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AN ALTERNATIVE APPROACH TO THE WICK ROTATION THROUGH PICARD-LEFSCHETZ THEORY

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Abstract

Quantum field theory (QFT) is the foundational toolbox of contemporary physics because of its amazing track record against experiments. QFT's powerful path integral formalism allows for extraordinary perturbative calculations, however it is computationally based on the Wick rotation. The Wick rotation transforms the time coordinate to imaginary time which changes the symmetry transformations of our spacetime. This thesis examines the application of Picard-Lefschetz theory to the computation of path integrals and their correlation functions without changing the symmetries of the spacetime.

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Chapter 1

Path Integrals in Quantum Theory

1.1 Derivation of the Path Integral

Quantum mechanics does not predict concrete paths like classical mechanics, however the time evolution of the system is still determined by its Hamiltonian H. We promote the Hamiltonian to an operator, $\hat{H}$, which allows us to evolve states according to the schrödinger equation:

$$i\partial_t|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{1.1}$$

Since the Schrödinger equation is first order, we can formally solve it:

$$|\psi(t)\rangle = \exp(-i\hat{H}t)|\psi(0)\rangle \tag{1.2}$$

for some normalized initial state $|\psi(0)\rangle$. The operator $\exp(-i\hat{H}t)$ describes the time evolution of a system. The transition amplitude for a quantum particle to go from a position state $|q_i\rangle$ at time t_i to $|q_f\rangle$ at time t_2 is given by:

$$\mathcal{W} = \langle q_f|\exp(-i\hat{H}(t_f - t_i))|q_i\rangle \tag{1.3}$$

The Dirac method [1] was devised as a method for computing transition amplitudes by taking a sum over all possible paths from q_i to q_f. This is the conceptual basis for the path integral formalism of quantum mechanics.

Imagine evolving the particle from t_i to some intermediate time t' then finishing the evolution to t_f. This intermediate evolution can be incorporated into the amplitude by inserting a complete set of states for $|q'\rangle$:

$$\mathcal{W} = \int \langle q_f|\exp(-i\hat{H}(t_f - t'))|q'\rangle\langle q'|\exp(-i\hat{H}(t' - t_i))|q_i\rangle dq' \tag{1.4}$$

What if, instead, we propagate to another intermediate time t'' between t' and t_f? Or, propagate to any number of intermediate times between t_i and t_f? Let's formalize this idea.

Consider a uniform partition of our time interval into N pieces of length δt. This allows us to insert N complete sets of states into our amplitude:

$$\mathcal{W} = \int \langle q_f|\exp(-i\hat{H}\delta t)|q_{N-1}\rangle\langle q_{N-1}|\exp(-i\hat{H}\delta t)|q_{N-2}\rangle \dots \langle q_1|\exp(-i\hat{H}\delta t)|q_i\rangle \prod_{j=1}^{N-1} dq_j \tag{1.5}$$

For each intermediate transition amplitude, we insert a complete set of momentum states:

$$\langle q_{j+1}|\exp(-i\hat{H}\delta t)|q_j\rangle = \int e^{-ip_j q_j}\langle q_{j+1}|\exp(-i\hat{H}\delta t)|p_j\rangle \frac{dp_j}{2\pi} \tag{1.6}$$

We cannot directly act the position and momentum states on the time evolution operator. However, if we assume N large enough, we can take δt to be infinitesimal and approximate $\exp(-i\hat{H}\delta t) = 1 - i\hat{H}\delta t$. We can act then the states on the Hamiltonian directly:

$$\langle q_{j+1}|\exp(-i\hat{H}\delta t)|q_j\rangle = \int e^{ip_j(q_{j+1}-q_j)}(1 - iH\delta t)\frac{dp_j}{2\pi} \tag{1.7}$$

This expression allows us to express our amplitude as:

$$\mathcal{W} = \int e^{i\sum_{j=1}^{N-1} p_j \frac{q_{j+1}-q_j}{\delta t}\delta t}(1 + \frac{-iH(t_f - t_i)}{N})^N \prod_{j=1}^{N-1} \frac{dq_j dp_j}{2\pi} \tag{1.8}$$

When we take the limit as $N \to \infty$ and $\delta t \to 0$ such that $t_f - t_i$ remains constant, we recover an integral in the exponential:

$$\mathcal{W} = \int \exp(i \int_{t_i}^{t_f} p\dot{q} - H dt)\mathcal{D}[q]\mathcal{D}[p] \tag{1.9}$$

We can recognise the exponential as the Lagrangian since it is the Legendre transform of the Hamiltonian. Under the Legendre transform, the integral loses all momentum dependence, so we can absorb the momentum integral into the position measure. This allows us to express the transition amplitude as an integral over the space of paths between $q_i(t_i)$ to $q_f(t_f)$. Hence, we have the path integral:

$$\mathcal{W} = \int \exp(iS[q])\mathcal{D}[q] \tag{1.10}$$

It should be noted that the action is classical. The integral can be thought of as a sum of all possible paths between q_i and q_f weighted by a factor $e^{iS[q]}$. The path integral also allows for easy access to the classical limit. If we determine the path that is the extremum of the action, this gives the dominating factor of the integral. This dominating path corresponds exactly to $\delta S = 0$ which yields the classical equations of motion.

The path integral is useful in computing observables [1]. In the traditional formalisms of quantum mechanics, observables are computed by projecting a hermitian operator onto the states:

$$\langle \hat{\mathcal{O}} \rangle = \langle q_f | \hat{\mathcal{O}} | q_i \rangle \tag{1.11}$$

We can translate this into the language of path integrals:

$$\langle \hat{\mathcal{O}} \rangle = \frac{1}{\mathcal{W}} \int \mathcal{O}[q] e^{iS[q]} \mathcal{D}[q] \tag{1.12}$$

Using the path integral to compute observables makes its visually similar to the partition function of statistical mechanics. We will explore this relationship further later.

Quantum mechanics has its short comings when dealing with processes that do not conserve particle number. The theory that supersedes quantum mechanics is quantum field theory. In order to generalize, we can use the path integral and upgrade the action from classical mechanics to classical field theory. For quantum field theory[1], we can write our path integral as:

$$\mathcal{Z} = \int \exp(iS[\phi])\mathcal{D}[\phi] \tag{1.13}$$

In quantum field theory, the observable quantities are related to time ordered correlation functions. These correlation functions are related to the path integral by:

$$\langle 0 | T\{\phi(x_1) \dots \phi(x_n)\} | 0 \rangle = \frac{1}{\mathcal{Z}} \int \phi(x_1) \dots \phi(x_n) e^{iS[\phi]} \mathcal{D}[\phi] \tag{1.14}$$

[1]For this thesis, our discussion will stay in the realm of scalar field theory.

We can inject a fictitious term $i \int J(x)\phi(x)d^4x$ and simplify the expression with functional derivatives:

$$\frac{1}{\mathcal{Z}} \int \phi(x_1)\ldots\phi(x_n)e^{iS[\phi]}\mathcal{D}[\phi] = \frac{(-i)^n}{\mathcal{Z}[0]} \frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)} \int e^{iS[\phi]+i\int J(x)\phi(x)d^4x}\mathcal{D}[\phi]|_{J=0} \tag{1.15}$$

If we could compute the path integral, we would be able to compute arbitrary n-point correlation functions. Unfortunately, the oscillatory nature of the path integral makes computation difficult. The direct application of numerical methods, particularly Monte Carlo methods, break down and give non-convergent results. The standard approach is to introduce a coordinate transformation known as the Wick rotation. The Wick rotation is an imaginary time transformation $t \to it$. The path integral transforms simply by effectively changing a Minkowski metric to a Euclidean metric:

$$\mathcal{Z} = \int e^{-S[\phi]}\mathcal{D}[q] \tag{1.16}$$

While the Wick rotation has the advantage of turning our oscillatory integral into a damped exponential, the coordinate transformation fails to be completely justified. The Minkowski metric is invariant under the action of the Poincaré group[2]: translations, rotations, and Lorentz boosts. Likewise, the Euclidean metric is invariant under the action of the Euclidean group[3]: translations and rotations. Even though these groups have the same number of generators, the symmetry transformations of the spaces are different. For there to be a proper relationship between the metrics, the symmetry transformations must be isomorphic.

The goal of this thesis is to provide an alternative method to the Wick rotation. This method is based on Picard-Lefschetz theory [2] and we seek to provide an analysis of the technique. We will examine its application to the free Klein-Gordon theory, a quartic field theory, and a toy model of gravity in $1+1$ spacetime dimensions. Each of these theories will lay the foundation for the possible viability of Picard-Lefschetz theory in a path integral formalism of quantum gravity.

[2]Formally, the Poincaré group is the semidirect product of the translation and orthogonal group $T(3,1) \rtimes O(1,3)$.
[3]Formally, the Euclidean group is the semidirect product of translations and orthogonal group $T(4) \rtimes O(4)$.

Chapter 2

Picard-Lefschetz Theory

2.1 Contour Integration

Picard-Lefschetz theory seeks to understand the properties of complex manifolds by studying the critical points of holomorphic functions on them. We want to use Picard-Lefschetz theory to change the integration surface of our path integral in order to better its convergence properties and to demand less of the intrinsic geometry. In order to see how Picard-Lefschetz theory can remove the need for the Wick rotation, consider an integral of the form:

$$\int_0^\infty e^{if(x)}dx \tag{2.1}$$

we assume that f is a well-behaved and without poles.

To properly situate this integral in the complex plane [3], we analytically continue the variable of integration, $x \to z = x + iy$. This continuation guarantees that $f(z)$ is holomorphic, complex differentiable, since $\partial_{\bar{z}} f = 0$. Since f is holomorphic, we can take any closed curve $\mathcal{C} \subset \mathbb{C}$ and apply Cauchy's theorem:

$$\oint_{\mathcal{C}} e^{if(z)}dz = 0 \tag{2.2}$$

Holomorphicity also allows for us to deform $\mathcal{C}$ to any other closed contour $\mathcal{C}'$ by the deformation theorem.

Therefore, let us pick a closed curve that bounds a sector of a circle centered at the origin. Part of the curve goes along the real axis out to some radius R, the next part is an arc around the circle by some angle θ, and the curve is closed by a path coming from the arc to the origin as depicted in Figure 1. This splits our Cauchy integral into three parts:

$$\int_0^R e^{if(x)}dx + \int_0^\theta e^{if(Re^{i\phi})+i\phi} iRd\phi + \int_R^0 e^{if(\gamma(t))}\gamma'(t)dt = 0 \tag{2.3}$$

where $\gamma(t)$ is the path that takes the arc back to the origin. We will use Picard-Lefschetz theory to determine $\gamma(t)$ [4, 2]. This third integral is oscillatory and we would like to find a contour that keeps the imaginary part of the exponential constant. We split $f(z)$ into real and imaginary parts:

$$f(x,y) = H(x,y) + ih(x,y) \tag{2.4}$$

We want to find the critical points of the real part of f which are given by:

$$\partial_z H(x_0,y_0) = \partial_x H(x_0,y_0) - i\partial_y H(x_0,y_0) = 0 \tag{2.5}$$

Since f is holomorphic, it forces the real part of f to satisfy the Laplace equation:

$$(\partial_x^2 + \partial_y^2)H = 0 \tag{2.6}$$

This allows us to conclude that the critical points of H are always saddle points. Picard-Lefschetz theory defines our integration contour $\gamma(x) = x + iy(x)$ by:

$$H(x,y) = H(x_0,y_0) \tag{2.7}$$

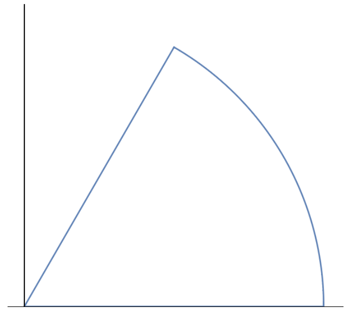

Figure 2.1: This figure depicts a generic contour that was utilized to compute our integrals.

This contour forces $H(x, y(x))$ to always be the fixed value of $H(x_0, y_0)$. This in turn forces the imaginary part of if to be constant along this contour, removing all imaginary contributions to the integral. We can apply this contour to our Cauchy integral:

$$\int_0^R e^{if(x)}dx + \int_0^\theta e^{if(Re^{i\phi})+i\phi} iRd\phi = e^{iH(x_0,y_0)} \int_0^R e^{-h(\gamma(t))}\gamma'(t)dt \tag{2.8}$$

When we take the limit as $R \to \infty$, we use Jordan's lemma to see that this integral vanishes as long as $R\exp\bigl(-\mathrm{Im}[f(Re^{i\phi})]\bigr) \to 0$[1], leading to our original integral related to a damped exponential:

$$\int_0^\infty e^{if(x)}dx = e^{iH(x_0,y_0)} \int_0^\infty e^{-h(\gamma(t))}\gamma'(t)dt \tag{2.9}$$

Generically, H can have multiple critical points and multiple contours. In the example given in section 2.2.1, we can see that H has one critical point that produces multiple contours. We exclude one of these contours on the basis that it would make the exponential of our Gaussian positive, and therefore a divergent integral. However, there is another reason why we can exclude this contour: the arc connecting the contour to the positive real axis does not satisfy the condition that allows the angular integral to vanish.

2.2 Worked Examples

In this section, we will derive two primary integrals of interest that will be useful in calculating path integrals and demonstrate how to apply Picard-Lefschetz theory in practice.

2.2.1 Gaussian Integrals

The primary example of our program in action is the Gaussian integral with an imaginary coefficient [1]:

$$\int_0^\infty e^{iax^2}dx \tag{2.10}$$

[1] If the integration takes place in the lower half plane, $i \to -i$, we flip the sign on the imaginary part. We can work in the upper half plane without loss of generality.

When we pass $x \to z$, the real and imaginary parts of z^2 are given by:

$$H(x, y) = a(x^2 - y^2) \tag{2.11}$$

$$h(x, y) = 2axy \tag{2.12}$$

The critical point of the real part is given by $(x_0, y_0) = (0, 0)$ which gives two relations $y(x) = \pm x$. Therefore, our possible curves are $\gamma(t) = (1 \pm i)t$. We pick the positive solution since it keeps the resulting integral convergent:

$$\int_0^\infty e^{iax^2} dx = \int_0^\infty e^{-2at^2}(1 + i)dt = \frac{1}{2}\sqrt{\frac{i\pi}{a}} \tag{2.13}$$

We can generalize this by adding a linear term ibx to the integral:

$$\int_0^\infty e^{i(ax^2+bx)} dx = \frac{1}{2}\sqrt{\frac{i\pi}{a}} e^{-i\frac{b^2}{4a}} \tag{2.14}$$

This integral can be derived by completing the square and using the same contour as above.

2.2.2 Gaussian Moment Integrals

While the Gaussian integral is important from a pedagogical point of view, the following generalization is useful for perturbative calculations:

$$\int_0^\infty x^\alpha e^{ix^\beta} dx \tag{2.15}$$

We will still analytically continue $x \to z$, but we will make use of the polar representation of complex numbers to simplify much of the algebra. The real part and imaginary part of x^β is given by:

$$H(r, \theta) = r^\beta \cos(\beta\theta) \tag{2.16}$$

$$h(r, \theta) = r^\beta \sin(\beta\theta) \tag{2.17}$$

The critical points of H have a degeneracy because any $r = 0$ always vanishes both derivatives but no value of θ will vanish either by itself. However, this is expected since $r = 0$ does not have a well defined polar angle.[2] Using $r = 0$, we can still devise a contour by a fixed $\theta = (n + \frac{1}{2})\frac{\pi}{\beta}$. In Cartesian coordinates, the contour is given by:

$$y(x) = \tan((n + \frac{1}{2})\frac{\pi}{\beta})x \tag{2.18}$$

The imaginary part is now given by:

$$h(r) = (-1)^n r^\beta \tag{2.19}$$

[2] If we stuck with Cartesian coordinates, we would see that system of partial derivatives does not have a solution. Polar coordinates shows us the solution should be $(0, 0)$.

In order to keep our integral convergent, we restrict ourselves to $n = 2k$. We also have to worry about the angular integral that connects $\gamma(t) = (1 + i\tan[(4k+1)\frac{\pi}{2\beta}])t$ to the positive real line. Since $\tan x$ is π-periodic, different values of k define an equivalence class of contours. It is easy to check that when $\beta = 1$ the tangent diverges, which corresponds to our segment returning to the positive real axis from the positive imaginary axis. When $\beta > 1$, the only value of k which always gives a positive sign is $k = 0$. When we put everything into our integral:

$$\int_0^\infty x^\alpha e^{ix^\beta} dx = (1 + i\tan(\frac{\pi}{2\beta}))^{\alpha+1} \int_0^\infty x^\alpha e^{-(\sec[\frac{\pi}{2\beta}]x)^\beta} dx \tag{2.20}$$

When we rescale the integral $x \to \cos[\frac{\pi}{2\beta}]x$, the unwieldy coefficient simplifies very nicely:

$$\int_0^\infty x^\alpha e^{ix^\beta} dx = i^{\frac{\alpha+1}{\beta}} \int_0^\infty x^\alpha e^{-x^\beta} dx \tag{2.21}$$

This final integral is a Gamma function in disguise:

$$\int_0^\infty x^\alpha e^{ix^\beta} dx = i^{\frac{\alpha+1}{\beta}} \frac{1}{\beta}\Gamma(\frac{\alpha+1}{\beta}) \tag{2.22}$$

Since the coefficient is power of i, it can easily be a root. When we picked $k = 0$, this fixed the value of the roots of i to be their value on the principle branch of $\mathbb{C}$.

2.3 Integration Surface Stability

Through the previous sections, we developed the application of Picard-Lefschetz theory to compute complex integrals of a given form. We only developed this theory for one-dimensional integrals. Our end goal is to apply this method to path integrals, which are infinite dimensional in a sense. This opens the question: if we have a multidimensional integral, does it matter what integration variable we analytically continue first? The short answer is no it does not matter, which we will prove.

2.3.1 Two-Dimensional Case

Our proof will be inductive. We will first show that a two-dimensional integral does not depend on the order of analytic continuation. We will then show that it holds for N-dimensions. Consider a general two-dimensional integral:

$$\iint e^{if(x,y)} dxdy \tag{2.23}$$

We do not need to work with the function itself, but only infinitesimally around the origin. Therefore, we use the Taylor series of f:

$$f(x,y) \approx f(0) + f_x(0)x + f_y(0)y + \frac{1}{2}f_{xx}(0)x^2 + f_{xy}(0)xy + \frac{1}{2}f_{yy}(0)y^2 \tag{2.24}$$

We drop the functional dependence to simplify the expressions. The integral we will be analyzing is:

$$\iint \exp(i(f_x x + f_y y + \frac{1}{2}f_{xx}x^2 + f_{xy}xy + \frac{1}{2}f_{yy}y^2))dxdy \tag{2.25}$$

The constant term will not affect our analysis, so we will drop it.

First, we analytically continue $x \to x + ix'$. The real and imaginary parts of the exponential are given by:

$$H(x, x', y) = f_x x + f_{xx}(x^2 - x'^2) + f_y y + f_{xy} xy + f_{yy} y^2 \tag{2.26}$$

$$h(x, x', y) = -f_x x' - f_{xx} x x' - f_{xy} y x' \tag{2.27}$$

We can solve for the critical point of H to be $(-\frac{1}{2f_{xx}}(f_{xy}y + f_x), 0, 0)$ which yields a contour in the $xx'y$ space:

$$x'(x, y) = \pm \frac{1}{2f_{xx}}(f_x + 2f_{xx}x + f_{xy}y) \tag{2.28}$$

With our $xx'y$ contour in hand, let's consider our integral and analytically continue $y \to y + iy'$. The real and imaginary parts are given by:

$$G(x, y, y') = f_x x + f_{xx} x^2 + f_y y + f_{xy} xy + f_{yy}(y^2 - y'^2) \tag{2.29}$$

$$g(x, y, y') = (-f_y - f_{xy} x - f_{xy} f_{yy}) y' \tag{2.30}$$

The critical point of G is given by $(0, -\frac{1}{2f_{yy}}(f_y + f_{xy}x), 0)$ which yields a contour in xyy' space:

$$y'(x, y) = \pm \frac{1}{2f_{yy}}(f_y + 2f_{yy}y + f_{xy}x) \tag{2.31}$$

In the whole analytically continued space $xx'yy'$, we have two hypersurfaces in $\mathbb{C}^2$. These hypersurfaces contain the one parameter family of curves determined by the contours.

$$\gamma^y(t) = (t, \pm \frac{1}{2f_{xx}}(f_x + 2f_{xx}t + f_{xy}y), y, 0) \tag{2.32}$$

$$\gamma^x(t) = (x, 0, t, \pm \frac{1}{2f_{yy}}(f_y + 2f_{yy}t + f_{xy}x)) \tag{2.33}$$

There are tangent vectors at each point along these curves for any $(x, y) \in \mathbb{R}^2$. These tangent vectors allow us to define a vector field on each of the hypersurfaces.

$$\mathcal{X} = \pm \frac{f_{xy}}{2f_{xx}} \partial_{x'} + \partial_y \tag{2.34}$$

$$\mathcal{Y} = \partial_x \pm \frac{f_{xy}}{2f_{yy}} \partial_{y'} \tag{2.35}$$

These vector fields have integral curves that correspond to the possible integration contours. If these two vector fields commute, then their integral curves flow together [5]. This means that if we follow γ^y then go along γ^x, this is equivalent to going along γ^x then going along γ^y, hence answering our stability question.

The commutator of two vector fields, also known as the Lie derivative, is given by:

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X} \tag{2.36}$$

Since our vector fields are only of coordinate derivatives with no coordinate dependence, the Lie derivative is trivial:

$$[\mathcal{X}, \mathcal{Y}] = 0 \tag{2.37}$$

Therefore, our vector fields commute, the integral curves flow together and we have stability in two dimensions.

2.3.2 N-Dimensional Case

The N-dimensional case follows from an observation that when we have an N-dimensional integral, we can generate N vector fields in our $\mathbb{R}^{2N}$ space. Each of these vector fields mutually commute since they are two-dimensional subspaces:

$$[\mathcal{X}_a, \mathcal{X}_b] = 0 \tag{2.38}$$

Therefore, we are agnostic about the order of analytic continuation in any number of dimensions. Additionally, any i'th vector field commutes with the product of any number of other vector fields since they all mutually commute:

$$[\mathcal{X}_i, \prod_j \mathcal{X}_j] = \mathcal{X}_i \prod_j \mathcal{X}_j - \prod_j \mathcal{X}_j \mathcal{X}_i = 0 \tag{2.39}$$

Since we have commutativity in N dimensions, then we can reasonably conclude that our Picard-Lefschetz program holds for functional integrals.

Chapter 3

Application to Path Integrals

3.1 Introduction

In the previous two chapters, we have developed both the theory of path integrals and Picard-Lefschetz theory. With both theories in hand, we now have all of the tools at our disposal to compute a path integral. In this chapter, we will discuss the calculation of the path integral and the correlation functions for three theories: the free field theory, a quartic field theory, and $1+1$ dilaton gravity.

3.2 Klein-Gordon Theory

The simplest relativistic field theory is described by single scalar field $\phi(x)$ of mass m is known as Klein-Gordon theory. For our future convenience, we will introduce a fictitious source field $J(x)$[1]:

$$S[\phi] = \int \Big(-\frac{1}{2}\partial^\mu \phi(x)\partial_\mu \phi(x) - \frac{1}{2}m^2\phi^2(x) - J(x)\phi(x)\Big) d^4x \tag{3.1}$$

We can recognize that the action is quadratic in field. The simplicity of this theory manifests from the path integral being quadratic in the field, which allows us to utilize our already developed theory of integration with Picard-Lefschetz theory:

$$\mathcal{Z} = \int \exp\Big(-i\int \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + J\phi d^4x\Big)\mathcal{D}[\phi] \tag{3.2}$$

The trick to evaluating path integrals is to approximate the action as sum over a lattice with spacing a. On the lattice, the derivative operator becomes a finite linear operator, which we can put in its eigenbasis. This lattice approximation turns the infinite dimensional path integral into an N dimensional Gaussian integral:

$$\mathcal{Z}_N = \int \exp\Big(-\frac{i}{2}a^4\sum_{n\in\mathbb{R}^4}^{N}(\lambda_n + m^2)\phi_n^2 + 2J_n\phi_n\Big)d\phi_n \tag{3.3}$$

We pull the sum out of the exponential and integral because the value of the field at two distinct points are independent. Therefore, our path integral is a product of N Gaussian integrals:

$$\mathcal{Z}_N = \prod_{n\in\mathbb{R}^4}^{N}\int \exp\Big(-\frac{i}{2}a^4([\lambda_n + m^2]\phi_n^2) + 2J_n\phi_n\Big)d\phi_n \tag{3.4}$$

We have previously evaluated this integral in section 2.1.1. We have also shown that all of these integrals are, in fact, independent after analytic continuation via our result of section 2.2. This allows us to write our path integral as a product of exponentials:

$$\mathcal{Z}_N = \prod_{n\in\mathbb{R}^4}^{N}\sqrt{\frac{-2\pi i}{a^4(\lambda_n + m^2)}}\exp\big(i\frac{J_n^2}{2(\lambda_n + m^2)}a^4\big) \tag{3.5}$$

[1] We will drop the functional dependence on x from here on out.

When we take the continuum limit, which is equivalent to the limits $N \to \infty$ and $a \to 0$. The eigenvalues of the derivative operator, $(\lambda_n + m^2)$, limits to the Klein-Gordon operator $\mathcal{M} = \partial^\mu \partial_\mu - m^2$. This allows us to write the path integral in terms of the Green's function $\mathcal{M}^{-1}$:

$$\mathcal{Z} = \sqrt{\det(-2\pi i \mathcal{M}^{-1})} \exp\big(\frac{i}{2} \int J \mathcal{M}^{-1} J d^4x\big) \tag{3.6}$$

As we have previously derived in section 1, the correlation function can be written in terms of functional derivatives of the path integral. The two point correlation function defines the propagator of our quantum field theory:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = -\frac{1}{\mathcal{Z}[0]} \frac{\delta^2}{\delta J(x) \delta J(y)} \mathcal{Z}|_{J=0} \tag{3.7}$$

When we take the derivatives and set $J = 0$, we recover the same propagator that the Wick rotation would yield:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i\mathcal{M}^{-1}(x - y) \tag{3.8}$$

While this result does not show anything new [1], it allows us to establish that our Picard-Lefschetz theory approach agrees with the canonically accepted approach of deriving the propagator from the path integral.

3.2.1 Klein-Gordon Theory on Curved Spacetime

Previously, we have considered the Klein-Gordon theory on flat Minkowski spacetime. This specific example was to illustrate how our Picard-Lefschetz approach agrees with standard methods. However, our approach opens up the ability to compute correlation functions on curved spacetime.

Consider some manifold $\mathcal{M}$ with a Lorentzian metric $g_{\mu\nu}$ [6]. In order to accommodate our new metric, we have to promote our partial derivatives to covariant derivatives[2] and correct our integration measure:

$$S[\phi] = \int_{\mathcal{M}} (-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - J\phi) \sqrt{-g} d^4x \tag{3.9}$$

Our old method of passing to a lattice in order to compute the path integral runs into the highly nontrivial problem of discretizing our curved spacetime and then passing back to the manifold. Instead, we can directly apply Picard-Lefschetz theory to the path integral. First, we want to prepare our action by integrating by parts:

$$S[\phi] = \int_{\mathcal{M}} (\frac{1}{2} \phi [g^{\mu\nu} \nabla_\mu \nabla_\nu - m^2] \phi - J\phi) \sqrt{-g} d^4x \tag{3.10}$$

We neglect the boundary term by assuming the field vanishes at infinity. While this action does not lead to a well posed variational problem, it allows us to more easily access the quadratic nature of the integral. Denote the differential operator by $\mathcal{A}$, we can complete the square:

$$S[\phi] = \int_{\mathcal{M}} \frac{1}{2} (\phi + \mathcal{A}^{-1} J) \mathcal{A} (\phi + \mathcal{A}^{-1} J) \sqrt{-g} d^4x + \int_{\mathcal{M}} \frac{1}{2} J \mathcal{A}^{-1} J \sqrt{-g} d^4x \tag{3.11}$$

[2] While the covariant derivative of a scalar is precisely the ordinary partial derivative, we will keep the derivatives covariant for consistency.

This reduction allows us to succinctly write our path integral. Since the path integral is translation invariant over the field, we can shift the field variable $\phi \to \phi - \mathcal{A}^{-1}J$:

$$\mathcal{Z} = \exp\Big(\frac{i}{2}\int_{\mathcal{M}} J\mathcal{A}^{-1}J\sqrt{-g}d^4x\Big)\int \exp\Big(\frac{i}{2}\int \phi\mathcal{A}\phi\sqrt{-g}d^4x\Big)\mathcal{D}[\phi] \tag{3.12}$$

We can directly apply Picard-Lefschetz theory to our path integral and use the contour to compute our path integral:

$$\mathcal{Z} = \sqrt{\det(\frac{2\pi i}{\sqrt{-g}}\mathcal{A}^{-1})}\exp\Big(\frac{i}{2}\int_{\mathcal{M}} J\mathcal{A}^{-1}J\sqrt{-g}d^4x\Big) \tag{3.13}$$

We can then compute the functional derivatives of our path integral to give our correlation function:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i\sqrt{-g}\mathcal{A}^{-1}(x-y) \tag{3.14}$$

3.3 Quartic Field Theory

Our next theory of interest is known as scalar ϕ^4 theory. This theory is characterized by a quartic potential:

$$S[\phi] = \int -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 d^4x \tag{3.15}$$

In comparison with the free field, this theory is much more difficult to solve. We will not be including a fictitious source term J, since it leads to an integral that is intractable. However, the source term does lend itself to the perturbative techniques that we will discuss. As we have previously shown, we can similarly reduce this path integral into a product of integrals:

$$\mathcal{Z}_N = \prod_{n\in\mathbb{R}^4}^{N}\int e^{-\frac{ia^4}{2}(\lambda_n\phi_n^2 + \frac{g}{12}\phi_n^4)}d\phi_n \tag{3.16}$$

Here, λ_n are the eigenvalues of the matrix associated with the discretization of $\mathcal{M} = \partial_\mu\partial^\mu - m^2$. In theory, this integral yields to Picard-Lefschetz. While it does give valid contours and it does produce a real integral out of our complex exponential, however this integral is difficult to handle with non-numerical methods. We will first look at the perturbative features of this integral and then we will discuss the full nonperturbative result.

3.3.1 Asymptotic Methods

Instead of dealing with our ϕ^4 integral directly, we will work with a more general integral:

$$\int e^{-i(ax^4+bx^2)}dx \tag{3.17}$$

Formally, this integral is a function of a and b. Since it is a function of two variables, we can express the integral as a power series. First, we take a to our expansion parameter and express the quartic part of the exponential as a power series:

$$\int e^{-i(ax^4+bx^2)}dx = \sum_{n=0}^{\infty}\frac{(-ia)^n}{n!}\int x^{4n}e^{-ibx^2}dx \tag{3.18}$$

We can scale out the b factor $x \to \frac{1}{\sqrt{b}}x$. The rescaled integral is twice of the Gaussian moment integral derived in section 2.1.2 where $\alpha = 4n$ and $\beta = 2$

$$\int e^{-i(ax^4+bx^2)}dx = \sqrt{\frac{-i}{b}} \sum_{n=0}^{\infty} \frac{\Gamma(2n+\frac{1}{2})}{n!} (\frac{ia}{b^2})^n \tag{3.19}$$

This series expansion reproduces the imaginary Gaussian when $a = 0$. If we pay attention to the coefficient of the series, we can notice that $\frac{\Gamma(2n+\frac{1}{2})}{n!} \to \infty$ as $n \to \infty$. This series must diverge, which at first glance, appears to be a useless result. However, this is an expected result since our series is an example of a asymptotic series [7]. The series has a zero radius of convergence because our integral is fundamentally multi-valued on $\mathbb{C}$. The integral picks up points that are on different branches. However, the series provides a good approximation for the integral if we truncate the series at some optimal value of n.

For a given function, the asymptotic series is unique. However, a given asymptotic series does not have a unique functional representation. For example, $f(x)$ and $f(x) + e^{-x}$ would both give the same series as $x \to \infty$. We know that function representation of our integral will give the series we derived about a. In order to characterize the function properly, we should consider the expansion about b.

$$\int e^{-i(ax^4+bx^2)}dx = \sum_{n=0}^{\infty} \frac{(-ib)^n}{n!} \int x^{2n} e^{-iax^4} dx \tag{3.20}$$

After rescaling $x \to \frac{1}{a^{\frac{1}{4}}}x$, we can recognize another Gaussian moment integral with $\alpha = 2n$ and $\beta = 4$:

$$\int e^{-i(ax^4+bx^2)}dx = \frac{1}{2} \frac{1}{(ia)^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n}{2}+\frac{1}{4})}{n!} (\sqrt{\frac{ib^2}{a}})^n \tag{3.21}$$

As expected, this series reproduces the imaginary quartic when $b = 0$. In contrast to our first series, this series converges for all $\sqrt{\frac{ib^2}{a}}$.

Since this sum converges, we can use Wolfram Mathematica to sum the series which yields a functional expression for our integral:

$$\int e^{-i(ax^4+bx^2)}dx = \sqrt{\frac{b}{4a}} e^{i\frac{b^2}{8a}} K_{\frac{1}{4}}(i\frac{b^2}{8a}) \tag{3.22}$$

Here, K_ν is the modified Bessel function of the second kind. With this expression, we can verify that its asymptotic series about a recovers the same series that we derived. Since we have a functional expression and a convergent series expression, we can use our result to express the path integral.

3.3.2 Path Integral

As we have derived earlier, the discretized path integral for ϕ^4 theory is given by:

$$\mathcal{Z}_N = \prod_{n\in\mathbb{R}^4}^{N} \int e^{-\frac{ia^4}{2}(\lambda_n\phi_n^2+\frac{g}{12}\phi_n^4)} d\phi_n \tag{3.23}$$

We can equate coefficients with our quartic integral and summation expression to aid in taking the continuum limit:

$$\mathcal{Z} = \sqrt{\det(\frac{3}{2g}\mathcal{M}e^{\frac{3i}{4g}\mathcal{M}^2})}K_{\frac{1}{4}}(\det(\frac{3i}{8g}\mathcal{M}^2)) \tag{3.24}$$

As we have previously discussed, the power of the path integral is its ability to generate correlation functions. In the free field case, we were able to inject a fictitious field that allows us to utilize the path integral as a generating functional. In the quartic case, the integral with an amended linear term cannot be computed with current methods[3]. However, the correlation function of primary interest is the two-point correlation function. We can use the Klein-Gordon operator to take derivatives:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \frac{-i}{\mathcal{Z}}\frac{\delta}{\delta\mathcal{M}(x-y)}\mathcal{Z} \tag{3.25}$$

This derivative requires operator derivative techniques, but can readily be computed:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = -i(\mathcal{M}^{-1}+\frac{3i}{4g}\mathcal{M})+i(\frac{K_{-\frac{3}{4}}(\det(\frac{3i}{8g}\mathcal{M}^2))}{K_{\frac{1}{4}}(\det(\frac{3i}{8g}\mathcal{M}^2))}+\frac{K_{\frac{5}{4}}(\det(\frac{3i}{8g}\mathcal{M}^2))}{K_{\frac{1}{4}}(\det(\frac{3i}{8g}\mathcal{M}^2))})\det(\frac{3i}{8g}\mathcal{M})\mathcal{M}^{-1} \tag{3.26}$$

It is nontrivial to see, but if we take the limit as $g \to 0$, we recover the free field propagator.

3.4 Dilaton Gravity

Einstein's theory of gravity in $3+1$ spacetime dimensions has evaded many efforts to be quantized into a complete quantum theory. The covariant action for this theory is given by:

$$S[g_{\mu\nu}] = \int R\sqrt{-g}\; d^4x \tag{3.27}$$

In order to better understand the quantum nature of gravity, we can investigate the path integral of a gravitational action in fewer spacetime dimensions. In $1+1$ spacetime dimensions, the geometry is determined completely by the Ricci scalar since we can express the Riemann tensor as [8]:

$$R_{abcd} = (g_{ac}g_{bd} - g_{ad}g_{bc})R \tag{3.28}$$

In two dimensions, we can always choose appropriate coordinates such that the metric takes the form:

$$g_{\mu\nu}dx^\mu dx^\nu = 2dx^0dx^1 + k(x)(dx^1)^2 \tag{3.29}$$

This metric determines the Ricci scalar $R = \partial_0^2 k(x)$. Therefore, we can conclude that the geometry is completely determined by a gauge function $k(x)$.

In order to have a well defined variational problem, we add a non-dynamical scalar field called the dilaton field:

$$S = \int \left(\phi R - V(\phi)\right)\sqrt{-g}d^2x \tag{3.30}$$

$V(\phi)$ is a potential associated with the dilaton field. We will consider the case where $V(\phi) = 0$, but we can handle different potentials perturbatively. In our path integral, we would integrate over

[3]This integral is similarly intractable to Mathematica.

both of the configuration variables ϕ and k. In the appropriate coordinates, our path integral is given by:

$$\mathcal{Z} = \int \exp(i \int \phi \partial_0^2 k d^2x) \mathcal{D}[\phi]\mathcal{D}[k] \tag{3.31}$$

While the action is quadratic, both of the fields are linear. If we blindly apply Picard-Lefschetz theory, then we would not be able to find a suitable contour. This is because the imaginary part of the analytical continued fields does not couple to the real part. However, we can avoid this issue by discretizing our path integral and doing a coordinate transformation in the field variables.

$$\mathcal{Z}_N = \prod_{n \in \mathbb{R}^2}^{N} \int e^{ia^4 \phi_n \lambda_n k_n} d\phi_n dk_n \tag{3.32}$$

Here, λ_n are the eigenvalues of the discritized second derivative operator $\mathcal{B} = \partial_0^2$. We can make the quadratic nature of the exponential manifest by changing to polar coordinates and scaling out the constants:

$$\frac{1}{a^4 \lambda_n} \int_0^{2\pi} \int_0^{\infty} e^{ir^2 \sin\theta \cos\theta} r dr d\theta \tag{3.33}$$

This integral can be computed by relating the angular integral to the zeroth Bessel function of the first kind. The radial integral becomes trivial since the Bessel functions are normalized on the interval $(0, \infty)$.

$$\int e^{ia^4 \phi_n \lambda_n k_n} d\phi_n dk_n = \frac{2\pi}{a^4 \lambda_n} \tag{3.34}$$

In this form, the continuum limit to the path integral is trivial:

$$\mathcal{Z} = \det(2\pi \mathcal{B}^{-1}) \tag{3.35}$$

In contrast to our previous field theories, dilaton gravity has three distinct two point correlation functions:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \frac{1}{\mathcal{Z}} \int \phi^2 e^{i\int \phi \partial_0^2 k d^2x} \mathcal{D}[\phi]\mathcal{D}[k] \tag{3.36}$$

$$\langle 0|T\{\phi(x)k(x)\}|0\rangle = \frac{1}{\mathcal{Z}} \int \phi k e^{i\int \phi \partial_0^2 k d^2x} \mathcal{D}[\phi]\mathcal{D}[k] \tag{3.37}$$

$$\langle 0|T\{k(x)k(y)\}|0\rangle = \frac{1}{\mathcal{Z}} \int k^2 e^{i\int \phi \partial_0^2 k d^2x} \mathcal{D}[\phi]\mathcal{D}[k] \tag{3.38}$$

All of these correlation functions are susceptible to source term methods. The most straightforward example of this is in the second correlation function. In this correlation function, the source term effectively shifts the operator $\mathcal{B} = \partial_0^2 k \to \mathcal{B}[J] = \partial_0^2 k + J.$:

$$\langle 0|T\{\phi(x)k(x)\}|0\rangle = \frac{-i}{\det(2\pi\mathcal{B}^{-1}[0])} \frac{\delta}{\delta J(x)} \det(2\pi \mathcal{B}^{-1}[J])_{J=0} = -i\mathcal{B}^{-1}(x) \tag{3.39}$$

When we insert the source term into the other two correlation functions:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \frac{1}{Z[0]} \int \phi^2 e^{i\int \phi \partial_0^2 k + \phi J \, d^2x} \mathcal{D}[\phi]\mathcal{D}[k] \tag{3.40}$$

$$\langle 0|T\{k(x)k(y)\}|0\rangle = \frac{1}{Z[0]} \int k^2 e^{i\int \phi \partial_0^2 k + kJ \, d^2x} \mathcal{D}[\phi]\mathcal{D}[k] \tag{3.41}$$

This source term does not affect the value of the path integral, since it can be removed from the exponential with a change of variables. Therefore, when we take the derivatives with respect to the source J, the resulting integral is J independent. This allows us to conclude that each of these correlation functions must vanish.

3.4.1 Quadratic Dilaton Potential

Previously, we have only considered vacuum propagators for the dilaton theory. In this section, we will analyze the correlation functions of dilaton gravity with a quadratic potential $V(\phi) = a\phi^2$, where a is some unknown coupling constant. The action for this theory is given by:

$$S = \int (\phi \partial_0^2 k - a\phi^2)\ d^2x \tag{3.42}$$

Since this action is manifestly quadratic in the dilaton field, its path integral is susceptible to the methods of Picard-Lefschetz theory we have been developing. If we put the this path integral on its Picard-Lefschetz integration surface, we see that the two fields decouple:

$$\mathcal{Z} = \int 2\exp(-\int \frac{1}{2a}(\partial_0^2 k)^2 + 2a\phi^2\ d^2x)\mathcal{D}[\phi]\mathcal{D}[k] \tag{3.43}$$

Since this integral is decoupled Gaussians, the partition functional is straight forward:

$$\mathcal{Z} = \det(2\pi\mathcal{B}^{-1}) \tag{3.44}$$

It is curious to note that this partition functional is identical to the vacuum partition functional given in (3.35). We can readily give the correlation functions with the appropriate source terms:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = -\frac{1}{\mathcal{Z}[0]}\frac{\delta^2}{\delta J(x)\delta J(y)}\int \exp(i\int \phi(\partial_0^2 k + J) - a\phi^2)\mathcal{D}[\phi]\mathcal{D}[k] \tag{3.45}$$

$$\langle 0|T\{k(x)\phi(x)\}|0\rangle = \frac{i}{\mathcal{Z}[0]}\frac{\delta}{\delta J(x)}\int \exp(i\int \phi(\partial_0^2 + J)k - a\phi^2)\mathcal{D}[\phi]\mathcal{D}[k] \tag{3.46}$$

$$\langle 0|T\{k(x)k(y)\}|0\rangle = -\frac{1}{\mathcal{Z}[0]}\frac{\delta^2}{\delta J(x)\delta J(y)}\int \exp(i\int (\phi\partial_0^2 + J)k - a\phi^2)\mathcal{D}[\phi]\mathcal{D}[k] \tag{3.47}$$

The first two of these are trivial. The dilaton correlation function vanishes because we can remove the source field from the integral with a field transformation. The dilation metric correlation function has the effect of shifting the operator $\mathcal{B} = \partial_0^2 \to \mathcal{B}[J] = \partial_0^2 + J$. This amounts to the same propagator as in the vacuum case. However, in contrast to the vacuum case, the metric correlation function does not obviously vanish. We can do the calculation and see that it, in fact, does not vanish:

$$\langle 0|T\{k(x)k(y)\}|0\rangle = 2ai\mathcal{B}^{-2}(x-y) \tag{3.48}$$

Within the framework of perturbative QFT, the correlation functions tell us how the fields propagate through space. Since the metric correlation function does not vanish, we can see that the metric interacts with itself. Additionally, the proportionality to a tells us that this self interaction term relies on there being a potential $V(\phi)$. However, when we wrote down the action in (3.42), the action manifestly only had an interaction between the dilaton field and the metric.

The self interaction becomes manifest in (3.43) when we put the path integral on its Picard-Lefschetz integration surface. Since the interaction has a derivative coupling, it resembles the kinetic term of simpler theories like the scalar field theory. In this form, it is clear to see why the dilaton correlation function vanishes; the field carries no kinetic term in the action. By situating the action on its Picard-Lefschetz surface, we can make nontrivial observations about the non-classical behavior of the system. If we consider the dilaton action with a quadratic potential and amend a higher curvature term with some unknown coupling constant:

$$S = \int \phi \partial_0^2 k - a\phi^2 - b(\partial_0^2 k)^2 \; d^2x \tag{3.49}$$

In order for this classical action to have a well defined quantum theory, we would expect it's path integral to converge. By putting this theory on its Picard-Lefschetz integration surface:

$$\mathcal{Z} = \int 2\exp\Big(-2\int (\frac{1}{4a} - b)(\partial_0^2 k)^2 + a\phi^2 \; d^2x\Big)\mathcal{D}[k]\mathcal{D}[\phi] \tag{3.50}$$

We can see that, assuming $b > 0$, the path integral converges for $b \leq \frac{1}{4a}$ which provides a succinct upper bound for b.

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