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GEOMETRIC CONDITIONS OF HYPERSURFACE DEFORMATIONS FOR CANONICAL GRAVITY THEORIES

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Abstract

Traditionally, canonical computations do not include lapse and shift functions inside of the Poisson brackets, as they do not change the equations of motion at first order. When working with theories that have higher order time derivatives, one needs to include the lapse and shift inside the Poisson brackets as they do add additional terms to the equations of motion. However, there still exists an ambiguity as to whether the lapse and shift should be inside or outside the brackets. We investigate if the canonical methods can describe a geometric theory. We compute the phase space dependence of the hypersurface deformation by computing the lapse and shift inside the Poisson brackets. We use the geometric formulation to derive conditions placed on the canonical formulation. We find that the canonical formulation, when it considers the phase space dependence of the lapse and shift and the deformation of the normal vector, leads not only to the sought after full consistency with the gauge functions inside the brackets, but also a method to obtain new modified gravity theories altogether.

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Chapter 1

Introduction

In traditional canonical theories one does not compute the lapse and shift functions inside of Poisson brackets[1], because the first-order equations of motion do not change by phase space dependence of the lapse and shift on-shell. Full consistency requires us to check whether this applies to higher order time derivatives given by consecutive brackets. While [1] defines the lapse and shift outside the consecutive brackets, the Gowdy model uses second-order time derivatives to find the equations of motion[2], so the phase space dependence must be accounted for, even on-shell. When the phase space dependence is considered the equations of motion take a different form: $\ddot{f} = \{\{f, H[N]\}, H[N]\} = \{N, \{f, H\}H[N]\} + ... = \{N, H\}\{f, H\}N + ...$, where the "..." are the rest of the terms. The ones shown here do not appear if the lapse, N, is phase space independent and these terms, in general, do not vanish on-shell when it is phase space dependent[2]. The authors of [2] included the lapse and shift inside the bracket, but they did not elaborate on that at all, despite the definition of the classic earlier work [1] of having the lapse and shift outside of the bracket. Thus the matter is not yet settled, and, in general, not disscussed.

We wanted to investigate how phase space dependence affects the hypersurface deformation algebra and find what conditions are needed to satisfy both the geometric and covariance conditions. We have indeed found how the phase space dependence affects the algebra and certain transformation properties of the gauge functions themselves, thus resolving the consistency issue described earlier. This leads to the geometric conditions, and these in turn can be used to obtain a new constraint algebra with a new structure function. Which implies a new spacetime metric, but this new metric is not necessarily covariant.

The organization of this thesis is as follows. In chapter 2 we work to understand the geometric meaning of the canonical hypersurface deformation algebra. We then consider phase space dependent lapse and shifts and arrive at a geometric condition that must be met. An example of a covariant and geometrically consistent lapse and shift, that differs from the classical one, is given. In chapter 3 we derive from the geometric formulation the geometric conditions that the canonical formulation must satisfy. Finally, in chapter 4 we discuss the implications of the new geometric condition.

Chapter 2

Canonical Formulation

2.1 Canonical theory

We begin with a foliated globally hyperbolic topology $M = \Sigma \times \mathbb{R}$ with a spacetime metric

$$ds^2 = \sigma N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \ , \tag{2.1}$$

a time-evolution vector field

$$t^\mu = N n^\mu + N^a s^\mu_a \ , \tag{2.2}$$

where n^μ is a normalized vector normal to Σ_t and s^μ_a are the three spatial basis vectors tangent to Σ. We also have the a lapse, N, the normal projection of t^μ on Σ, and a shift, N^a, the projection of t^μ on Σ. The Hamiltonian is independent of time derivatives of the lapse and shift functions giving two constraints, the Hamiltonian constraint and the diffeomorphism constraint. The total Hamiltonian is given by the linear combination of the constraints and Lagrange multipliers: $H[N] + H_a[N^a] = 0$, where H is the Hamiltonian constraint and H_a is the diffeomorphism constraint. Using Poisson brackets and the diffeomorphism constraint a closed constraint algebra emerges

$$\{H_a[N], H_a[M^a]\} = H_a[\mathcal{L}_{N^a} M^a] \ , \tag{2.3}$$

$$\{H_a[N^a], H[N]\} = H[N^a \partial_a N] \ , \tag{2.4}$$

$$\{H[N], H[M]\} = -H_a[q^{ab}(N\partial_b M - M\partial_b)] \ , \tag{2.5}$$

where q^{ab} is a structure function. This algebra is only valid for phase space independent smearing functions. The gauge transform of the phase space independent lapse and shift are

$$\delta_\epsilon N = \dot{\epsilon}^0 + \epsilon^a \partial_a N - N^a \partial_a \epsilon^0 \ , \tag{2.6}$$

$$\delta_\epsilon N^a = \dot{\epsilon}^a + \epsilon^b \partial_b N^a - N^b \partial_b \epsilon^a - \sigma q^{ab}\left(\epsilon^0 \partial_b N - N \partial_b \epsilon^0\right) \ , \tag{2.7}$$

if the Jacobiator of the constraints vanishes. In section 2.3 we will show a more general derivation that includes phase dependence. The constraint algebra in this deformation form ensures a general covariance in General Relativity [1], but further conditions beyond the hypersurface deformation algebra must be met for modified gravity theories. The on-shell gauge transformations of 3-metric correspond to diffeomorphisms

$$\{q_{ab}, \vec{H}[\vec{\epsilon}]\}\big|_{\text{O.S.}} = \mathcal{L}_{\vec{\epsilon}} q_{ab} \ , \tag{2.8}$$

$$\{q_{ab}, H[\epsilon^0]\}\big|_{\text{O.S.}} = \mathcal{L}_{\epsilon^0 n} q_{ab} \ . \tag{2.9}$$

The gauge transformation of the 3-metric, lapse, and shift give the diffeomorphism of the full metric

$$\delta_\epsilon g_{\mu\nu}\big|_{\text{O.S.}} = \mathcal{L}_\xi g_{\mu\nu} \ , \tag{2.10}$$

where $\mathcal{L}_\xi$ is a Lie derivative along the 4-vector field ξ. The components of the vector field are in terms of the gauge generators and the lapse and shift

$$\xi^\mu = \epsilon^0 n^\mu + \epsilon^a s^\mu_a = \xi^t t^\mu + \xi^a s^\mu_a \ , \tag{2.11}$$

$$\xi^t = \frac{\epsilon^0}{N} \ , \tag{2.12}$$

$$\xi^a = \epsilon^a - \frac{\epsilon^0}{N} N^a \ . \tag{2.13}$$

2.2 Geometric meaning of the hypersurface deformation algebra

2.2.1 Transformation of the normal vector

$$g^{\mu\nu} = q^{ab} s^\mu_a s^\nu_a + \frac{\sigma}{N^2} \left(t^\mu - N^a s^\mu_a\right) \left(t^\nu - N^b s^\nu_b\right) \ . \tag{2.14}$$

After a hypursurface deformation the normal vector and the 3-metric transform to $n^\mu + \delta_\epsilon n^\mu$ and $q_{ab} + \delta q_{ab}$ respectively. The normal vector remains normal if and only if $\delta_\epsilon(N n^\mu q_{\mu\nu}) = 0$ where $q_{\mu\nu} = g_{ab} s^a_\mu s^b_\nu$. Using the full metric is more convenient as we can see how all components must transform

$$\delta_\epsilon(N n^\mu g_{\mu\nu}) dx^\nu = 2\sigma N \delta_\epsilon N dt \ . \tag{2.15}$$

The left-hand side (l.h.s) is expanded using the Leibnitz rule, a change of basis $n^\mu = N^{-1}(t^\mu - N^a s^\mu_a)$, and the component $g^{t\mu}$ which gives

$$\delta_\epsilon n^\mu = -\frac{1}{N} \delta_\epsilon N n^\mu - \frac{1}{N} \delta_\epsilon N^a s^\mu_a \ . \tag{2.16}$$

Using this and the normal condition, $\delta_\epsilon(n^\mu q_{\mu\nu})$, shows that the gauge transformation of the 3-metric can not be orthogonal to the normal vector

$$N n^\nu \delta_\epsilon q_{\mu\nu} = q_{\mu b} \delta_\epsilon N^b \ . \tag{2.17}$$

Consider a spatial basis vector in the form

$$\delta_\epsilon s^\mu_b := A_b n^\mu + B s^\mu_b \ . \tag{2.18}$$

Requiring that $\delta_\epsilon(g_{\mu\nu} n^\mu s^\nu_b) = 0$ along with (2.17) and (2.1) determines

$$A_b = g_{\mu b} \delta_\epsilon n^\mu + n^\mu \delta_\epsilon g_{\mu b} = 0 \ . \tag{2.19}$$

Using $\delta_\epsilon q_{ab} = \delta_\epsilon(g_{\mu\nu} s^\mu_a s^\nu_b)$ determines $B = 0$ which means that the spatial basis vector does not suffer a deformation.

2.2.2 Transformation of the normal generator

A normal generator deformed by passive deformation, then subjected to a deformation generated by ϵ_1^μ, then another deformation generated by the lapse and shift becomes

$$\begin{aligned} H\left[N,\vec{N}\right] \to\ & H\left[N-\delta_{\epsilon_1}N,\vec{N}-\delta_{\epsilon_1}\vec{N}\right] \\ &= H\left[N-\delta_{\epsilon_1}N\right]-\vec{H}\left[\delta_{\epsilon_1}\vec{N}\right]+\vec{H}\left[\vec{N}\right] \\ &=:\ H^{\delta_{\epsilon_1}}[N]+\vec{H}\left[\vec{N}\right]\ , \end{aligned} \tag{2.20}$$

where $H^{\delta_{\epsilon_1}}$ is defined as the new normal generator that keeps the spatial generator undeformed

$$H^{\delta_{\epsilon_1}} = H\left(1-\frac{\delta_{\epsilon_1}N}{N}\right) - H_a\frac{\delta_{\epsilon_1}N^a}{N}\ . \tag{2.21}$$

2.2.3 Transformation of phase space independent gauge functions

We follow the usual derivation, up to the use of the deformed normal constraint, of phase space independent gauge functions. The geometric meaning of the constraints can be understood using deformation generators. We first consider an observable, A, a general deformation that can act on gauge functions, δ_{ϵ_1}, and a deformation that acts only on observables $\bar{\delta}_{\epsilon_1} = \{\cdot, H[\epsilon_1]\}$. We need a deformation that acts only on the observable since the gauge functions can be independent of the phase space. If the observable A depends on a gauge function ϵ_2, the full deformation is

$$\delta_{\epsilon_1}A = \bar{\delta}_{\epsilon_1}A + \int\frac{\delta A}{\delta\epsilon_2}\delta_{\epsilon_1}\epsilon_2\ . \tag{2.22}$$

Consider an observable, $\mathcal{O}$, and two deformations generators, ϵ_1 and ϵ_2. Acting on the observable with the first generator, ϵ_1, produces

$$\mathcal{O}^{(1)} \equiv\ \mathcal{O}+\delta_{\epsilon_1}\mathcal{O} = \mathcal{O}+\{\mathcal{O},H[\epsilon_1]\}\ . \tag{2.23}$$

Then, acting on $\mathcal{O}^{(1)}$ with ϵ_2 produces

$$\begin{aligned} \mathcal{O}^{(1,2)} \equiv\ & \mathcal{O}^{(1)}+\delta_{\epsilon_2}^{\delta_{\epsilon_1}}\mathcal{O}^{(1)} \\ &= \mathcal{O}+\{\mathcal{O},H[\epsilon_1]\}+\{\mathcal{O},H^{\delta_{\epsilon_1}}[\epsilon_2]\}+\{\{\mathcal{O},H[\epsilon_1]\},H[\epsilon_2]\} \\ &= \mathcal{O}+\{\mathcal{O},H[\epsilon_1]\}+\{\mathcal{O},H[\epsilon_2]\}+\{\{\mathcal{O},H[\epsilon_1]\},H[\epsilon_2]\}-\left\{\mathcal{O},H_T\left[\epsilon_2^0\frac{\delta_{\epsilon_1}N}{N},\epsilon_2^0\frac{\delta_{\epsilon_1}\vec{N}}{N}\right]\right\}\ . \end{aligned} \tag{2.24}$$

The order of deformations is reversed and the two are commuted which gives

$$\begin{aligned} \mathcal{O}^{[1,2]} \equiv\ & \mathcal{O}^{(1,2)}-\mathcal{O}^{(2,1)} \\ &= \{\{\mathcal{O},H[\epsilon_1]\},H[\epsilon_2]\}-\{\{\mathcal{O},H[\epsilon_2]\},H[\epsilon_1]\} \\ &\quad+\left\{\mathcal{O},H_T\left[\epsilon_1^0\frac{\delta_{\epsilon_2}N}{N}-\epsilon_2^0\frac{\delta_{\epsilon_1}N}{N},\epsilon_1^0\frac{\delta_{\epsilon_2}\vec{N}}{N}-\epsilon_2^0\frac{\delta_{\epsilon_1}\vec{N}}{N}\right]\right\} \\ &= \{\mathcal{O},\{H[\epsilon_1],H[\epsilon_2]\}\}+\{\mathcal{O},H[\epsilon_1],H[\epsilon_2]\} \\ &\quad+\left\{\mathcal{O},H_T\left[\epsilon_1^0\frac{\delta_{\epsilon_2}N}{N}-\epsilon_2^0\frac{\delta_{\epsilon_1}N}{N},\epsilon_1^0\frac{\delta_{\epsilon_2}\vec{N}}{N}-\epsilon_2^0\frac{\delta_{\epsilon_1}\vec{N}}{N}\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&= \{\mathcal{O}, H[\Delta_{12}]\} + \{\mathcal{O}, H[\epsilon_1], H[\epsilon_2]\} \\
&+ \left\{\mathcal{O}, H_T\left[\epsilon_1^0\frac{\delta_{\epsilon_2}N}{N} - \epsilon_2^0\frac{\delta_{\epsilon_1}N}{N}, \epsilon_1^0\frac{\delta_{\epsilon_2}\vec{N}}{N} - \epsilon_2^0\frac{\delta_{\epsilon_1}\vec{N}}{N}\right]\right\} ,
\end{aligned} \tag{2.25}
$$

where we used the Jacobiator

$$
\{A, B, C\} \equiv \{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} , \tag{2.26}
$$

and where Δ_{12} is

$$
\Delta_{12}^0 = \epsilon_{(1)}^b \partial_b \epsilon_{(2)}^0 - \epsilon_{(2)}^b \partial_b \epsilon_{(1)}^0 , \tag{2.27}
$$

$$
\Delta_{12}^a = \epsilon_{(1)}^b \partial_b \epsilon_{(2)}^a - \epsilon_{(2)}^b \partial_b \epsilon_{(1)}^a + \sigma q^{ab}\left(\epsilon_{(1)}^0 \partial_b \epsilon_{(2)}^0 - \epsilon_{(2)}^0 \partial_b \epsilon_{(1)}^0\right) . \tag{2.28}
$$

We repeat the process but deform the gauge functions instead of the constraints

$$
\mathcal{O}^{(1,2)} \equiv \mathcal{O} + \delta_{\epsilon_1}\mathcal{O} + \delta^{\delta_{\epsilon_1}}_{\epsilon_2 + \delta_{\epsilon_1}\epsilon_2}\mathcal{O} . \tag{2.29}
$$

Reversing the order of the deformations and commuting the operation gives

$$
\mathcal{O}^{[1,2]} \equiv \{\mathcal{O}, H[\delta_{\epsilon_1}\epsilon_2 - \delta_{\epsilon_2}\epsilon_1]\} + \left\{\mathcal{O}, H_T\left[\epsilon_1^0\frac{\delta_{\epsilon_2}N}{N} - \epsilon_2^0\frac{\delta_{\epsilon_1}N}{N}, \epsilon_1^0\frac{\delta_{\epsilon_2}\vec{N}}{N} - \epsilon_2^0\frac{\delta_{\epsilon_1}\vec{N}}{N}\right]\right\} . \tag{2.30}
$$

By comparing these two procedures we find, provided the Jacobiator vanishes, that

$$
\delta_{\epsilon_1}\epsilon^\mu_{(2)} - \delta_{\epsilon_2}\epsilon^\mu_{(1)} = \Delta^\mu_{12} \tag{2.31}
$$

By substituting $\epsilon_1 \to \epsilon$, $\epsilon_2 \to N$ and setting $\delta_{N,\vec{N}}\epsilon^\mu = \dot{\epsilon}^\mu$ we arrive back at (2.6) and (2.7).

2.2.4 Geometric meaning of a vanishing Jacobiator

We consider the same system as the one above. The commutation can be rewritten using the ADM decomposition of the Lie derivatives

$$
\mathcal{O}^{[1,2]} = \{\mathcal{O}, H[\mathcal{L}_{\xi_1}\xi^0_{(2)}] + H_a[\mathcal{L}_{\xi_1}\xi^a_{(2)}]\} + \{\mathcal{O}, H[\epsilon_1], H[\epsilon_2]\} . \tag{2.32}
$$

What we see is that the canonical methods can reproduce the geometric theory if the Jacobiator is vanishing and the deformation of the normal vector is taken into account, otherwise it does not produce the correct normal deformation. This has been overlooked in all previous canonical treatments as far as we know. We can further examine the Jacobiator by using (2.4) and (2.5) to find a condition that must be satisfied for the Jacobiator to be zero. We start by permuting $\{\{H[\epsilon_1^0], H[\epsilon_2^0]\}, H[\epsilon_3^0]\}$ which gives

$$
\begin{aligned}
\left\{\left\{H[\epsilon_1^0], H[\epsilon_2^0]\right\}, H[\epsilon_3^0]\right\} &= \left\{\vec{H}\left[\sigma q^{ab}\left(\epsilon_1^0\partial_b\epsilon_2^0 - \epsilon_2^0\partial_b\epsilon_1^0\right)\right], H[\epsilon_3^0]\right\} \\
&= H\left[\sigma q^{ab}\left(\epsilon_1^0\partial_b\epsilon_2^0 - \epsilon_2^0\partial_b\epsilon_1^0\right)\partial_a\epsilon_3^0\right] \\
&\quad + \int d^3x\, \sigma H_a(x)\left(\epsilon_1^0(x)\partial_b\epsilon_2^0(x) - \epsilon_2^0(x)\partial_b\epsilon_1^0(x)\right)\left\{q^{ab}(x), H[\epsilon_3^0]\right\} \\
&= H\left[\sigma q^{ab}\left(\epsilon_1^0\partial_b\epsilon_2^0 - \epsilon_2^0\partial_b\epsilon_1^0\right)\partial_a\epsilon_3^0\right] \\
&\quad + H_a\left[\sigma\left(\epsilon_1^0\partial_b\epsilon_2^0 - \epsilon_2^0\partial_b\epsilon_1^0\right)\left(Q^{ab}\epsilon_3^0 + Q^{abc}\partial_c\epsilon_3^0 + Q^{abcd}\partial_c\partial_d\epsilon_3^0\right)\right] ,
\end{aligned} \tag{2.33}
$$

$$
\left\{\left\{H[\epsilon_2^0], H[\epsilon_3^0]\right\}, H[\epsilon_1^0]\right\} = H\left[\sigma q^{ab}\left(\epsilon_2^0\partial_b\epsilon_3^0 - \epsilon_3^0\partial_b\epsilon_2^0\right)\partial_a\epsilon_1^0\right]
$$

$$+ H_a \left[\sigma \left(\epsilon_2^0 \partial_b \epsilon_3^0 - \epsilon_3^0 \partial_b \epsilon_2^0\right) \left(Q^{ab}\epsilon_1^0 + Q^{abc}\partial_c \epsilon_1^0 + Q^{abcd}\partial_c \partial_d \epsilon_1^0\right)\right] \;, \quad (2.34)$$

$$\left\{\left\{H[\epsilon_3^0], H[\epsilon_1^0]\right\}, H[\epsilon_2^0]\right\} = H\left[\sigma q^{ab}\left(\epsilon_3^0 \partial_b \epsilon_1^0 - \epsilon_1^0 \partial_b \epsilon_3^0\right)\partial_a \epsilon_2^0\right] + H_a \left[\sigma \left(\epsilon_3^0 \partial_b \epsilon_1^0 - \epsilon_1^0 \partial_b \epsilon_3^0\right) \left(Q^{ab}\epsilon_2^0 + Q^{abc}\partial_c \epsilon_2^0 + Q^{abcd}\partial_c \partial_d \epsilon_2^0\right)\right] \;, \quad (2.35)$$

where the Q tensors are the generic form of the Poisson bracket of q^{ab}. Higher order Q terms have been ignored but it is straightforward to consider them. Summing the permutations gives

$$\left\{\left\{H[\epsilon_1^0], H[\epsilon_2^0]\right\}, H[\epsilon_3^0]\right\} + \left\{\left\{H[\epsilon_2^0], H[\epsilon_3^0]\right\}, H[\epsilon_1^0]\right\} + \left\{\left\{H[\epsilon_3^0], H[\epsilon_1^0]\right\}, H[\epsilon_2^0]\right\} = \sigma H_a \left[Q^{abcd}\left(\left(\epsilon_1^0 \partial_b \epsilon_2^0 - \epsilon_2^0 \partial_b \epsilon_1^0\right)\partial_c \partial_d \epsilon_3^0 + \left(\epsilon_2^0 \partial_b \epsilon_3^0 - \epsilon_3^0 \partial_b \epsilon_2^0\right)\partial_c \partial_d \epsilon_1^0 + \left(\epsilon_3^0 \partial_b \epsilon_1^0 - \epsilon_1^0 \partial_b \epsilon_3^0\right)\partial_c \partial_d \epsilon_2^0 \right)\right] . \quad (2.36)$$

We see that if $Q^{abcd} = 0$ the Jacobi identity is satisfied. If higher orders of Q appear, they could make the Jacobiator non-vanishing

2.3 Phase space dependent generators

2.3.1 Poisson brackets of phase dependent phase gauges

If we have gauge functions that depend on the phase space then there will be changes to the constraint algebra. We find these new constraints to be

$$\{\vec{H}[\vec{N}], \vec{H}[\vec{M}]\} = -\vec{H}[\mathcal{L}_{\vec{M}}\vec{N}] + H_a[\{N^a, \vec{H}[\vec{M}]\} - \{M^a, H_b[N^b]\} - H_b[\{N^b, M^a\}]] \;, \quad (2.37)$$

$$\{H[N], \vec{H}[\vec{M}]\} = -H[M^b \partial_b N - \{N, \vec{H}[\vec{M}]\}] - H_b[\{M^b, H[N]\} + H[\{N, M^b\}]] \;, \quad (2.38)$$

$$\{H[N], H[M]\} = \vec{H}[q^{ab}(N\partial_b M - M\partial_b N)] + H\left[\{N, H[M]\} - \{M, H[N]\} - H[\{N, M\}]\right] \;, \quad (2.39)$$

where we have suppressed certain integrals and here, the smearings are implied by the contracted indices. Evaluating the Poisson brackets of the constraints reveals anomalous terms in the algebra.

2.3.2 Transformation of phase dependant generators

We follow the same derivation steps as in 2.2.3 along with the effects of the phase space dependence of the gauge functions. We have an observable that is acted upon, consecutively, by two deformations. The gauge transformation is defined as

$$\delta_\epsilon \mathcal{O} \equiv \int d^3x \; \left(\epsilon(x)\{\mathcal{O}, H(x)\} + H(x)\{\mathcal{O}, \epsilon(x)\}\right) =: \not{\delta}_\epsilon \mathcal{O} + \int d^3x \; H(x)\{\mathcal{O}, \epsilon(x)\} \,. \quad (2.40)$$

The deformation generated by ϵ_1 on observable $\mathcal{O}$ is

$$\mathcal{O}^{(1)} \equiv \mathcal{O} + \delta_{\epsilon_1}\mathcal{O} = \mathcal{O} + \{\mathcal{O}, H[\epsilon_1]\} \,. \quad (2.41)$$

After the second deformation the observable takes the form

$$\mathcal{O}^{(1,2)} \equiv \mathcal{O}^{(1)} + \delta_{\epsilon_2}^{\delta_{\epsilon_1}}\mathcal{O}^{(1)} = \mathcal{O} + \{\mathcal{O}, H[\epsilon_1]\} + \{\mathcal{O}, H[\epsilon_2]\} + \{\{\mathcal{O}, H[\epsilon_1]\}, H[\epsilon_2]\} - \left\{\mathcal{O}, H_T\left[\epsilon_2^0 \frac{\delta_{\epsilon_1} N}{N}, \epsilon_2^0 \frac{\delta_{\epsilon_1}\vec{N}}{N}\right]\right\} . \quad (2.42)$$

The commutator of the two deformations is

$$\begin{aligned}
\mathcal{O}^{[1,2]} \equiv\ & \mathcal{O}^{(1,2)} - \mathcal{O}^{(2,1)}\big|_{\text{O.S.}} \\
=\ & \{\{\mathcal{O}, H_T[\epsilon_1]\}, H_T[\epsilon_2]\} - \{\{\mathcal{O}, H_T[\epsilon_2]\}, H_T[\epsilon_1]\} \\
& + \left\{\mathcal{O}, H_T\left[\epsilon_1^0 \frac{\delta_{\epsilon_2} N}{N} - \epsilon_2^0 \frac{\delta_{\epsilon_1} N}{N}, \epsilon_1^0 \frac{\delta_{\epsilon_2} \vec{N}}{N} - \epsilon_2^0 \frac{\delta_{\epsilon_1} \vec{N}}{N}\right]\right\}\Big|_{\text{O.S.}} \\
=\ & \{\mathcal{O}, \{H_T[\epsilon_1], H_T[\epsilon_2]\}\} + \{\mathcal{O}, H_T[\epsilon_1], H_T[\epsilon_2]\} \\
& + \left\{\mathcal{O}, H_T\left[\epsilon_1^0 \frac{\delta_{\epsilon_2} N}{N} - \epsilon_2^0 \frac{\delta_{\epsilon_1} N}{N}, \epsilon_1^0 \frac{\delta_{\epsilon_2} \vec{N}}{N} - \epsilon_2^0 \frac{\delta_{\epsilon_1} \vec{N}}{N}\right]\right\}\Big|_{\text{O.S.}} \\
=\ & \left\{\mathcal{O}, H_I\left[\Delta_{12}^I - \{\epsilon_2^I, H_J[\epsilon_1^J]\} + \{\epsilon_1^I, H_J[\epsilon_2^J]\} - H_J[\{\epsilon_1^I, \epsilon_2^J\}]\right]\right\} \\
& + \{\mathcal{O}, H_T[\epsilon_1], H_T[\epsilon_2]\} + \left\{\mathcal{O}, H_T\left[\epsilon_1^0 \frac{\delta_{\epsilon_2} N}{N} - \epsilon_2^0 \frac{\delta_{\epsilon_1} N}{N}, \epsilon_1^0 \frac{\delta_{\epsilon_2} \vec{N}}{N} - \epsilon_2^0 \frac{\delta_{\epsilon_1} \vec{N}}{N}\right]\right\} ,
\end{aligned} \tag{2.43}$$

where we use the anomalous constraints instead of the anomaly-free one. The process is repeated for the deformation of the gauge functions instead of the constraints which gives

$$\mathcal{O}^{(1,2)} \equiv\ \mathcal{O} + \delta_{\epsilon_1}\mathcal{O} + \delta^{\delta_{\epsilon_1}}_{\epsilon_2 + \not\delta_{\epsilon_1}\epsilon_2}\mathcal{O} , \tag{2.44}$$

where $\not\delta_{\epsilon_1}\delta_2$ means that the transformation only considers the phase independent part of ϵ_2. Commuting the observable gives

$$\mathcal{O}^{[1,2]} \equiv \left\{\mathcal{O}, H_T\left[\epsilon_1^0 \frac{\delta_{\epsilon_2} N}{N} - \epsilon_2^0 \frac{\delta_{\epsilon_1} N}{N}, \epsilon_1^0 \frac{\delta_{\epsilon_2} \vec{N}}{N} - \epsilon_2^0 \frac{\delta_{\epsilon_1} \vec{N}}{N}\right]\right\} + \{\mathcal{O}, H[\not\delta_{\epsilon_1}\epsilon_2 - \not\delta_{\epsilon_2}\epsilon_1]\} . \tag{2.45}$$

Comparing the two procedures gives

$$\begin{aligned}
\{\mathcal{O}, H_I[\not\delta_{\epsilon_1}\epsilon_2^I - \not\delta_{\epsilon_2}\epsilon_1^I]\} = & \left\{\mathcal{O}, H_I\left[\Delta_{12}^I + \{\epsilon_1^I, H_J[\epsilon_2^J]\} - \{\epsilon_2^I, H_J[\epsilon_1^J]\} - H_J[\{\epsilon_1^I, \epsilon_2^J\}]\right]\right\} \\
& + \{\mathcal{O}, H_I[\epsilon_1^I], H_I[\epsilon_2^I]\} ,
\end{aligned} \tag{2.46}$$

or

$$\{\mathcal{O}, H_I[\delta_{\epsilon_1}\epsilon_2^I - \delta_{\epsilon_2}\epsilon_1^I]\} = \{\mathcal{O}, H_I[\Delta_{12}^I - H_J[\{\epsilon_1^I, \epsilon_2^J\}]]\} + \{\mathcal{O}, H_I[\epsilon_1^I], H_I[\epsilon_2^I]\} , \tag{2.47}$$

where the full gauge transformations is defined as

$$\delta_{\epsilon_1}\epsilon_2^I \equiv\ \not\delta_{\epsilon_1}\epsilon_2^I + \{\epsilon_2^I, H_J[\epsilon_1^J]\} . \tag{2.48}$$

The final transformation, provided the Jacobiator vanishes, is

$$\delta_{\epsilon_1}\epsilon_2^I - \delta_{\epsilon_2}\epsilon_1^I = \Delta_{12}^I - H_J[\{\epsilon_1^I, \epsilon_2^J\}] . \tag{2.49}$$

On-shell, this is the exact transformation we expect arrive at with the geometric approach. This means that even phase space dependent gauge functions transform consistently.

2.3.3 New geometric condition

Replacing the generic gauge function in (2.47) with the lapse and shift implies a new geometric condition

$$\not\delta_\epsilon N(x) + \{N(x), H_T[\epsilon]\} = \frac{\partial \epsilon^0(x)}{\partial t} + \{\epsilon^0(x), H_T[N]\} + \epsilon^a\partial_a N - N^a\partial_a\epsilon^0 , \tag{2.50}$$

$$\oint_\epsilon N^a(x)+\{N^a(x),H_T[\epsilon]\} = \frac{\partial\epsilon^a(x)}{\partial t}+\{\epsilon^a(x),H_T[N]\}+\epsilon^b\partial_b N^a-N^b\partial_b\epsilon^a-\sigma q^{ab}\left(\epsilon^0\partial_b N-N\partial_b\epsilon^0\right) . \tag{2.51}$$

If we take the phase dependant lapse and shift and consider when $\oint_\epsilon N=\oint_\epsilon N^a=0$, then an arbitrary phase independent gauge ϵ can not produce the right-hand-side of the equations. This shows that we must, in general, retain a phase independent portion of the lapse and shift to complete the geometric condition.

2.4 Linear combination example: A new modified gravity theory

2.4.1 Geometric condition

Consider a lapse $N=B\bar{N}$, and shift $N^a=A^a\bar{N}+\bar{N}^a$ and $\epsilon^0=B\bar{\epsilon}^0$, $\epsilon^a=A^a\bar{\epsilon}^0+\bar{\epsilon}^a$. The spatial scalar, B and spatial vector, A^a are phase dependent and the barred functions are phase independent. The on-shell deformations are

$$\begin{aligned}\oint_\epsilon\bar{N}\big|_{\text{O.S.}} = &\frac{\partial\bar{\epsilon}^0}{\partial t}+\left(A^a\bar{\epsilon}^0+\bar{\epsilon}^a\right)\partial_a\bar{N}-\left(A^a\bar{N}+\bar{N}^a\right)\partial_a\bar{\epsilon}^0\\ &+B^{-1}\left(\bar{\epsilon}^0\{B,H[B\bar{N}]\}-\bar{N}\{B,H[B\bar{\epsilon}^0]\}\right)\big|_{\text{O.S.}} ,\end{aligned} \tag{2.52}$$

$$\begin{aligned}\oint_\epsilon\bar{N}^a\big|_{\text{O.S.}} = &\frac{\partial\bar{\epsilon}^a}{\partial t}+\bar{\epsilon}^b\partial_b\bar{N}^a-\bar{N}^b\partial_b\bar{\epsilon}^a-\sigma B^2q^{ab}\left(\bar{\epsilon}^0\partial_b\bar{N}-\bar{N}\partial_b\bar{\epsilon}^0\right)\\ &+\bar{\epsilon}^0\{A^a,H[B\bar{N}]\}-\bar{N}\{A^a,H[B\bar{\epsilon}^0]\}\\ &-A^a\left(A^b\bar{\epsilon}^0\partial_b\bar{N}-A^b\bar{N}\partial_b\bar{\epsilon}^0+\bar{\epsilon}^0\{B,H[B\bar{N}]\}-\bar{N}\{B,H[B\bar{\epsilon}^0]\}\right)\big|_{\text{O.S.}} .\end{aligned} \tag{2.53}$$

The barred lapse can transform like the original lapse only if

$$0=A^b\bar{\epsilon}^0\partial_b\bar{N}-A^b\bar{N}\partial_b\bar{\epsilon}^0+\bar{\epsilon}^0\{B,H[B\bar{N}]\}-\bar{N}\{B,H[B\bar{\epsilon}^0]\}\big|_{\text{O.S.}} , \tag{2.54}$$

which is only true if

$$\{B,H[\bar{\epsilon}^0]\}\big|_{\text{O.S.}}=\mathcal{B}\bar{\epsilon}^0+\mathcal{B}^a\partial_a\bar{\epsilon}^0\big|_{\text{O.S.}} , \tag{2.55}$$

with the higher order terms vanishing and where

$$A^a\big|_{\text{O.S.}}=-\mathcal{B}^a\big|_{\text{O.S.}} . \tag{2.56}$$

The shift can transform by following a similar method

$$\{A^a,H[\bar{\epsilon}^0]\}=\mathcal{A}^a\bar{\epsilon}^0+\mathcal{A}^{ab}\partial_b\bar{\epsilon}^0 , \tag{2.57}$$

with higher order terms vanishing and the structure function replaced by

$$\bar{q}^{ab}=B^2q^{ab}-\sigma B\mathcal{A}^{ab} , \tag{2.58}$$

this imposes that $\mathcal{A}^{ab}$ must be symmetric so the structure function can remain symmetric. The barred objects give a consistent spacetime metric

$$d\bar{s}^2=\sigma\bar{N}^2dt^2+\bar{q}_{ab}(dx^a+\bar{N}^adt)(dx^b+\bar{N}^bdt) , \tag{2.59}$$

and the constraint generators are smeared with the barred gauge functions

$$\bar{H}=BH+A^aH_a , \tag{2.60}$$
$$\bar{H}_a=H_a . \tag{2.61}$$

2.4.2 Anomaly-free condition

Using the new constraint generators we can find the new constraint algebra

$$\begin{aligned}\{\bar{H}[\bar{N}],\vec{\bar{H}}[\vec{\bar{M}}]\} &= \{H[B\bar{N}]+H_b[A^b\bar{N}],\bar{H}_a[\bar{M}^a]\}\\ &= -H\left[B\bar{M}^a\partial_a\bar{N}\right]-H_a\left[A^a\bar{M}^b\partial_b\bar{N}\right]\\ &= -\bar{H}\left[\bar{M}^b\partial_b\bar{N}\right] \ , \end{aligned} \tag{2.62}$$

$$\begin{aligned}\{\bar{H}[\bar{N}],\bar{H}[\bar{M}]\} &= \{H[B\bar{N}]+H_a[A^a\bar{N}],H[B\bar{M}]+H_a[A^a\bar{M}]\}\\ &= \bar{H}_a\left[\sigma B^2q^{ab}\left(\bar{M}\partial_b\bar{N}-\bar{N}\partial_b\bar{M}\right)\right]+H\left[BA^b\left(\bar{N}\partial_b\bar{M}-\bar{M}\partial_b\bar{N}\right)\right]\\ &+\int d^3xd^3y\ H(x)\Big(\{B(x),H(y)\}\left(\bar{N}(x)B(y)\bar{M}(y)-\bar{M}(x)B(y)\bar{N}(y)\right)\\ &+\{B(x),B(y)\}\bar{N}(x)H(y)\bar{M}(y)\\ &-\{A^a(y),H(x)\}B(x)\left(\bar{N}(x)H_a(y)\bar{M}(y)-\bar{M}(x)H_a(y)\bar{N}(y)\right)\\ &+\{B(x),A^a(y)\}H(x)\left(\bar{N}(x)H_a(y)\bar{M}(y)-\bar{M}(x)H_a(y)\bar{N}(y)\right)\\ &+\{A^a(x),A^b(y)\}H_a(x)\bar{N}(x)H_b(y)\bar{M}(y)\Big) \ . \end{aligned} \tag{2.63}$$

We used the following expression in determining the constraint algebra

$$\begin{aligned}\{H[B\bar{N}],\bar{H}_a[\bar{M}^a]\} &= \int d^3xd^3y\ \left(\{H(x),H_a(y)\}B(x)\bar{N}(x)\bar{M}^a(y)\right.\\ &\left.+\{B(x),\bar{H}_a(y)\}H(x)\bar{N}(x)\bar{M}^a(y)\right)\\ &= -H\left[\bar{M}^a\partial_a\left(B\bar{N}\right)\right]+H[\bar{M}^a\partial_aB\bar{N}]\\ &= -H\left[B\bar{M}^a\partial_a\bar{N}\right] \ , \end{aligned} \tag{2.64}$$

$$\begin{aligned}\{H_b[A^b\bar{N}],\bar{H}_a[\bar{M}^a]\} &= \int d^3xd^3y\ \left(\{H_b(x),\bar{H}_a(y)\}A^b(x)\bar{N}(x)\bar{M}^a(y)\right.\\ &\left.+\{A^b(x),\bar{H}_a(y)\}H_b(x)\bar{N}(x)\bar{M}^a(y)\right)\\ &= -H_a\left[\mathcal{L}_{\vec{\bar{M}}}\left(A^a\bar{N}\right)\right]+H_b[\mathcal{L}_{\vec{\bar{M}}}A^b\bar{N}]\\ &= -H_a\left[A^a\bar{M}^b\partial_b\bar{N}\right] \ , \end{aligned} \tag{2.65}$$

$$\begin{aligned}\{H[B\bar{N}],H[B\bar{M}]\} &= \int d^3xd^3y\ \Big(\{H(x),H(y)\}B(x)\bar{N}(x)B(y)\bar{M}(y)\\ &+\{H(x),B(y)\}B(x)\bar{N}(x)H(y)\bar{M}(y)+H(x)\{B(x),H(y)\}\bar{N}(x)B(y)\bar{M}(y)\\ &+H(x)\{B(x),B(y)\}\bar{N}(x)H(y)\bar{M}(y)\Big)\\ &= H_a\left[\sigma q^{ab}\left((B\bar{M})\partial_b(B\bar{N})-(B\bar{N})\partial_b(B\bar{M})\right)\right]\\ &+\int d^3xd^3y\ H(x)\Big(\{B(x),H(y)\}\left(\bar{N}(x)B(y)\bar{M}(y)-\bar{M}(x)B(y)\bar{N}(y)\right)\\ &+\{B(x),B(y)\}\bar{N}(x)H(y)\bar{M}(y)\Big) \ , \end{aligned} \tag{2.66}$$

$$\{H[B\bar{N}],H_a[A^a\bar{M}]\} = \int d^3xd^3y\ \Big(\{H(x),H_a(y)\}B(x)\bar{N}(x)A^a(y)\bar{M}(y)$$

$$
\begin{aligned}
&+\{H(x),A^a(y)\}B(x)\bar{N}(x)H_a(y)\bar{M}(y)\\
&+\{B(x),H_a(y)\}H(x)\bar{N}(x)A^a(y)\bar{M}(y)\\
&+\{B(x),A^a(y)\}H(x)\bar{N}(x)H_a(y)\bar{M}(y)\Big)\\
=&-H\left[(A^a\bar{M})\partial_b(B\bar{N})\right]+H\left[(A^a\bar{M})\partial_a B\bar{N}\right]\\
&+\int d^3x d^3y\,\Big(-\{A^a(y),H(x)\}B(x)\bar{N}(x)H_a(y)\bar{M}(y)\\
&+\{B(x),A^a(y)\}H(x)\bar{N}(x)H_a(y)\bar{M}(y)\Big)\\
=&-H\left[BA^a\bar{M}\partial_b\bar{N}\right]+\int d^3x d^3y\,\Big(\\
&-\{A^a(y),H(x)\}B(x)\bar{N}(x)H_a(y)\bar{M}(y)\\
&+\{B(x),A^a(y)\}H(x)\bar{N}(x)H_a(y)\bar{M}(y)\Big)\,,
\end{aligned}
\tag{2.67}
$$

$$
\begin{aligned}
\{H_a[A^a\bar{N}],H_b[A^b\bar{M}]\}=&\int d^3x d^3y\,\Big(\{H_a(x),H_b(y)\}A^a(x)\bar{N}(x)A^b(y)\bar{M}(y)\\
&+\{H_a(x),A^b(y)\}A^a(x)\bar{N}(x)H_b(y)\bar{M}(y)\\
&+\{A^a(x),H_b(y)\}H_a(x)\bar{N}(x)A^b(y)\bar{M}(y)\\
&+\{A^a(x),A^b(y)\}H_a(x)\bar{N}(x)H_b(y)\bar{M}(y)\Big)\\
=&\,H_b\left[\mathcal{L}_{\vec{A}\bar{N}}(A^b\bar{M})\right]-H_b[\bar{M}\mathcal{L}_{\vec{A}\bar{N}}A^b]+H_b[\bar{N}\mathcal{L}_{\vec{A}\bar{M}}A^b]\\
&+\int d^3x d^3y\,\{A^a(x),A^b(y)\}H_a(x)\bar{N}(x)H_b(y)\bar{M}(y)\\
=&\int d^3x d^3y\,\{A^a(x),A^b(y)\}H_a(x)\bar{N}(x)H_b(y)\bar{M}(y)\,.
\end{aligned}
\tag{2.68}
$$

For the algebra to be anaomaly-free, the H term in the second line needs to be canceled by other terms. The off-shell extensions of (2.55), (2.56), and (2.57) cancel this extra term

$$
\begin{aligned}
\{\bar{H}[\bar{N}],\bar{H}[\bar{M}]\}=&\,\bar{H}_a\left[\sigma\left(B^2q^{ab}-\sigma B\mathcal{A}^{ab}\right)\left(\bar{M}\partial_b\bar{N}-\bar{N}\partial_b\bar{M}\right)\right]\\
&-H\left[B\left(A^b+\mathcal{B}^b\right)\left(\bar{M}\partial_b\bar{N}-\bar{N}\partial_b\bar{M}\right)\right]\\
&+\int d^3x d^3y\,\Big(H(x)\{B(x),B(y)\}\bar{N}(x)H(y)\bar{M}(y)\\
&+\{B(x),A^a(y)\}H(x)\left(\bar{N}(x)H_a(y)\bar{M}(y)-\bar{M}(x)H_a(y)\bar{N}(y)\right)\\
&+\{A^a(x),A^b(y)\}H_a(x)\bar{N}(x)H_b(y)\bar{M}(y)\Big)\\
=&\,\bar{H}_a\left[\sigma\bar{q}^{ab}\left(\bar{M}\partial_b\bar{N}-\bar{N}\partial_b\bar{M}\right)\right]\\
&+\int d^3x d^3y\,\Big(H(x)\{B(x),B(y)\}\bar{N}(x)H(y)\bar{M}(y)\\
&+\{B(x),A^a(y)\}H(x)\left(\bar{N}(x)H_a(y)\bar{M}(y)-\bar{M}(x)H_a(y)\bar{N}(y)\right)\\
&+\{A^a(x),A^b(y)\}H_a(x)\bar{N}(x)H_b(y)\bar{M}(y)\Big)\,,
\end{aligned}
\tag{2.69}
$$

producing an anomaly-free hypersurface deformation algebra, provided the last three lines vanish off-shell. In spherically symmetric models the functions are such that the last three lines do vanish[3].

2.4.3 Covariance condition

We need to check if the new algebra satisfies the covariance condition

$$\frac{1}{\bar{\epsilon}^0}\{\bar{q}^{ab},\bar{H}[\bar{\epsilon}^0]\}\big|_{\text{O.S.}} = \frac{1}{\bar{N}}\{\bar{q}^{ab},\bar{H}[\bar{N}]\}\big|_{\text{O.S.}} \,. \tag{2.70}$$

This condition comes from $\delta_\epsilon \bar{q}^{ab} = \mathcal{L}_\xi \bar{q}^{ab}$ and using Poisson brackets to replace the δ_ϵ on the l.h.s. and the time derivative on the right-hand side (r.h.s.)[3]. We use the classical result

$$\{q^{ab},H[\bar{\epsilon}^0]\} =: \; Q^{ab}\bar{\epsilon}^0 \,, \tag{2.71}$$

and expand the l.h.s

$$\begin{aligned}
\{\bar{q}^{ab},\bar{H}[\bar{\epsilon}^0]\}\big|_{\text{O.S.}} &= \{\bar{q}^{ab},H[B\bar{\epsilon}^0]+H_c[A^c\bar{\epsilon}^0]\}\big|_{\text{O.S.}}\\
&= \{\bar{q}^{ab},H[B\bar{\epsilon}^0]\}+\mathcal{L}_{\vec{A}\bar{\epsilon}^0}\bar{q}^{ab}\big|_{\text{O.S.}}\\
&= \{B^2q^{ab}-\sigma B\mathcal{A}^{ab},H[B\bar{\epsilon}^0]\}+\mathcal{L}_{\vec{A}\bar{\epsilon}^0}\bar{q}^{ab}\big|_{\text{O.S.}}\\
&= B^2\{q^{ab},H[B\bar{\epsilon}^0]\}+q^{ab}\{B^2,H[B\bar{\epsilon}^0]\}-\sigma B\{\mathcal{A}^{ab},H[B\bar{\epsilon}^0]\}\\
&\quad -\sigma\mathcal{A}^{ab}\{B,H[B\bar{\epsilon}^0]\}+\mathcal{L}_{\vec{A}\bar{\epsilon}^0}\bar{q}^{ab}\big|_{\text{O.S.}}\\
&= \mathcal{L}_{\vec{A}\bar{\epsilon}^0}\bar{q}^{ab}+\int d^3y\,\Big(B^2\left(B(y)\bar{\epsilon}^0(y)\{q^{ab},H(y)\}\right)\\
&\quad +\left(2Bq^{ab}-\sigma\mathcal{A}^{ab}\right)\left(B(y)\bar{\epsilon}^0(y)\{B,H(y)\}\right)\\
&\quad -\sigma B\left(B(y)\bar{\epsilon}^0(y)\{\mathcal{A}^{ab},H(y)\}\right)\Big)\Big|_{\text{O.S.}}\\
&= \mathcal{L}_{\vec{A}\bar{\epsilon}^0}\bar{q}^{ab}+B^3Q^{ab}\bar{\epsilon}^0+\left(2Bq^{ab}-\sigma\mathcal{A}^{ab}\right)\left(B\mathcal{B}\bar{\epsilon}^0+\mathcal{B}^c\partial_c\left(B\bar{\epsilon}^0\right)\right)\\
&\quad -\sigma B\int d^3y\;B(y)\bar{\epsilon}^0(y)\{\mathcal{A}^{ab},H(y)\}\big|_{\text{O.S.}}\,,
\end{aligned} \tag{2.72}$$

where we define

$$\{\mathcal{A}^{ab},H[\bar{\epsilon}^0]\} = \Lambda^{ab}\bar{\epsilon}^0+\Lambda^{abc}\partial_c\bar{\epsilon}^0 \,. \tag{2.73}$$

Finally, the covariance condition requires

$$\begin{aligned}
\Lambda^{abc}\big|_{\text{O.S.}} &= \sigma B^{-2}\left(A^b\left(B^2q^{ac}-\sigma B\mathcal{A}^{ac}\right)+A^a\left(B^2q^{bc}-\sigma B\mathcal{A}^{bc}\right)+\left(2Bq^{ab}-\sigma\mathcal{A}^{ab}\right)B\mathcal{B}^c\right)\big|_{\text{O.S.}}\\
&= B^{-1}\left(\mathcal{B}^a\left(\mathcal{A}^{bc}-\sigma Bq^{bc}\right)+\mathcal{B}^b\left(\mathcal{A}^{ca}-\sigma Bq^{ca}\right)-\mathcal{B}^c\left(\mathcal{A}^{ab}-2\sigma Bq^{ab}\right)\right)\big|_{\text{O.S.}}\,.
\end{aligned} \tag{2.74}$$

This condition also forces the Jacobiator to vanish since only higher order derivative terms can give a non-vanishing Jacobiator and 2.70 and 2.71 ensure that no derivatives of ϵ^0 appear.

Chapter 3

Gemetrodynamical Formulation

3.1 Geometric theory

The following derivations in this chapter are to provide the geometric conditions that we imposed on the canonical theory. We begin with the same building blocks, a foliated globally hyperbolic topology, $M = \Sigma \times \mathbb{R}$, a spacetime metric

$$ds^2 = \sigma N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \ , \tag{3.1}$$

and a time-evolution vector field

$$t^\mu = N n^\mu + N^a s_a^\mu \ . \tag{3.2}$$

We also have an inverse metric

$$g^{\mu\nu} = q^{ab} s_a^\mu s_a^\nu + \frac{\sigma}{N^2} \left(t^\mu - N^a s_a^\mu\right) \left(t^\nu - N^b s_b^\nu\right) \ , \tag{3.3}$$

and an ADM decomposition of $\mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu \xi^\alpha + g_{\nu\alpha} \partial_\mu \xi^\alpha$

$$\mathcal{L}_\xi g_{ab} = \frac{\epsilon^0}{N} \dot{q}_{ab} + \epsilon^c \partial_c q_{ab} + q_{ca} \partial_b \epsilon^c + q_{cb} \partial_a \epsilon^c - \frac{\epsilon^0}{N} \left(N^c \partial_c q_{ab} + q_{ca} \partial_b N^c + q_{cb} \partial_a N^c\right) \ , \tag{3.4}$$

$$\begin{aligned} \mathcal{L}_\xi g_{ta} &= N^b \mathcal{L}_\xi g_{ba} + q_{ab} \left(\dot{\epsilon}^b + \epsilon^c \partial_c N^b - N^c \partial_c \epsilon^b - \sigma q^{bc} \left(\epsilon^0 \partial_c N - N \partial_c \epsilon^0\right)\right) \\ &=: \ N^b \mathcal{L}_\xi g_{ba} + q_{ab} \delta_\epsilon N^b \ , \end{aligned} \tag{3.5}$$

$$\begin{aligned} \mathcal{L}_\xi g_{tt} &= \sigma 2N \left(\dot{\epsilon}^0 + \epsilon^a \partial_a N - N^a \partial_a \epsilon^0\right) + N^a N^b \mathcal{L}_\xi g_{ab} \\ &\quad + 2 q_{ab} N^a \left(\dot{\epsilon}^b + \epsilon^c \partial_c N^b - N^c \partial_c \epsilon^b - \sigma q^{bc} \left(\epsilon^0 \partial_c N - N \partial_c \epsilon^0\right)\right) \\ &=: \ \sigma 2N \delta_\epsilon N + N^a N^b \mathcal{L}_\xi g_{ab} + 2 q_{ab} N^a \delta_\epsilon N^b \ , \end{aligned} \tag{3.6}$$

$$\begin{aligned} \mathcal{L}_\xi g^{t\mu} &= \ \xi^\alpha \partial_\alpha g^{t\mu} - g^{\alpha\mu} \partial_\alpha \xi^t - g^{t\alpha} \partial_\alpha \xi^\mu \\ &= \xi^t \partial_t g^{t\mu} - g^{t\mu} \partial_t \xi^t - g^{tt} \partial_t \xi^\mu + \xi^c \partial_c g^{t\mu} - g^{c\mu} \partial_c \xi^t - g^{tc} \partial_c \xi^\mu \end{aligned}$$

$$= \frac{\epsilon^0}{N}\partial_t g^{t\mu} - g^{t\mu}\partial_t\left(\frac{\epsilon^0}{N}\right) - g^{c\mu}\partial_c\left(\frac{\epsilon^0}{N}\right)$$

$$- \frac{\sigma}{N^2}\partial_t\xi^\mu + \left(\epsilon^c - \frac{\epsilon^0}{N}N^c\right)\partial_c g^{t\mu} + \frac{\sigma}{N^2}N^c\partial_c\xi^\mu\ , \tag{3.7}$$

$$\mathcal{L}_\xi g^{tt} = -\sigma\frac{2}{N^3}\left(\dot{\epsilon}^0 + \epsilon^c\partial_c N - N^c\partial_c\epsilon^0\right) = -\sigma\frac{2}{N^3}\delta_\epsilon N\ , \tag{3.8}$$

$$\mathcal{L}_\xi g^{ta} = -\sigma\frac{1}{N^2}\left(\delta_\epsilon N^a - \frac{2}{N}N^a\delta_\epsilon N\right)\ . \tag{3.9}$$

We want to check if the geometrodynamical formulation matches the canonocal formulation, and if they do not match, what conditions are needed so that they do.

3.2 Deformation of the lapse and shift

We find the expressions for the deformation of the lapse and shift by setting $\delta_\epsilon g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$ and using the spacetime metric and the ADM decomposition of the Lie derivatie of the metric

$$\delta_\epsilon N = \dot{\epsilon}^0 + \epsilon^a\partial_a N - N^a\partial_a\epsilon^0\ , \tag{3.10}$$

$$\delta_\epsilon N^a = \dot{\epsilon}^a + \epsilon^c\partial_c N^a - N^b\partial_b\epsilon^a - \sigma q^{ab}\left(\epsilon^0\partial_b N - N\partial_b\epsilon^0\right)\ , \tag{3.11}$$

$$\delta_\epsilon q_{ab} = \frac{\epsilon^0}{N}\dot{q}_{ab} + \epsilon^c\partial_c q_{ab} + q_{ca}\partial_b\epsilon^c + q_{cb}\partial_a\epsilon^c - \frac{\epsilon^0}{N}\left(N^c\partial_c q_{ab} + q_{ca}\partial_b N^c + q_{cb}\partial_a N^c\right)\ , \tag{3.12}$$

Here, the δ is an abstract transformation due to hypersurface deformations. We use the same δ as the canonical gauge because we will require that the canonical gauge transformation, (2.6) and (2.7), match the hypersurface deformations.

An arbitrary deformation generator must also transform similarly

$$\delta_{\epsilon_1}\epsilon^\mu_{(2)} - \delta_{\epsilon_2}\epsilon^\mu_{(1)} = \Delta^\mu\ , \tag{3.13}$$

$$\Delta^0 = \epsilon^b_{(1)}\partial_b\epsilon^0_{(2)} - \epsilon^b_{(2)}\partial_b\epsilon^0_{(1)}\ , \tag{3.14}$$

$$\Delta^a = \epsilon^b_{(1)}\partial_b\epsilon^a_{(2)} - \epsilon^b_{(2)}\partial_b\epsilon^a_{(1)} + \sigma q^{ab}\left(\epsilon_{(1)}\partial_b\epsilon^0_{(2)} - \epsilon_{(2)}\partial_b\epsilon^0_{(1)}\right)\ . \tag{3.15}$$

The deformation of the generators will be expanded upon in section 3.6

3.3 Deformation of the normal vector

The the normal vector to the foliated hypersurface is given by

$$n^\mu = \frac{\sigma}{\|\cdot\|}g^{\mu\alpha}\epsilon_{\alpha\beta\gamma\delta}\frac{\partial x^\beta}{\partial y^a}\frac{\partial x^\gamma}{\partial y^b}\frac{\partial x^\delta}{\partial y^c}\frac{\epsilon^{abc}}{3!}. \tag{3.16}$$

The following embedding can be used to evaluate infinitesimal deformations to the hypersurface

$$x^\mu = x^\mu + \epsilon^0 n^\mu + \epsilon^b s^\mu_b. \tag{3.17}$$

Substituting (3.17) into the derivatives of (3.16) gives a new normal vector up to first order in the generators

$$n^\mu_{(t1)} = \frac{\sigma}{\|\cdot\|_{(t1)}}\left(g^{t\mu}(x_{(t1)}) - q^{\mu b}\frac{\partial\epsilon^0}{\partial y^b} + X^\mu\epsilon^0 + Y^{\mu b}_a\frac{\partial\epsilon^\alpha}{\partial y^b}\right) + O\left((\epsilon^0)^2, (\epsilon^\alpha)^2, \epsilon^0\epsilon^a\right)\ , \tag{3.18}$$

where

$$X^t = -\frac{\sigma}{N^3}\partial_a N^a \ , \tag{3.19}$$

$$X^a = \frac{\sigma}{N^3}N^a\partial_b N^b \ , \tag{3.20}$$

$$Y_a^{\mu b} = g^{t\mu}\delta_b^a = \frac{\sigma}{N^2}(t^\mu - N^c s_c^\mu)\delta_a^b \ , \tag{3.21}$$

$$Y_a^{tb} = \frac{\sigma}{N^2}\delta_a^b \ , \tag{3.22}$$

$$Y_a^{cb} = -\frac{\sigma}{N^2}N^c\delta_a^b \ . \tag{3.23}$$

By performing the embedding we must also take into account a coordinate transformation in the metric components

$$\delta_\epsilon \bar{n}^\mu_{(t)} = \mathcal{L}_\xi g^{t\mu} - q^{\mu b}\frac{\partial \epsilon^0}{\partial y^b} + X^\mu \epsilon^0 + Y_a^{\mu b}\frac{\partial \epsilon^a}{\partial y^b} \ , \tag{3.24}$$

$$\delta_\epsilon \bar{n}^t_{(t)} = \sigma\left[-\frac{2\delta_\epsilon N}{N^3} + \frac{1}{N^3}\left(N\partial_a\epsilon^a - \epsilon^0\partial_a N^a\right) + \frac{1}{N^2}N^a\partial_a\epsilon^0\right] \ , \tag{3.25}$$

$$\begin{aligned}\delta_\epsilon \bar{n}^a_{(t)} &= \sigma\left[-N^a\left(-\frac{2\delta_\epsilon N}{N^3} + \frac{1}{N^3}\left(N\partial_b\epsilon^b - \epsilon^0\partial_b N^b\right) + \frac{1}{N^2}N^b\partial_b\epsilon^0\right) - \frac{1}{N^2}\delta_\epsilon N^a\right] \\ &= -N^a\delta_\epsilon\bar{n}^t - \frac{\sigma}{N^2}\delta_\epsilon N^a \ .\end{aligned} \tag{3.26}$$

The embedding also causes the normalization to be deformed, $\|\cdot\|_{(t1)} = \|\cdot\|_{(t)} + \delta_\epsilon \|\cdot\|_{(t)}$. The new normalization can be found using

$$\begin{aligned}\sigma &= g^{(t1)}_{\mu\nu} n^\mu_{(t1)} n^\nu_{(t1)} \\ &= \frac{1}{\|\cdot\|^2_{(t1)}}\left(g^{(t)} + \mathcal{L}_\xi g^{(t)}_{\mu\nu}\right)\left(\bar{n}^\mu_{(t)} + \delta_\epsilon\bar{n}^\mu_{(t)}\right)\left(\bar{n}^\nu_{(t)} + \delta_\epsilon\bar{n}^\nu_{(t)}\right) + O\left((\epsilon^0)^2\right)\right) \\ &= \frac{1}{\|\cdot\|^2_{(t1)}}\left(g^{(t)}_\mu \nu\bar{n}^\nu_{(t)}\bar{n}^\nu_{(t)} + 2g^{(t)}_{\mu\nu}\bar{n}^\mu_{(t)}\delta_\epsilon\bar{n}^\nu_{(t)} + \bar{n}^\mu_{(t)}\bar{n}^\nu_{(t)}\mathcal{L}_\xi g^{(t)}_{\mu\nu}\right) + O(\epsilon^2) \\ &= \frac{1}{\|\cdot\|^2_{(t1)}}\left(\sigma\|\cdot\|^2_{(t)} + 2g^{(t)}_{\mu\nu}\bar{n}^\mu_{(t)}\delta_\epsilon\bar{n}^\nu_{(t)} + \bar{n}^\mu_{(t)}\bar{n}^\nu_{(t)}\mathcal{L}_\xi g^{(t)}_{\mu\nu}\right) + O(\epsilon^2).\end{aligned} \tag{3.27}$$

Using the deformed normalization and the above equation we find

$$\begin{aligned}\delta_\epsilon\|\cdot\|_{(t)} &= \frac{\sigma}{\|\cdot\|_{(t)}}\left(\bar{n}^\mu_{(t)}g^{(t)}_{\mu\nu}\delta_\epsilon\bar{n}^\nu_{(t)} + \frac{1}{2}\bar{n}^\mu_{(t)}\bar{n}^\nu_{(t)}\mathcal{L}_\xi g^{(t)}_{\mu\nu}\right) + O(\epsilon^2) \\ &= n^\mu_{(t)}g^{(t)}_{\mu\nu}\delta_\epsilon\bar{n}^\nu_{(t)} + \frac{\sigma}{2N}n^\mu_{(t)}n^\nu_{(t)}\mathcal{L}_\xi g^{(t)}_{\mu\nu} + O(\epsilon^2) \\ &= n^\mu_{(t)}g^{(t)}_{\mu\nu}\delta_\epsilon\bar{n}^\nu_{(t)} + \frac{\sigma}{2n^3}\left(\mathcal{L}_\xi g^{(t)}_{tt} - 2n^b\mathcal{L}_\xi g^{(t)}_{bt} + N^bN^c\mathcal{L}_\xi g^{(t)}_{bc}\right) + O(\epsilon^2) \\ &= \sigma N\delta_\epsilon\bar{n}^t + \frac{1}{N^2}\delta_\epsilon N + O(\epsilon^2) \ .\end{aligned} \tag{3.28}$$

With all the above pieces the normal vector can be reassembled

$$n^\mu_{(t1)} = \frac{\sigma}{\|\cdot\|_{(t)}}\left(1 - \frac{\delta_\epsilon\|\cdot\|_{(t)}}{\|\cdot\|_{(t)}}\right)\left(\bar{n}^\nu_{(t)} + \delta_\epsilon\bar{n}^\mu_{(t)}\right) + O((\epsilon^0)^2)$$

$$
\begin{aligned}
&= \frac{\sigma}{\|\cdot\|_{(t)}} \left(\bar{n}^{\mu}_{(t)} - \frac{\delta_\epsilon \|\cdot\|_{(t)}}{\|\cdot\|_{(t)}} \bar{n}^{\mu}_{(t)} + \delta_\epsilon \bar{n}^{\mu}_{(t)} \right) + O((\epsilon^0)^2) \\
&= n^{\mu}_{(t)} + \left(-n^{\mu}_{(t)} N \delta_\epsilon \|\cdot\|_{(t)} + \sigma N \delta_\epsilon \bar{n}^{\mu}_{(t)} \right) + O((\epsilon^0)^2) \\
&=: \; n^{\mu}_{(t)} + \delta_\epsilon n^{\mu}_{(t)} + O(e^2). \qquad (3.29)
\end{aligned}
$$

We see that

$$
\begin{aligned}
\delta_\epsilon n^t &= -\frac{1}{N^2} \delta_\epsilon N \;, \qquad (3.30) \\
\delta_\epsilon n^a &= \frac{N^a}{N^2} \delta_\epsilon N - \frac{1}{N} \delta_\epsilon N^a \;, \qquad (3.31) \\
\delta_\epsilon n^\mu &= \delta_\epsilon n^t t^\mu + \delta_\epsilon n^a s^\mu_a \\
&= N \delta_\epsilon n^t n^\mu + \left(N^a \delta_\epsilon n^t + \delta_\epsilon n^a \right) s^\mu_a \\
&= -\frac{\delta_\epsilon N}{N} n^\mu - \frac{\delta_\epsilon N^a}{N} s^\mu_a. \qquad (3.32)
\end{aligned}
$$

This agrees with the canonical prediction (2.16).

3.4 Deformation of the spatial basis vector

It is important to check how the basis vectors change under the deformation. The condition the spatial basis vector must satisfy is

$$
\begin{aligned}
0 &= g^{(t1)}_{\mu\nu} N^{\mu}_{(t1)} s^{\nu}_{a(t1)} \\
&= \left(g^{(t1)}_{\mu\nu} + \mathcal{L}_\xi g^{(t1)}_{\mu\nu} \right) \left(n^{\mu}_{(t)} + \delta_\epsilon n^{\mu}_{(t)} \right) \left(s^{\nu}_{a(t)} + \delta_\epsilon s^{\nu}_{a(t1)} \right) + O((\epsilon^0)^2) \\
&= g^{(t)}_{\mu\nu} n^{\mu}_{(t)} \delta_\epsilon s^{\nu}_{a(t)} + g^{(t)}_{\mu\nu} s^{\nu}_{a(t)} \delta_\epsilon n^{\mu}_{(t)} + n^{\mu}_{(t)} s^{\nu}_{a(t)} \mathcal{L} - \xi g^{(t)}_{\mu\nu} + O(\epsilon^2) \;. \qquad (3.33)
\end{aligned}
$$

Which gives

$$
\begin{aligned}
\left(g^{(t)}_{t\nu} - N^b g^{(t)}_{b\nu} \right) \delta_\epsilon s^{\nu}_{a(t)} &= -N g^{(t)}_{\mu a} \delta_\epsilon n^{\mu}_{(t)} - N n^{\mu}_{(t)} \mathcal{L}_\xi g^{(t)}_{\mu a} \\
&= \left(\frac{1}{N} g^{(t)}_{ta} - \frac{N^b}{N} g^{(t)}_{ba} \right) \delta_\epsilon N + g^{(t)}_{ba} \delta_\epsilon N^b - \mathcal{L}_\xi g^{(t)}_{ta} + N^b \mathcal{L}_\xi g^{(t)}_{ba} \\
&= q^{(t)}_{ba} \delta_\epsilon N^b - \mathcal{L}_\xi g^{(t)}_{ta} + N^b \mathcal{L}_\xi g^{(t)}_{ba} \\
&= 0 \;. \qquad (3.34)
\end{aligned}
$$

The time component in the parenthesis on the l.h.s is non-zero, thus $\delta_\epsilon s^t_{a(t)} = 0$, but it is zero for the spatial components. The spatial basis remains spatial, $\delta_\epsilon s^c_{a(t)} = 0$, by requiring $q^{(t1)}_{ab} = g^{(t1)}_{\mu\nu} s^{\mu}_{a(t1)} s^{\nu}_{b(t1)}$. Confirming that the basis vectors remain unchanged is needed to ensure any physical conclusion drawn from this geometrodynamical method are valid.

3.5 Deformation of the normal flow, time flow, and the time evolution vector field

Other vector fields will be affected by the deformation. The normal flow transforms like

$$
n^\mu \to \; n^\mu + \delta_\epsilon n^\mu = \left(1 - \frac{\delta_\epsilon N}{N} \right) n^\mu - \frac{\delta_\epsilon N^a}{N} s^\mu_a \;, \qquad (3.35)
$$

as a consequence of the normal vector changing. However, the time evolution vector field, $t^\mu = Nn^\mu + N^a s_a^\mu$, does not change

$$\begin{aligned} t^\mu \to\ t^\mu + \delta_\epsilon t^\mu &= t^\mu + \delta\left(Nn^\mu + N^a s_a^\mu\right) \\ &= t^\mu + N\left(\delta_\epsilon n^\mu + \frac{\delta_\epsilon N}{N} n^\mu + \frac{\delta_\epsilon N^a}{N} s_a^\mu\right) \\ &= t^\mu \,. \end{aligned} \tag{3.36}$$

If $H[\epsilon^0]$ and $H_a[\epsilon^a]$ are abstract generators for normal and spatial deformations respectively, then the time evolution abstract generator, $H[N] + H_a[N^a]$, remains unchanged. The normal abstract generator, H, does deform, as a consequence of the deformation of the normal vector, to

$$H \to\ H - H\frac{\delta_\epsilon N}{N} - H_a\frac{\delta_\epsilon N^a}{N} =: H^{\delta_\epsilon}\,, \tag{3.37}$$

which matches with the canonical prediction 2.21.

3.6 Deformation of the gauge functions

We have two different vector fields $\xi^\mu_{(1)} = \xi^t_{(1)} t^\mu + \xi^a_{(1)} s^\mu_a = \epsilon^0_{(1)} n^\mu + \epsilon^a_{(1)} s^\mu_a$ and $\xi^\mu_{(2)} = \xi^t_{(2)} t^\mu + \xi^a_{(2)} s^\mu_a = \epsilon^0_{(2)} n^\mu + \epsilon^a_{(2)} s^\mu_a$. After a deformation generated by $\xi^\mu_{(1)}$, the components of $\xi^\mu_{(2)}$ will be deformed, $\mathcal{L}_{\xi_1}\xi^\mu_{(2)} = \xi^\nu_{(1)}\partial_\nu \xi^\mu_{(2)} - \xi^\nu_{(2)}\partial_\nu \xi^\mu_{(1)}$. The ADM decomposition of the components are

$$\begin{aligned} \mathcal{L}_{\xi_1}\xi^t_{(2)} &= \xi^\nu_{(1)}\partial_\nu \xi^\mu_{(2)} - \xi^\nu_{(2)}\partial_\nu \xi^\mu_{(1)} \\ &= \xi^t_{(1)}\partial_t \xi^t_{(2)} - \xi^t_{(2)}\partial_t \xi^t_{(1)} + \xi^a_{(1)}\partial_a \xi^t_{(2)} - \xi^a_{(2)}\partial_a \xi^t_{(1)} \\ &= \frac{\epsilon^0_{(1)}}{N}\partial_t\left(\frac{\epsilon^0_{(2)}}{N}\right) - \frac{\epsilon^0_{(2)}}{N}\partial_t\left(\frac{\epsilon^0_{(1)}}{N}\right) + \left(\epsilon^a_{(1)} - \frac{\epsilon^0_{(1)}}{N}N^a\right)\partial_a\left(\frac{\epsilon^0_{(2)}}{N}\right) \\ &\quad - \left(\epsilon^a_{(2)} - \frac{\epsilon^0_{(2)}}{N}N^a\right)\partial_a\left(\frac{\epsilon^0_{(1)}}{N}\right) \\ &= \frac{1}{N^2}\left(\epsilon^0_{(1)}\dot\epsilon^0_{(2)} - \epsilon^0_{(2)}\dot\epsilon^0_{(1)} - N^a\left(\epsilon^0_{(1)}\partial_a\epsilon^0_{(2)} - \epsilon^0_{(2)}\partial_a\epsilon^0_{(1)}\right) + \epsilon^0_{(1)}\epsilon^a_{(2)}\partial_a N - \epsilon^0_{(2)}\epsilon^a_{(1)}\partial_a N\right) \\ &\quad + \frac{1}{N}\left(\epsilon^a_{(1)}\partial_a\epsilon^0_{(2)} - \epsilon^a_{(2)}\partial_a\epsilon^0_{(1)}\right) \\ &= \frac{1}{N^2}\left(\epsilon^0_{(1)}\delta_{\epsilon_2}N - \epsilon^0_{(2)}\delta_{\epsilon_1}N\right) + \frac{1}{N}\left(\epsilon^a_{(1)}\partial_a\epsilon^0_{(2)} - \epsilon^a_{(2)}\partial_a\epsilon^0_{(1)}\right)\,, \end{aligned} \tag{3.38}$$

$$\begin{aligned} \mathcal{L}_{\xi_1}\xi^a_{(2)} &= \frac{1}{N}\left(\epsilon^0_{(1)}\left(\dot\epsilon^a_{(2)} + \epsilon^b_{(2)}\partial_b N^a - N^b\partial_b\epsilon^a_{(2)}\right) - \epsilon^0_{(2)}\left(\dot\epsilon^a_{(1)} + \epsilon^b_{(1)}\partial_b N^a - N^b\partial_b\epsilon^a_{(1)}\right)\right) \\ &\quad + \epsilon^b_{(1)}\partial_b\epsilon^a_{(2)} - \epsilon^b_{(2)}\partial_b\epsilon^a_{(1)} - N^a\mathcal{L}_{\xi_1}\xi^t_{(2)} \\ &= \frac{1}{N}\Big(\epsilon^0_{(1)}\left(\delta_{\epsilon_2}N^a + \sigma q^{ab}\left(\epsilon^0_{(2)}\partial_b N - N\partial_b\epsilon^0_{(2)}\right)\right) \\ &\quad - \epsilon^0_{(2)}\left(\delta_{\epsilon_1}N^a + \sigma q^{ab}\left(\epsilon^0_{(1)}\partial_b N - N\partial_b\epsilon^0_{(1)}\right)\right)\Big) \\ &\quad + \epsilon^b_{(1)}\partial_b\epsilon^a_{(2)} - \epsilon^b_{(2)}\partial_b\epsilon^a_{(1)} - N^a\mathcal{L}_{\xi_1}\xi^t_{(2)}\,, \end{aligned} \tag{3.39}$$

$$\begin{aligned} \mathcal{L}_{\xi_1}\xi^\mu_{(2)} &= \left(\mathcal{L}_{\xi_1}\xi^t_{(2)}\right)t^\mu + \left(\mathcal{L}_{\xi_1}\xi^a_{(2)}\right)s^\mu_a \\ &= \left(\epsilon^a_{(1)}\partial_a\epsilon^0_{(2)} - \epsilon^a_{(2)}\partial_a\epsilon^0_{(1)}\right)n^\mu + \left(\epsilon^b_{(1)}\partial_b\epsilon^a_{(2)} - \epsilon^b_{(2)}\partial_b\epsilon^a_{(1)} - \sigma q^{ab}\left(\epsilon^0_{(1)}\partial_b\epsilon^0_{(2)} - \epsilon^0_{(2)}\partial_b\epsilon^0_{(1)}\right)\right)s^\mu_a \end{aligned}$$

$$
\begin{aligned}
&+ \epsilon^0_{(2)}\delta_{\epsilon_1} n^\mu - \epsilon^0_{(1)}\delta_{\epsilon_2} n^\mu \\
=:&\ \Delta^0 n^\mu + \Delta^a s^\mu_a + \epsilon^0_{(2)}\delta_{\epsilon_1} n^\mu - \epsilon^0_{(1)}\delta_{\epsilon_2} n^\mu \ .
\end{aligned} \tag{3.40}
$$

Commuting $\delta_{\epsilon_1}\xi^\mu_{(2)} - \delta_{\epsilon_2}\xi^\mu_{(1)}$ gives

$$
\begin{aligned}
\delta_{\epsilon_1}\xi^\mu_{(2)} - \delta_{\epsilon_2}\xi^\mu_{(1)} &= \left(\delta_{\epsilon_1}\epsilon^0_{(2)} - \delta_{\epsilon_2}\epsilon^0_{(1)}\right) n^\mu + \left(\delta_{\epsilon_1}\epsilon^a_{(2)} - \delta_{\epsilon_2}\epsilon^a_{(1)}\right) s^\mu_a + \epsilon^0_{(2)}\delta_{\epsilon_1} n^\mu - \epsilon^0_{(1)}\delta_{\epsilon_2} n^\mu \\
&= \mathcal{L}_{\xi_1}\xi^\mu_{(2)} \ .
\end{aligned} \tag{3.41}
$$

The r.h.s of the equation shows that there are extra terms compared to the canonical transformation of the gauge functions. These extra terms can be accounted for in the canonical formulation by including the deformation of the normal constraint, which would represent the deformation of the normal vector in (3.41).

Chapter 4

Conclusion

We have shown that the canonical formulation for the hypersurface deformation algebra is not complete without considering phase space dependence of the lapse and shift and the transformation of the normal vector. We have indeed achieved our goal of taking the phase space dependence of the gauge generators into account , and have showed that the theory is consistent with having them inside the brackets. We have also studied the geometric formulation, from which we derived the conditions that canonical theories must satisfy for them to describe a geometric and covariant theory of spacetime. If the canonical theory does not comply with the these geometric conditions, then it is only a gauge theory, not a spacetime theory. We also provided an example of a new modified gravity theory that is covariant. While it is too difficult to apply to the full 4D theory, it represents the generalization of a procedure previously applied to spherical symmetry, containing a singularity-free black hole solution[3][4], and it can be similarly applied to other symmetry reduced models. We plan to apply these conditions to the polarized Gowdy model, where work in progress shows that one obtains a new modified gravity theory.

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