϕ_2, and it takes her $T + \Delta t$ time via ϕ_1 for some positive Δt, as per the paragraph above. Then, let the lady use strategy ϕ_2 from $t = 0$ to $t = T$, and then try to move as far away angularly from the expected position of the monster under strategy γ as possible with her remaining Δt time. Call this strategy described above as ϕ_3. Then we see that in the end, after time $T + \Delta t$, strategy ϕ_3 pays strictly better than ϕ_1, unless ϕ_1 guarantees maximum payoff for the lady $|\theta| = \pi$ after time T. Lastly, even when ϕ_1 guarantees maximum payoff for the lady $|\theta| = \pi$ after time T, the lady can still move further away from the monster radially under the final Δt time of strategy ϕ_3, which will give her an edge later in the game and allow her to get ashore faster. Thus, strategy ϕ_3 almost strictly dominates ϕ_1, excluding it from the optimal strategy space; and even when in extreme cases when ϕ_1 can still be optimal, it is still non-relevant compared to ϕ_3. Thus, ϕ_1 cannot be a relevant Nash Equilibrium strategy for the lady.

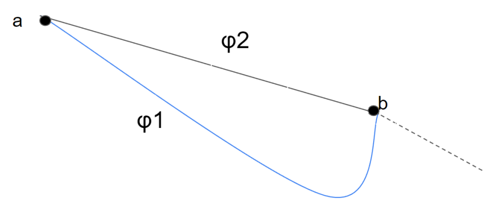

Figure 1.3: Visuals for ϕ_1 and ϕ_2

Thus, in general, any strategy that involves "turning back" cannot be a relevant Nash Equilibrium strategy for the lady. Thus, all relevant optimal strategies involve no "turn back". Thus theorem is proved.

□

With this theorem, we can now limit our scope of investigation exclusively to those strategies that don't "turn back", which makes intuitive sense. The relevant strategy space is reduced from $(-\pi, \pi]$ to $(-\pi/2, \pi/2)$. Further, no "turning back" means that there is a bijection between t and r. Since $r(t)$ is smooth (due to the nature of the game), we can deduce that its inverse, given that it exists, is also smooth. Thus, inside the Nash Equilibrium strategy space of the lady, there is a smooth bijection between r and t. This fact will be very useful later when we perform analysis on the dynamics of the game. Namely, this fact will allow us to use change of variables.

Chapter 2

Analysis and Solution: Stage 1 and 2

2.1 Analysis: Stage One

In this chapter, we will seek to find an optimal strategy for the lady to escape. In the following, we will discover that the game is naturally divided into two stages, and so is the lady's optimal strategy. Then we will find the lady's optimal strategy in each stage separately.

Since the lady already starts at maximum angular distance, it is natural that we first study under what conditions can the lady maintain this angular distance π. One condition which allows the lady to maintain angular distance at any instance is when $\frac{d|\theta|}{dt} = 0$. Plugging this into Equation (1.12) we have

$$\frac{v_2 \sin(|\phi|)}{r} - 1/R = 0 \tag{2.1}$$

$$\frac{v_2 \sin(|\phi|)}{r} = 1/R \tag{2.2}$$

$$\sin(|\phi|) = \frac{r}{v_2 R} \tag{2.3}$$

By symmetry, without loss of generality, let ϕ be positive, because the lady can go left or right with the same overall strategy. Since the magnitude of the sine function is always bounded by 1, this equation only has solution when $r \leq v_2 R$. Thus, the game is naturally divided into two stages: stage one is when the above condition holds, i.e. $r \leq v_2 R$. Stage two is when $r > v_2 R$. (Stage two has to exist because $v_2 < 1$). Further, in stage one, since the lady gets to choose any angle ϕ at any instance, by choosing $\phi(t) = \arcsin(\frac{r(t)}{v_2 R})$, the lady maintains $\frac{d\theta}{dt} = 0$, thus keeping θ at her optimal payoff $\theta_0 = \pi$. This means that $\phi(t) = \arcsin(\frac{r(t)}{v_2 R})$ is indeed her optimal strategy for stage one. In reality, this means that when r is small enough, she can manage to always stay diagonally opposed to the monster, thus keeping maximum angular distance possible.

Theorem 2. *The condition of stage 1, $r \leq v_2 R$, is not only the sufficient condition, but also the necessary condition for the lady to have a strategy that maintains maximum payoff π.*

Proof. First, since we have already derived a specific path of the lady, namely, $\phi(t) = \arcsin(\frac{r(t)}{v_2 R})$, which guarantees maximum payoff for the lady in stage 1, this proves the sufficient part. In other words, we have found how the lady could maintain maximum distance in stage 1, so stage 1 is a sufficient condition for such strategy to exist.

Next, we need to prove the necessary part. The condition of stage 1, $r \leq v_2 R$, implies $1/R \leq v_2/r$. Geometrically, notice that $1/R$ is the monster's angular speed, while v_2/r is the lady's angular speed. Thus, in stage one, the lady is "faster" in terms of angular speed. On the other hand, in stage two, the lady is "slower" in angular speed. Therefore, even if the monster keeps running in a fixed direction without changing strategy, he can reduce angular distance regardless of the lady's strategy in stage two. Thus, the lady cannot maintain angular distance in stage 2, regardless of her strategy, because she is always slower in angular speed. This means we indeed verify that the condition of stage one is the necessary and sufficient condition of maintaining a maximum optimal angular distance of π. □

But obviously, r cannot remain small forever because the lady wants to go ashore. So eventually we will enter stage two, where r becomes large enough that this strategy of the lady no longer is optimal. Thus, it is natural to divide the optimal strategy of the lady ϕ^* into two regions, where

$$\phi^*(r,t) = \begin{cases} \phi_1^*(r,t) = \arcsin(\frac{r(t)}{v_2 R}) & for \quad r \leq v_2 R \\ \phi_2^*(r,t) & for \quad r > v_2 R \end{cases} \tag{2.4}$$

We have found ϕ_1^*. Our next goal is to find ϕ_2^*.

2.2 Analysis: Stage Two– A Trial Approach

In stage two, we start with the end condition of stage one, which is $\theta_1 = \theta_0 = \pi$ and $r_1 = v_2 R$. Given the new initial conditions at stage two, our updated $|\theta|$ payoff function is

$$|\theta| = \int_0^\infty \left(\frac{v_2 \sin(|\phi|)}{\int_0^t v_2 \cos(|\phi|) d\tilde{t} + v_2 R} - 1/R\right) dt + \pi \tag{2.5}$$

Here, if we do the same thing as we did in stage one, which is to try to find a function ϕ which keeps $|\theta|$ at its maximum $|\theta_1| = \pi$, we can quickly spot that $\phi = \pi/2$ is already a solution. This is true because if we plug in $\phi = \pi/2$ into Equation (2.5), we see that $\cos(|\phi|) = 0$, and thus the integral in the denominator disappears. Thus, what's left inside the overall integral is merely $1/R - 1/R$ after v_2 cancels out, which is 0. Thus, it seems like without breaking a sweat, we already found the optimizer ϕ^* of θ!

However, if we put it into the context of the game itself, we quickly see that this strategy fails. This is because choosing $\phi = \pi/2$ means that the lady always swims perpendicular to the line segment between her current position and the origin. In other words, she would keep going in circles! Of course, since she is still on the verge of stage one, she can keep the monster diagonally opposed by swimming in circles. But she will drown eventually according to the game assumptions. Thus, a naive approach like the one in stage one fails. The lesson here is that at stage two, instead of merely trying to maximize θ, the lady also wants to make progress in the growth of r. Without the latter assumption, she swims in circles forever.

2.3 Analysis: Stage Two

Per the last section, we learned that the best strategy for the lady in stage two has to not only optimize θ, but also make sure r grows as well. To account for this, it is best to shift gears now and start to investigate how θ changes with respect to r instead of t.

In section 1.6, we observed that in the Nash Equilibrium strategy space, there is a smooth bijection between r and t. Thus, we can invert Equation (1.13) to get $\frac{dt}{dr} = \frac{1}{v_2 \cos(|\phi|)}$. Then, we can apply a change of variable to Equation (1.14) and get

$$|\theta| = \int_0^{R-v_2R} \left[\frac{v_2 \sin(|\phi|)}{r + v_2R} - \frac{1}{R} \right] \frac{dt}{dr} dr + \pi \tag{2.6}$$

$$= \int_0^{R-v_2R} \left[\frac{v_2 \sin(|\phi|)}{r + v_2R} - \frac{1}{R} \right] \frac{1}{v_2 \cos(|\phi|)} dr + \pi \tag{2.7}$$

$$= \int_0^{R-v_2R} \frac{\tan(|\phi|)}{r + v_2R} - \frac{1}{v_2R\cos(|\phi|)} dr + \pi \tag{2.8}$$

$$= \int_{v_2R}^{R} \frac{\tan(|\phi|)}{r} - \frac{1}{v_2R\cos(|\phi|)} dr + \pi \tag{2.9}$$

An alternative way of deriving only the lady's part without the monster's part in this $|\theta|$ payoff function can be found in Appendix 1. Since we write $|\theta|$ as an integral of r now, we no longer need to worry about r not growing. Thus, in order to maximize the payoff $|\theta|$ with respect to the lady's strategy ϕ, we set $\frac{d|\theta|}{d\phi} = 0$ and solve for the optimizer function ϕ. Using Euler-Lagrange's formula for functional differentiation, we have

$$\frac{d|\theta|}{d\phi} = 0 \tag{2.10}$$

$$\frac{d}{d\phi} \left\{ \frac{\tan(|\phi|)}{r} - \frac{1}{v_2R\cos(|\phi|)} \right\} = 0 \tag{2.11}$$

$$\frac{d}{d\phi} \left\{ \frac{\pm\tan(\phi)}{r} - \frac{1}{v_2R\cos(\phi)} \right\} = 0 \tag{2.12}$$

$$\pm\frac{1}{r\cos^2(\phi)} - \frac{\sin(\phi)}{Rv_2\cos^2(\phi)} = 0 \tag{2.13}$$

we can simplify and get

$$\sin(\phi) = \pm\frac{Rv_2}{r} \tag{2.14}$$

$$\phi^* = \arcsin\left(\pm\frac{Rv_2}{r}\right) \tag{2.15}$$

$$= \pm\arcsin\left(\frac{Rv_2}{r}\right) \tag{2.16}$$

Again, by symmetry, we can allow ourselves to only consider ϕ positive. Notice that the lady's optimum strategy depends only on r. Further, notice that if we plot it, $\sin(\phi) = \frac{Rv_2}{r}$ is a straight line perpendicular to the lady-monster line PE at the time when $r = v_2R$!

So far we have only shown that $\phi^* = \arcsin(\frac{Rv_2}{r})$ is the only local extremum, we can verify that it is indeed a maximizer, as we expected, analytically. Since ϕ^* is the only extremum, it suffices to find one other strategy ϕ_0 with no "turn back" such that ϕ_0 returns a lesser payoff, i.e. $|\theta(\phi_0)| < |\theta(\phi^*)|$, It is easy to imagine such an extreme ϕ_0. For example, the lady can decide to snake around the trajectory ϕ^* (which, remember, is a straight line), and we shall call this strategy ϕ_0. Then obviously, ϕ_0 arrives at the same arrival place as ϕ^* but gives the monster more time to catch up, which will produce a lesser final angular distance between the two players. Thus, we have found a ϕ_0 with no "turn back" such that $|\theta(\phi_0)| < |\theta(\phi^*)|$. Thus, ϕ^* is indeed a maximizer.

2.4 Conclusion

Thus, our solution to the Lady in the Lake problem, which is the lady's optimal strategy, is described by the following:

$$\phi^*(r,t) = \begin{cases} \arcsin(\frac{r}{v_2R}) & for \quad r \leq v_2R \\ \arcsin(\frac{Rv_2}{r}) & for \quad r > v_2R \end{cases} \tag{2.17}$$

where the monster's best strategy is $-1/R$. This is a Nash Equilibrium of the game. To find the value of the game (and whether the lady actually gets to escape or not), we plug in $\phi = \arcsin(\frac{Rv_2}{r})$ to Equation (2.9) and get

$$|\theta| = \int_{v_2 R}^{R} \frac{\tan(|\arcsin(\frac{Rv_2}{r})|)}{r} - \frac{1}{v_2 R \cos(|\arcsin(\frac{Rv_2}{r})|)} dr + \pi \tag{2.18}$$

$$= \int_{v_2 R}^{R} \frac{\tan(\arcsin(\frac{Rv_2}{r}))}{r} - \frac{1}{v_2 R \cos(\arcsin(\frac{Rv_2}{r}))} dr + \pi \tag{2.19}$$

$$= \int_{v_2 R}^{R} \frac{Rv_2/r}{\sqrt{1 - R^2 v_2^2/r^2}}/r - \frac{1}{Rv_2} \frac{1}{\sqrt{1 - R^2 v_2^2/r^2}} dr + \pi \tag{2.20}$$

$$= \int_{v_2 R}^{R} \frac{1}{\sqrt{1 - R^2 v_2^2/r^2}} (\frac{Rv_2}{r^2} - \frac{1}{Rv_2}) dr + \pi \tag{2.21}$$

$$= \arccos(v_2) - \frac{1}{v_2}\sqrt{1 - v_2^2} + \pi \tag{2.22}$$

If the value of the game is positive, the lady escapes; if the value is 0 or negative, the lady cannot escape. Notice that the value depends only on the relative advantage between the players, v_2, and not R. It has been shown that θ can be positive only when v_2 is large enough ($v_2 > 1/4.6033$, approximately. See [3]).

Chapter 3

Towards Finding the Fastest Optimal Strategy

3.1 An observation from Chapter 2

In this chapter, we will first observe that it naturally follows from Chapter 2 that the optimal strategy of the lady is non-unique. Then, we will define the "best" optimal strategy as the fastest optimal strategy, and try to develop a numerical algorithm to approximate this fastest optimal strategy.

In Section 2.1, we found an optimal strategy of the lady in stage 1. To arrive at that, we set $\frac{d|\theta|}{dt} = 0$ at every t. However, this is a sufficient but not necessary condition for an optimal strategy of the lady. Namely, the lady doesn't have to keep the payoff constant *at all times*–she only needs to make sure that it is at its maximum π once she reaches the border and gets ready for stage 2. We can easily image many such strategies, based upon the one we gave in Section 2.1. For example, if she perturbs a little bit in the early stage of her strategy, she can still adjust and make it to being opposed to the monster quickly, since she is that much faster than the monster angularly when r is small.

Mathematically, notice that the optimization process done in Section 2.3 should work not only for stage 2, but for the whole game itself. However, if we look at the result from that analysis in the context of stage 1, we see that the solution $\arcsin(\frac{Rv_2}{r})$ could be undefined, as $\frac{Rv_2}{r} \geq 1$. This happens probably because there is no local extrema in stage 1, given that we did not put a hard upper limit on the payoff $|\theta|$ in the model. In other words, a solution that gives $\theta = \pi$ in the end and another that gives $\theta = 3\pi$ are equally as good. Thus, the optimal strategy of the lady in stage 1 is not unique.

But does this mean that all the Nash Equilibrium strategies of the lady are equally as good? Probably not. You can imagine the lady going in small circles close to the origin for years before finally deciding to approach the borders of stage 1, and this could still be a Nash Equilibrium optimal strategy. But this is obviously nonrealistic, as the lady gets tired. She probably wants to be ashore as soon as possible. Thus, given there are many optimal strategies in terms of payoff $|\theta|$, maybe we can pick the "best" one or ones of them by T, the time it takes to reach the borders of stage 1. In other words, let's say that the fastest optimal strategy is the best optimal strategy of all, and let's see if we can find it.

3.2 A Numerical approach

This paper offers a numerical approach to bounding the fastest optimal strategy in stage 1. To do this, we need some new notations.

Given the set of all optimal strategies in stage 1, we have a set of all arrival times, defined as the time it takes the lady to read the borders of stage 1. Since this set is real and has a lower bound (0, for example), it has an infimum T. In other words, there always exists at least one optimal strategy that has an arrival time T or arbitrarily close to it. For the sake of this numerical argument, we will say that some optimal strategy has an arrival time of T. This is the fastest optimal strategy we are looking for, denote it $E(t)$. Define the independent time variable t as the remaining time until arrival. Denote the monster's position as $P(t)$, still. Denote the radius of the borders of stage 1 as R (v_2R in Chapter 2, but R now in Chapter 3 for short).

Finally, for the sake of the following argument, let's put the entire system on a complex plane. Each position inside stage 1 is expressed by a unique complex number $re^{i\Theta}$, where r is defined as the distance to the origin and Θ as π minus the angular distance between the lady and the monster, i.e. $\Theta = 0 - \theta$. We do this because it gives us more convenient notations to work with in the following. Notice that since we define Θ relative to the two players, the coordinate system is also a "rotating" system with respect to the standard Cartesian system.

For the following arguments, please refer to Figure 3.1. Now, fix an arbitrary time $0 < t_0 < T$, and fix the complex plane at t_0. This means that at this very moment, there is t_0 time left until T. Naturally, the angular distance from the monster to himself is 0, thus the monster's position is always on the right-most point, at $P(t) = -R$, where $\Theta = 0 - \pi = -\pi$. Then, with t_0 time remaining, the set of possible angular positions of the monster at time T is $P_\Theta(T) \in [-\pi - t_0/R, -\pi + t_0/R]$ (relative to coordinate system at t_0). Each point in this set is theoretically possible to be reached by the monster at time T. Since under the fastest optimal strategy $E(t)$, the lady wants to be radially opposed to the monster, this means that the lady's current position $E(t_0)$ has to be within reach to every point opposed to the arch $[-\pi - t_0/R, -\pi + t_0/R]$, which is the arch $[t_0/R, -t_0/R]$. But the lady's maximum speed is v_2, which means her maximum displacement in the remaining time t_0 is $v_2 t_0$. Thus, $E(t_0)$ has to be within distance $v_2 t_0$ to every point on the arch $[t_0/R, t_0/R]$ on the borders of the pond. Since the pond is continuous and the distance function is continuous, the lady can reach every point on the arch in time if and only if she can reach the two extreme points in time. Thus, in order to be radially opposed to the monster at time T at the borders of stage 1, the lady has to be within $v_2 t_0$ distance to both $A := Re^{it_0/R}$ and $B := Re^{-it_0/R}$. Further, we also need to make sure that this current position of the lady $E(t_0)$ is reachable itself from the origin. This means we need to make sure $|E(t_0) - 0| = |E(t_0)| \leq v_2(T - t_0)$. Under these conditions, $E(t_0)$ is both reachable from the beginning by the lady, and it is within reachability to being opposed to every possible position the monster can be in at the final time T. Thus, any point that satisfies these conditions is a point on some optimal strategy. Finally, for potentially many such $E(t_0)$s that satisfy, the one with the biggest magnitude (or distance to the origin) is the one that is the fastest, which means it is a candidate that we are looking for.

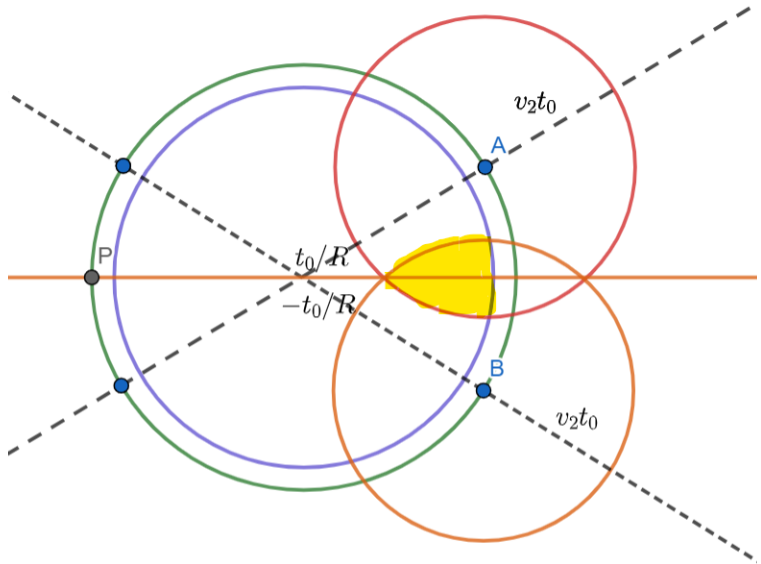

Figure 3.1: Region of optimization

Thus, if we write function $E(t)$ as $E(t) = E_r(t)e^{iE_\theta(t)}$, our problem can be written as a nonlinear programming problem in the following standard form:

Maximize $E_r(t_0)$

subject to

$$|E(t_0) - A|^2 = (E_r\cos(E_\Theta) - R\cos(t_0/R))^2 + (E_r\sin(E_\Theta) - R\sin(t_0/R))^2 \le v_2^2t_0^2 \tag{3.1}$$

$$|P(t_0) - B|^2 = (E_r\cos(E_\Theta) - R\cos(-t_0/R))^2 + (E_r\sin(E_\Theta) - R\sin(-t_0/R))^2 \tag{3.2}$$

$$= (E_r\cos(E_\Theta) - R\cos(t_0/R))^2 + (E_r\sin(E_\Theta) + R\sin(t_0/R))^2 \le v_2^2t_0^2 \tag{3.3}$$

$$E_r \le v_2(T - t_0) \tag{3.4}$$

$$0 < E_r \le R \tag{3.5}$$

$$-\pi < E_\Theta \le \pi \tag{3.6}$$

where we abbreviate $E_r(t_0)$ and $P_\Theta(t_0)$ to E_r and P_Θ.

This nonlinear programming problem seems intimidating algebraically, but it turns out to be an easy problem visually. Geometrically, the problem is equivalent to finding the point with the longest distance to origin inside the intersection of three circles. Visually, we can clearly see that every point on right arc of the borders of the intersection area satisfies (marked in red in Figure 3.2). Denote this arc as $Arc(t_0)$.

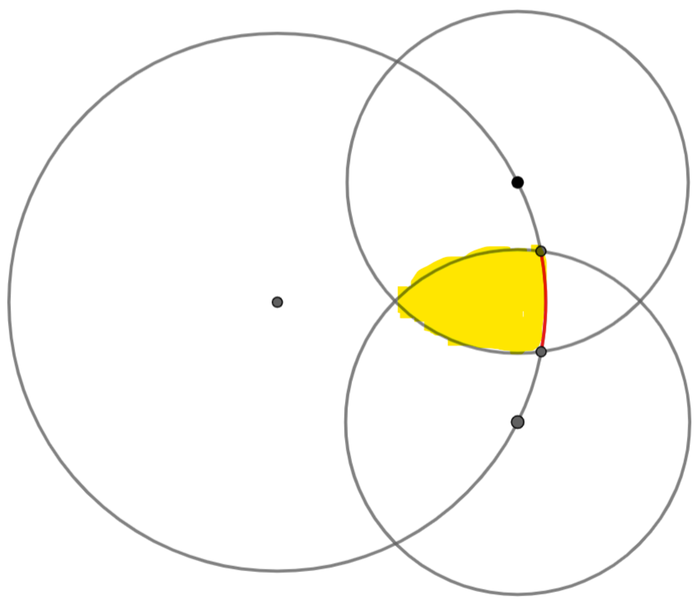

Figure 3.2: Candidates for $P(t_0)$

3.3 One additional caveat–why a naive approach fails

So far, it seems like we have located $E(t_0)$ at every t_0, and the only thing left is to connect them all to make $E(t)$. However, not every point on $Arc(t_0)$ can necessarily be a point on the *same* fastest optimal strategy E. This is because there is one additional condition that $E(t_0)$ needs to satisfy, which is reachability to and from every $E(t)$ for $0 < t \leq T$. Namely, $E(t_0)$ has to be reachable from other points on this strategy before, and it has to be able to reach later points on this strategy as well.

To illustrate, imagine if we superimpose $Arc(t_0)$ as well as $Arc(t_0 - \Delta t)$ (remember t is defined "backwards", as the time remaining) onto the coordinate system at $t = t_0$, we would get something like Figure 3.3. Notice there are two segments of $Arc(t_0 - \Delta t)$ because we don't know which direction the monster is going to move, a priori. Observe that not every point on $Arc(t_0)$ can reach every point on $Arc(t_0 - \Delta t)$. In fact, since the lady has max speed v_2 and the difference between the radii of the two arcs is $v_2 \Delta t$, for every point on $Arc(t_0)$, there is at most one point on $Arc(t_0 - \Delta t)$ that it can reach.

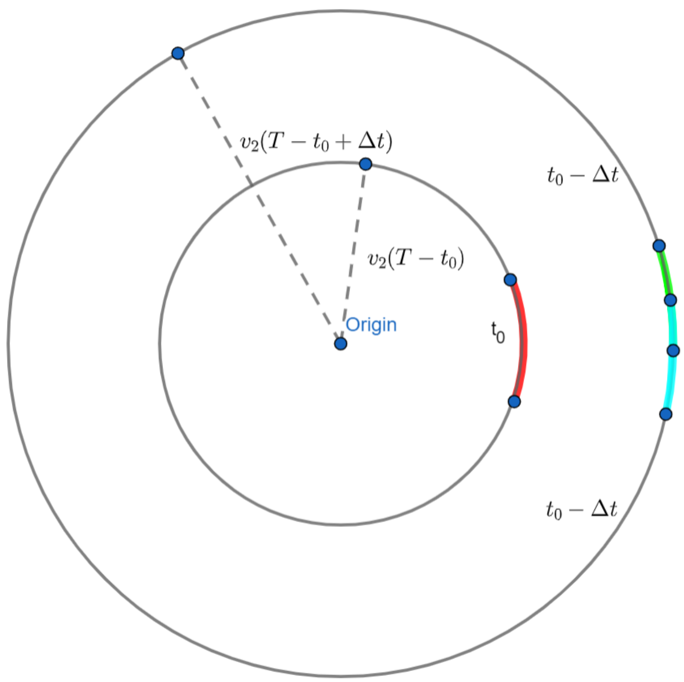

Figure 3.3: $Arc(t_0)$ and $Arc(t_0 - \Delta t)$ on coordinate system t_0

Thus, we cannot simply connect the arcs at each different time to get $E(t)$ because reachability between these arcs breaks. In other words, each point $p(t)$ on some $Arc(t)$ has to be a point on some pretty fast optimal strategy, but they don't have to (and probably won't) be on the same fastest optimal strategy $E(t)$. This is easy to verify: if every point of $E(t)$ is on some $Arc(t)$, then it means the lady is always traveling to the furthest radially (meeting the equality condition in Inequality (3.4)). But this only happens when she travels in a straight line, which we know is not an optimal strategy at all.

3.4 Reachability

Here we need to formally define "reachability" from a point on the fastest optimal strategy E. We will justify this definition after stating it first.

Definition 2. *Supposed $E(t_0)$ is a point on E, and we discretize the game into intervals of Δt. Then another point $z = z_r e^{iz_\Theta}$ is reachable from $E(t_0)$ by the time $t = t_0 - \Delta t$ if and only if $|z \cdot \exp(\textbf{\textit{sign}}(E_\Theta(t_0)) \cdot i\Delta t/R) - E(t_0)| \leq v_2 \Delta t$.*

Essentially, this definition means that after a certain rotation, z should be within $v_2\Delta t$ away from $E(t_0)$. This definition should follow logically from previously discussion, as it accounts for both the lady's movement and the monster's. Since we defined a "rotating" complex plain where the monster's position is still, we need to account for the monster's movement in time Δt by considering it as the lady's additional motion relative to the monster. Visually, the complex plane rotates by $\Delta t/R$ whenever the monster moves in time Δt. Remember since t is defined as remaining time, $t_0 - \Delta t$ is less remaining time and hence more forward in time. Further, the direction of the monster's movement is always towards the lady in his optimal strategy, which means the sign of rotation is always opposite to the sign of $E_\Theta(t_0)$. Thus, the projection of z back to the coordinate system at t_0 is $z \cdot \exp((-1)(-1)\textbf{\textit{sign}}(E_\Theta(t_0)) \cdot i\Delta t/R) = z \cdot \exp(\textbf{\textit{sign}}(E_\Theta(t_0)) \cdot i\Delta t/R)$ (two negative signs cancel out), which should be within $v_2\Delta t$ apart from $E(t_0)$ if it is reachable by the lady. In reality, we can expect the sign to be consistent, since we can reasonably assume that the lady stays in one half-plane only, by symmetry, given that Δt is small enough for her to always be able to navigate properly.

Once reachability is defined and considered, we have the final piece to finishing a complete description to the problem. In addition to the nonlinear programming problem that we stated in

Section 3.2, we need to add another recursive condition that $E(t_0)$ is reachable from its previous point, $E(t_0 + \Delta t)$. Namely, we should add the following constraint to the programming problem:

$$|E(t_0 - \Delta t)\exp(\textit{sign}(E_\Theta(t_0))i\Delta t/R) - E(t_0)| \leq v_2 \Delta t \tag{3.7}$$

3.5 The final algorithm

Thus, our final model for finding the fastest optimal strategy of the lady in stage 1 is the following recursive nonlinear programming problem:

Maximize $E_r(t_0)$

subject to

$$|E(t_0) - A|^2 = (E_r \cos(E_\Theta) - R\cos(t_0/R))^2 + (E_r \sin(E_\Theta) - R\sin(t_0/R))^2 \leq v_2^2 t_0^2 \tag{3.8}$$

$$|P(t_0) - B|^2 = (E_r \cos(E_\Theta) - R\cos(-t_0/R))^2 + (E_r \sin(E_\Theta) - R\sin(-t_0/R))^2 \tag{3.9}$$

$$= (E_r \cos(E_\Theta) - R\cos(t_0/R))^2 + (E_r \sin(E_\Theta) + R\sin(t_0/R))^2 \leq v_2^2 t_0^2 \tag{3.10}$$

$$E_r \leq v_2(T - t_0) \tag{3.11}$$

$$|E(t_0 - \Delta t)\exp(\textit{sign}(E_\Theta(t_0))i\Delta t/R) - E(t_0)| \leq v_2 \Delta t \tag{3.12}$$

$$0 < E_r \leq R \tag{3.13}$$

$$-\pi < E_\Theta \leq \pi \tag{3.14}$$

where we abbreviate $E_r(t_0)$ and $P_\Theta(t_0)$ to E_r and P_Θ.

To carry out this algorithm, we would start with $P(0) = R$, since the lady has to end with the maximum payoff where she is directly opposed to the monster. Then we can start the recursion by setting a step size Δt. At each step, perform the nonlinear programming problem numerically to find E_r and E_Θ, which will then be used in the next iteration as constants in the next nonlinear programming problem.

The final importance notice is with respect to T. A priori, we don't know what it is. But realistically, when carrying out the algorithm, we can start with a T value of R. This is because if the lady travels a straight line, the arrival time is R, which means it is a lower bound on T. If R is too small for T-value, the nonlinear programming problem should return false, since it would be an over-limiting case where the boundary area is empty. In this case, increase R by some step size ΔT until the programming problem returns a point.

For some additional notes with respect to potentially making the problem more easily computable, see Appendix 2.

Chapter 4

Conclusion and Comparison

4.1 Summary of results

In Chapter 2, we have found that a best strategy for the lady is

$$\phi^*(r,t) = \begin{cases} \arcsin(\frac{r}{v_2 R}) & for \quad r \leq v_2 R \\ \arcsin(\frac{R v_2}{r}) & for \quad r > v_2 R \end{cases} \tag{4.1}$$

where the monster's best strategy is $-1/R$. This is the Nash Equilibrium of the game, and the value of the game is

$$|\theta| = \arccos(v_2) - \frac{1}{v_2}\sqrt{1 - v_2^2} + \pi \tag{4.2}$$

Then, in Chapter 3, we gave a recursive nonlinear programming problem as a numerical way to determine the fastest optimal strategy of the lady in stage 1, which is the fastest strategy overall, since the optimal strategy in stage 2 is unique (up to symmetry).

4.2 Discussion

Classically, the Lady in the Lake problem is treated as a prototype two-player pursuit-evasive differential game. Thus, in the classic model, two strategies are involved as variables in the differential model, namely, the lady's strategy (u^2 in notation of [1]) and the monster's strategy (u^1). As a result, the problem is modeled as a minimax optimization problem with respect to u^2 and u^1, which requires a complicated tool like the Isaacs Equation. Only after solving the Isaacs Equation can one discover that the optimal strategy of the monster $u^1* = sign(\theta)$.

However, this paper notices that instead of having infinite strategies on $(-\pi, \pi]$ like the lady, the monster essentially only has two discrete instantaneous strategies, going clockwise or counterclockwise. Thus, if we can simplify the model by determining a fixed strategy for the monster, then the problem becomes a minimization problem with respect to only one variable u^1, which is a lot easier to solve. And the trick that enables this simplification turns out to be using $|\theta|$ as the payoff of the game, instead of θ. Another factor that makes this approach easier is that it proves and uses some very helpful intuitions. For example, we proved that it is never beneficial for the lady to turn back. By using these intuitions we made the minimization process much more straight-forward.

In addition, it has long been noticed that the optimal strategy in this game is non-unique (for example, by [2]). This paper takes an innovative numerical approach to look for the fastest of all the optimal strategies. This numerical algorithm is not perfect, since it relies on two step sizes Δt and ΔT from the recursion process and the nonlinear programming problem at each step, respectively. Both layers of algorithm could generate errors, and they could stack together. The dynamics of the stacked error could deserve further study. But the algorithm is both feasible (as techniques for nonlinear programming and recursion already exist) and computable.

Appdendix 1: A geometric way of determining the lady's payoff function with respect to dr

Referring back to the geometry in Figure 1.2 we have $AB \approx AD$. Thus, assuming all angles are positive, we have

$$AB = AD \tag{3}$$

$$d\theta k = DE \times \tan(\phi) \tag{4}$$

$$d\theta(r + dr) = [dr - AB\sin(little)]\tan(\phi) \tag{5}$$

$$d\theta r + d\theta dr = dr\tan(\phi) - AB\sin(d\theta/2)\tan(\phi) \tag{6}$$

We established before that $\angle little \approx d\theta/2$. As $d\theta$ approaches 0, we know that $lim\frac{\sin d\theta}{d\theta} = 1$, so we can approximate $\sin(little)$ as $d\theta/2$. And, since $AB = d\theta(r + dr)$, we plug them in and get

$$d\theta r + d\theta dr = dr\tan(\phi) - d\theta(r + dr)(d\theta/2)\tan(\phi) \tag{7}$$

As both $d\theta$ and dr approach 0, we can drop all those terms have have two or more delta terms. Thus, when we take the limit as $d\theta, dr \to 0$ we get

$$d\theta r = dr\tan(\phi) \tag{8}$$

$$\frac{d\theta}{dr} = \tan(\phi)/r \tag{9}$$

Thus, in stage two our payoff function (involving only the lady's choice) θ can be written as

$$\theta = \int_{v_2 R}^{R} \frac{\tan(\phi)}{r} dr \tag{10}$$

Appdendix 2: A potential method to make the numerical nonlinear programming problem more computable

In the final model given in Chapter 3, the nonlinear programming problem at each iteration involves a lot of constraints and thus maybe hard to compute numerically. A possible idea to make the problem more computable is to get ride of constraint (3.14), $-\pi < E_\Theta \leq \pi$, and instead change the objective function to maximize $100 \times E_r(t_0) - |E_\theta(t_0)|$. This would get rid of two constraints, and the numerical process might become easier, depending on the situation. This trick works because we give E_r a really big weight with respect to E_Θ, and thus adding $|E_\Theta|$ shouldn't change the final solution by much. However, by adding the $|E_\Theta|$ term in the objective function, we do make sure that the final solution of $|E_\Theta|$ has the smallest absolute value, which is between 0 and π, as we desire (since $-\pi$ to π is a full period).

Bibliography

[1] Chapter 8: Pursuit-evasion games. In Tamer Başar and Geert Jan Olsder, editors, *Dynamic Noncooperative Game Theory*, volume 160 of *Mathematics in Science and Engineering*, pages 344–398. Elsevier, 1982.

[2] Maurizio Falcone. Numerical methods for differential games based on partial differential equations. *International Game Theory Review (IGTR)*, 08:231–272, 06 2006.

[3] Henry (https://math.stackexchange.com/users/6460/henry). A lady and a monster. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/31129 (version: 2011-04-06).

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- The Shibley Award, 2022.
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- H. Freeman Stecker Scholarship, 2021-2022.
  ----Awarded to one of the top graduating students in mathematics
- The Evan Pugh Scholar Senior Award of Penn State, 2022.
  ----Awards 0.5% of the graduating class
- The President Sparks Award of Penn State, 2020-2021.
- The President's Freshman Award, 2019.
- Deans List, 2018-2022.

Selected Past Academic Projects in Math

1. Lady in the Lake Problem, 2022, Penn State. Click for link.
2. The Brachistochrone Problem with a functional calculus approach, 2022, Penn State. Click for link.
3. Coding Theory: Proof of column-wise formula for symbol error rate of a binary linear code, 2020, Penn State. Click for link.
4. Coding Theory: Maximum of the number of words of a linear binary (2d,d) code, 2020, Penn State. Click for link.
5. Coding Theory: Construction of linear binary code to correct specific errors, 2020, Penn State. Click for link.
6. Numerical Analysis: Code for solving definite integrals numerically using adaptive Simpson's rule, 2020, Penn State. Click for link.
7. Geometry: An Introduction to Hyperbolic Geometry, 2020, Penn State, for Math 497H student colloquium course. Click for link.
8. Using Laplace Transforms to Solve Wave Equations, With Jonah Glunt, 2019, Penn State. Click for link.

Selected Past Academic Papers in Philosophy

1. A Kantian Defense of "Fact of Reason" Against Social Constructivist's Critique, 2021, Penn State. Click for link.
2. Freedom and Human Specialness, 2020, Penn State. Click for link.
3. Two final essays in African American philosophy, 2021, Penn State. Click for link.
4. Addressing the Weird and the Alien Through the Fantastic and Waldenfels, 2019, Penn State. Click for link.
5. Hatred as a "Relativistic Encounter", 2019, Penn State. Click for link.

Leadership

English 138H, Public Deliberation Leader
2019

- Initiated a public deliberation on college mental health issues with my group
- Navigated the conversation in my discussion group
- Prepared the issue guide, infographs, questionnaires and notes for my discussion group
- Concluded the results and wrote a report

Northland Christian High School Debate Team, Assistant Coach
2019,2020

- Coached the debate team at national tournaments: Minneapolis in 2019, Berkeley in 2020, NSDA nationals 2020, as well as many online tournaments.

Relevant Courses Taken Before

- **Abstract math**
  Linear Algebra, Real Analysis (Honors), Classical Analysis, Complex Analysis, Elementary Topology, Mathematical Logic, Probabilities (Honors), Statistics, Numerical Analysis (Honors), Abstract Algebra, Foundations of Geometry, Discrete Math.
- **Applied math**
  Mathematical Modeling, Multivariable Calculus (Honors), Elementary ODE&PDE (Honors), Linear Programming, Game Theory.
- **Philosophy**
  Kant's Moral Theory, Modern Philosophy, 20^{th} Century Philosophy, 19^{th} Century Philosophy, German Philosophy, European Philosophy on Dialogue, African American Philosophy, Elementary Ancient Greek Philosophy, Philosophy of Technology, Intro to Epistemology.
- **CS-related**
  Python, Abstract Coding theory.
- **Science-related**
  Physics (Mechanics, Electro-magnetic, Elementary Fluids, Thermal, Elementary Quantum), Philosophy of Technology, Technical writing, Elementary Chemistry.
- **Other courses in different areas.**

Skills

Python, Matlab.
Fluent in Chinese and English.